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SUBGROUPS WITH THE CHARACTER RESTRICTION PROPERTY AND RELATED TOPICS

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ABSTRACT. Let $P \in \mathrm{Syl}_p(G)$, $N = \mathrm{N}_G(P)$. Isaacs (J. Algebra 100 (1986), 403–420, Theorem B) has proved that if N has the character restriction property in G, then G contains a normal subgroup M such that $N \cap M = \{1\}$ and $G = N \cdot M$ (it follows from the classification of finite simple groups, that M is solvable). We present a character free version of the theorem above (see Theorem 5) and give the different proof of another Isaacs' result (the cited paper, Theorem A). Proposition 7 is a weak form of a character free version of Theorem 2. A new case of the existence of normal complements to subgroups having the character restriction property yields Proposition 10.

Let G be a finite group, $H \leq G$. We will say that H has the *character restriction property* in G (H is a CR-subgroup of G or H has property CR in G) if every irreducible character of H is the restriction of some (irreducible) character of G (see [HH]). If H has a normal complement in G, it is a CR-subgroup of G (but the converse, in general, is not true: every subgroup of an abelian group G has property CR in G). If $H \leq G'$ and H has property CR in G, then $H' = H$.

Isaacs has proved the following two results:

Theorem 1. [I1, Theorem B] *Let $P \in \mathrm{Syl}_p(G)$, where p is a prime divisor of $|G|$. If $N = N_G(P)$ has property CR in G, then N has a normal complement in G.*

Besides, a normal complement in Theorem 1 is solvable (see the remark following Theorem 5).

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Theorem 2. [I1, Theorem A] *Let π be a set of primes. Let $H \leq G$ be a solvable π-subgroup and suppose H maximal with this property. If H satisfies CR in G, then G has a normal π-complement and H is a π-Hall subgroup of G.*

In this note we generalize Theorem 1 and give another proof of Theorem 2. Many related results are proved as well.

In what follows, we make use of the following known results:

Lemma 3. *(a) (Isaacs (*[I1, Lemma 2.2], *see also* [K, Lemma 27.5.3]*) Let H be a maximal solvable π-subgroup of G. If $L \leq G$ is such that $H \leq N_G(L)$ and $L \cap H = \{1\}$, then L is a π'-subgroup. In particular, if H is a maximal solvable subgroup of HL, then $L = \{1\}$.*

(b) (Tate [T]*; see also* [I2], *Theorem 6.31) Let H be normal in G and $P \in \mathrm{Syl}_p(G)$. If $H \cap P \leq \Phi(P)$ (where $\Phi(P)$ is the Frattini subgroup of P), then H is p-nilpotent.*

(c) (Clemens; see [C]*) Let a p-group P act on a p'-group G in such a way that $C_G(P) = \{1\}$. Then G is solvable.*

(d) (see [I1], *Proposition 1.1 or* [K], *Lemma 27.5.5) Suppose that $H \leq G$ satisfies CR in G. If K is normal in H and K^G is the normal closure of K in G, then $K^G \cap H = K$ and HK^G/K^G satisfies CR in G/K^G.*

(e) (Sah [S]*; see also* [K], *Theorem 26.2.2) Let H be a solvable Hall subgroup of G. If H satisfies CR in G, then H has a normal complement in G.*

Definition 1. A triple (G, H, K) is said to be *special* in G if $K \trianglelefteq H \leq G$ and $H \cap K^G = K$.

Definition 2. A subgroup H is said to be an NR-*subgroup* of G (Normal Restriction) if, whenever K is normal in H, the triple (G, H, K) is special in G.

Triples (G, H, H) and $(G, H, \{1\})$ are special in G for all $H \leq G$.

Lemma 4. *Let $K \trianglelefteq H \leq T \leq G$ and the triple (G, H, K) is special in G. Then*

(a) The triple (T, H, K) is special in T.

