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FRATTINI LIKE PROPERTIES OF KERNELS OF SOME CHARACTERS

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ABSTRACT. Let K be minimal among the kernels of the irreducible characters of a finite group G, and let Q/K be a normal subgroup of G/K. Then Q is π-closed, where π is a set of primes, if and only if Q/K is. This result improves Theorem 12.24 in I.M. Isaacs, Characters of Finite Groups, Academic Press, New York, 1976. In particular, G is π-closed if and only if G/K is. The analog of the result above holds for minimal quasikernels too. Some related results are proved.

Throughout this note only finite groups and their complex characters are considered. In what follows, π, π' are complementary sets of primes. A group G is *π-closed* if its maximal normal π-subgroup $O_\pi(G)$ is a π-Hall subgroup of G. Obviously, subgroups and epimorphic images of π-closed groups are π-closed. A group G is *p-nilpotent* if it is $\{p\}'$-closed. Let $\pi(G)$ be the set of prime divisors of $|G|$ and $\mathrm{Irr}(G)$, $\mathrm{Irr}_1(G)$ the sets of irreducible, nonlinear irreducible characters of G, respectively.

It is well known [G] that the Frattini subgroup $\Phi(G)$ of G has the following properties:

(Φ1) If N is normal in G, then N is π-closed if and only if $N\Phi(G)/\Phi(G)$ is.

(Φ2) If N is normal in G, then $\Phi(N) \leq \Phi(G)$.

(Φ3) $\pi(G/\Phi(G)) = \pi(G)$.

(Φ4) If N is normal in G and $N \leq \Phi(A)$ for some $A \leq G$, then $N \leq \Phi(G)$.

(Φ1) and (Φ2) are also true if N is subnormal in G.

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In this note we will show that the kernels of some characters have properties similar to (Φ1)–(Φ3).

Let K be minimal among the kernels of irreducible characters of a group G (in that case, we will call K a *minimal kernel* of G). By [I], Corollary 12.20, K is nilpotent. Using that result, we will prove the following

Theorem 1. *Let $K = \ker(\chi)$ be a minimal kernel of G and Q/K a normal subgroup of G/K. Then Q is π-closed if and only if Q/K is.*

PROOF. By [I], Corollary 12.20, K is nilpotent.

If Q is π-closed, then, obviously, Q/K is. Therefore, it remains to show that, if Q/K is π-closed, then Q is. In what follows we assume, therefore, that Q/K is π-closed. Let Q_0/K be a (normal) π-Hall subgroup of Q/K. Since Q_0/K is characteristic in Q/K, it follows that Q_0 is normal in G. Obviously, $|Q : Q_0|$ is a π'-number. Since K is nilpotent, it follows by Schur-Zassenhaus Theorem that Q_0 contains a π-Hall subgroup H (indeed, Q_0 contains a normal nilpotent π'-Hall subgroup that coincides with the π'-Hall subgroup of K). Obviously, H is a π-Hall subgroup of Q. Suppose that we have proved that H is normal in Q_0. Since H is characteristic in Q_0, it is normal in Q, and so Q is π-closed. Therefore we may assume, without loss of generality, that $Q_0 = Q$. Since, in that case, Q/K is a π-group, we get $Q = HK$. We have to prove that H is normal in G. Assume that this is false. Since all π-Hall subgroups of Q are conjugate by the Schur-Zassenhaus Theorem, $G = \mathrm{N}_G(H)Q = \mathrm{N}_G(H)HK = \mathrm{N}_G(H)K$, by Frattini's argument; $\mathrm{N}_G(H) < G$ since H is not normal in G. If M is a maximal subgroup of G containing $\mathrm{