(b) If $L/K^G \trianglelefteq HK^G/K^G$ and the triple $(G, H, L \cap H)$ is special in G, then the triple $(G/K^G, HK^G/K^G, L/K^G)$ is special in G/K^G (in particular, if $K \trianglelefteq G$, then the triple $(G/K, H/K, L/K)$ is special in G/K).

(c) If H is an NR-subgroup of G, then HK^G/K^G is an NR-subgroup in G/K^G.

(d) Every CR-subgroup is also an NR-subgroup.

Proof. The equality $K \leq H \cap K^T \leq H \cap K^G = K$ proves (a).

To prove (b), enough to show that $(L^G/K^G) \cap (HK^G/K^G) = L/K^G$ or, what is the same, $L^G \cap HK^G = L$. It follows from $K^G \leq L \leq K^G H$ that $L = K^G(L \cap H)$, by the modular law. Therefore, $L^G = K^G(L \cap H)^G = [K(L \cap H)]^G = (L \cap H)^G$ (since, by assumption, $K \leq L \cap H$). By the modular law, $L^G \cap HK^G = (L \cap H)^G \cap HK^G = K^G[(L \cap H)^G \cap H]$. The right hand-side of the last equality is equal to $K^G(L \cap H)$ (since $(G, H, L \cap H)$ is special by assumption; obviously, $L \cap H \trianglelefteq H$). By what was proved above, $K^G(L \cap H) = L$. Thus, $L^G \cap HK^G = L$, as desired. Obviously, (b) implies (c). (d) is known (see [I1]). $\square$

If H is a subgroup of prime order in a group G, then H is an NR-subgroup but not a CR-subgroup if $G = G'$. As Lemma 4 shows, NR-subgroups have some important properties of CR-subgroups (see Lemma 3(d)).

Recall that a group is *p-closed* (*p-nilpotent*) if it has a normal Sylow p-subgroup (a normal p-complement).

The following theorem is a character free version of Theorem 1:

Theorem 5. *Let $P \in \mathrm{Syl}_p(G)$, $N = N_G(P)$. If the triples (G, N, P) and $(G, N, \Phi(P))$ are special, then N has a normal complement in G.*

PROOF. Write $L = P^G$. By Frattini's Lemma, $LN = G$. Note that P is self-normalizing in L (in fact, $\mathrm{N}_L(P) = L \cap N = P^G \cap N = P$ since the triple (G, N, P) is special). If L has a normal p-complement, it is a normal complement to N in G (since, by what has just been said, $N \cap L = P$). Therefore, enough to show that L is p-nilpotent. Write $T = \Phi(P)^G$. Then $\Phi(P) \leq T \cap P \leq \Phi(P)^G \cap N = \Phi(P)$ (since the triple $(G, N, \Phi(P))$ is special), and so $T \cap P = \Phi(P)$. Note that the abelian subgroup PT/T is self-normalizing in L/T, and thus L/T has a normal p-complement H/T by Burnside's Normal p-Complement Theorem. Since $PH = L$ and $P \cap H = P \cap (PT \cap H) = P \cap T = \Phi(P)$, it follows from Lemma 3(b) that L has a normal p-complement, as desired. $\square$

Remark. If N is a normalizer of a Sylow p-subgroup of G, then every normal complement H to N in G is solvable. Indeed, then $\mathrm{N}_{PH}(P) = PH \cap N = P$, and so H is solvable by Lemma 3(c). Thus, the normal complement to N in Theorem 5 is solvable.

Corollary 6. *Let N be a normalizer of a Sylow p-subgroup P of G. If every irreducible character of N of p'-degree is the restriction of an irreducible character of G, then N has a solvable normal complement in G.*

PROOF. In fact, all irreducible characters of N/P and $N/\Phi(P)$ have p'-degrees by Ito's Theorem (see [I2, Theorem 6.15]). It follows that (G, N, P) and $(G, N, \Phi(P))$

are special triples in G (since the intersection of kernels of the irreducible characters of N/K is K/K, and so $K^G \cap N = K$, where $K \in \{P, \Phi(P)\}$; recall that all irreducible characters of N/K have extensions to G by what was said above and assumption). Therefore, Theorem 5 implies the result. $\square$

PROOF OF THEOREM 2 Assume that the theorem is proved for all groups of order $< |G|$. We may assume that $H > \{1\}$.

Let R be a minimal normal subgroup of H. Then R is an elementary abelian p-group for some $p \in \pi$. Write $K = R^G$. By Lemma 3(d), $K \cap H = R$ and HK/K has property CR in G/K.

Suppose that $HK < G$. Then the pair $H \leq HK$ satisfies the assumption of the theorem. By the induction hypothesis, $HK = H \cdot L$, where L is a normal π'-Hall subgroup of HK. Since HK/K is a π-group, it follows that $L \leq K$, and so $L \lhd G$ (since L is characteristic in $K \lhd G$). Suppose that $HK/K \leq F/K$, where F/K is a maximal solvable π-subgroup of G/K. Obviously, L is a normal π'-Hall subgroup of F. Therefore, by the theorem of Schur-Zassenhaus, F contains a π-Hall subgroup H_1 such that $H \leq H_1$. Since H_1 is a solvable π-subgroup, by assumption $H_1 = H$, and so $HK/H = H_1K/K = F/K$. Thus, H/K is a maximal solvable π-subgroup of G/K. Therefore, by the induction hypothesis, $G/K = HK/K \cdot S/K$, where $S \lhd G$ and $S \cap HK = K$. It follows that $HS = G$ and $S \cap H = S \cap (HK \cap H) = (S \cap HK) \cap H = K \cap H = R$. If $HK = G$, then $H \cap K = R$ (see the second paragraph of the proof). Thus, in any case, G contains a normal subgroup S such that $G = HS$ and $H \cap S = R$.

Suppose that $R \lhd G$ (in that case, $K = R$). Then H/R is a maximal solvable π-subgroup of G/R (see the previous paragraph), and hence, by Lemma 3(a), S/R is a π'-group. Then H is a π-Hall subgroup of G, and Lemma 3(e) shows that G has a normal π-complement.

Let $R \ntrianglelefteq G$. Then $N = \mathrm{N}_G(R) < G$ and $H \leq N$. Since the pair $H \leq N$ satisfies assumptions of the theorem, it follows by the induction hypothesis, that $N = H \cdot T$, where $H \cap T = \{1\}$ and T is a normal π'-Hall subgroup of N. Therefore, $[R, T] = \{1\}$. Now, $T \leq S$ since G/S is a π-group. By the modular law, $N = H(N \cap S)$. Now, $H \cap (N \cap S) = H \cap S = R$, so that $|N| = \frac{|H| \cdot |N \cap S|}{|R|}$. Thus $|N \cap S| = \frac{|N| \cdot |R|}{|H|} = |T| \cdot |R|$, and hence $R \times T = RT = N \cap S = \mathrm{N}_S(R)$. In particular, R is an abelian Sylow p-subgroup of S. By Burnside's Normal p-Complement Theorem, S has a normal p-complement U. It follows that $U \lhd G$ (since U is characteristic in $S \lhd G$), $U \cap H = \{1\}$ (since $(|H|, |U|) = 1$) and

$G = H \cdot U$. By Lemma 3(a), H is a π-Hall subgroup of G, completing the proof. □

It should be noticed that our proof of Theorem 2 depends on Lemma 3(e) while Isaacs' proof does not.

Note that if H is a maximal solvable π-subgroup and NR-subgroup of G, then H is not necessarily a Hall subgroup of G (for example, G is a direct product of PSL(2, 5) and the cyclic group of order 3, $\pi = \{3, 5\}$ and H is cyclic of order 15), i.e., a character free version of Theorem 2 is not true. But the following weaker result holds:

Proposition 7. *Let H be a maximal solvable subgroup of G. If H is an NR-subgroup of G, then $H = G$.*

PROOF. Suppose that G is a counterexample of minimal order. By Lemma 4(a) and the induction hypothesis, H is maximal in G. Clearly, $H > \{1\}$. Let K be a minimal normal subgroup of H.Then $K^G \cap H = K$ and K is a p-subgroup for some prime p. If $K^G = K$, then H/K is a maximal solvable subgroup of G/K and H/K is a NR-subgroup of G/K (Lemma 4(c)). By the induction hypothesis, $H/K = G/K$, contrary to the assumption $H < G$. Thus, $K^G > K$, and so $K^G \not\leq H$. Thus, $HK^G = G$ since H is maximal in G. It follows that $\mathrm{N}_G(K) = H$, and so $\mathrm{N}_{K^G}(K) = H \cap K^G = K$. Therefore, K is a self normalizing abelian Sylow p-subgroup of K^G. By Burnside's Normal p-Complement Theorem, K^G has a normal p-complement R. Then $G = HK^G = HKR = H \cdot R$ and R is normal in G (since R is characteristic in $K^G \lhd G$) and $R \cap H = \{1\}$, contrary to Lemma 3(a). □

Conjecture 1. *Let H be a Hall subgroup of G and let $N = \mathrm{N}_G(H)$. If N has property CR in G, then N has a normal complement in G (this is true if $N = H$* [F]*; the proof in* [F] *depends on the classification of finite simple groups).*

If N of Conjecture 1 is an NR-subgroup of G, then, in general, N has no normal complement in G (for example, $G = \mathrm{PSL}(2, 11)$, $H = \mathrm{PSL}(2, 5)$).

Corollary 8. *Let N be a normalizer of a Sylow p-subgroup P of a group G. Suppose that, for $K \in \{P, \Phi(P)\}$, there exists a subgroup L in G such that $N \cap L = K$ and $NL = G$. Then N has a normal complement in G.*

PROOF. By Theorem 5, it suffices to show that (G, N, P) and $(G, N, \Phi(P))$ are special triples in G. This follows from the following lemma. □

Lemma 9. *Let $K \trianglelefteq H \leq G$. If there exists a subgroup L such that $HL = G$ and $H \cap L = K$, then the triple (G, H, K) is special.*

PROOF. If $g \in G$, write $g = hl$ with $h \in H$ and $l \in L$. Then $K^g = (K^h)^l = K^l \leq L$, and thus $K^G \leq L$. Therefore $K \leq K^G \cap H \leq L \cap H = K$, and so $K^G \cap H = K$, as required. □

We are indebted to the referee for a shorter proof of Corollary 8. (Of course, Lemma 9 is not new: (it was proved in one of the papers of S.A. Chunikhin about 60 years ago.)

Conjecture 2. *If all maximal subgroups are NR-subgroups of G, then G is solvable.*

Proposition 10 generalizes and strengthens the following known result: if a Carter subgroup H of a solvable group G has property CR in G, then $H = G$.

Proposition 10. *Let H be a nilpotent subgroup of G such that $N_G(L) = L$ for all subgroups L of G containing H. If H is an NR-subgroup of G, then H has a normal complement in G, and this complement coincides with the last member of the lower central series of G.*

PROOF. [1] Write G^∞ to denote the last term of the lower central series of G. Note that $HG^\infty = G$ since H is self normalizing. What must be shown is that $H \cap G^\infty = \{1\}$. If this is false, let p be a prime divisor of $|H \cap G^\infty|$ and let N be the normal p-complement of H. Write $M = N^G$. By the NR property, $M \cap H = N$. By assumption, MH/M is self normalizing in G/M and by Lemma 4(c), MH/M has NR property in G/M. Therefore, by induction, $MH/M \cdot T/M = G/M$, where T is normal in G and $MH \cap T = M$. We get $G = HMT = HT$, and so $G^\infty \leq T$. Thus,

$$G^\infty \cap H \leq T \cap H = T \cap (MH \cap H) = (T \cap MH) \cap H = M \cap H = N,$$

a contradiction since p does not divide $|N|$ and so it does not divide $|G^\infty \cap H|$. □

Proposition 11. *If all Sylow subgroups of G are NR-subgroups, then G is supersolvable.*

PROOF. Let p be the smallest prime divisor of $|G|$, $P \in \mathrm{Syl}_p(G)$. We will prove by induction on $|G|$ that if P is an NR-subgroup of G, then G is p-nilpotent. Let R be a normal subgroup of order p in P. Then $P \cap R^G = R$ and PR^G/R^G is an NR-subgroup of G/R^G (Lemma 4(c)). By the induction hypothesis, $G/R^G = PR^G/R^G \cdot S/R^G$, where S/R^G is a normal p-complement of G/R^G. Then $G = PS$ and $P \cap S = (P \cap PR^G) \cap S = P \cap (PR^G \cap S) = P \cap R^G = R$, and so R is a Sylow

[1] This proof is due to the referee.

subgroup of order p in S. Since p is the smallest prime divisor of $|S|$, it follows that S has a normal p-complement H. Since H normal in G and $G = P \cdot H$, the result follows.

We will prove that G has an ordered Sylow tower such as every supersolvable group has. Let p be the smallest prime divisor of G. By the result of the previous paragraph, G has a normal p-complement H. Let $Q \in \mathrm{Syl}(H)$. By Lemma 4(a), Q is an NR-subgroup of H. Therefore, H is supersolvable by induction (in particular, H has the desired Sylow tower), proving the claim.

Let q be the largest prime divisor of G, $Q \in \mathrm{Syl}_q(G)$; by the result of the previous paragraph, Q is normal in G. If Q_0 is a normal subgroup of Q, then $Q_0^G = Q_0$ since $Q_0^G \leq Q$, and so Q_0 is normal in G. Let $G = T \cdot Q$, where T is a q-complement of G. By Lemma 4(a) and induction, T is supersolvable. Since $G/Q \cong T$ is supersolvable and the indices of the segment $Q > Q_1 > \cdots > \{1\}$ of a principal series of G are equal to q (by what we have just been proved), all is done. $\square$

If all Sylow subgroups of G are cyclic, it satisfies the assumption of Proposition 11 (so G of that proposition is not necessarily nilpotent). However, it follows from Lemma 3(e) that G is nilpotent if all its Sylow subgroups are CR-subgroups.

Question 1. *Let $G' < G$. Study the structure of G if G' has property CR in G (in that case $G'' = G'$).*

Question 2. *Let H be the solvable residual of G. Study the structure of G if H has the property CR in G.*

For some related results, see [F] and [HH].

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References

[C] K-H. Clemens, Fixpunktfreie Automorphismen endlicher Gruppen, Diplomarbeit, Mathematisches Institut der Albert-Ludwigs-Universität, Freiburg i. Br., 1978.

[F] P. Ferguson, Character restriction and relative normal complements, J. Algebra **124** (1989), 47–53.

[G] W. Gaschütz, Über die Φ-Untergruppe endlicher Gruppen, Math. Z. **58** (1953), 160–170.

[H] B. Huppert, Endliche Gruppen, Bd. 1, Springer, Berlin, 1967.

[HH] T.O. Hawkes and J.F. Humphreys, A character-theoretic criterion for the existence of normal complements to subgroups of finite groups, J. Algebra **94** (1985), 382–387.

[I1] I.M. Isaacs, Subgroups with the character restriction property, J. Algebra **100** (1986), 403–420.

[I2] I.M. Isaacs, Character Theory of Finite Groups, Academic Press, New York, 1976.

[K] G. Karpilovsky, Group Representations, Volume 1, Part B, North-Holland, Amsterdam, 1992.

[S] C.H. Sah, Existence of normal complements and extensions of characters in finite groups, Illinois J. Math. **6, 2** (1962), 282–291.

[T] J. Tate, Nilpotent quotient groups, Topology **3** (1964), 109–111.

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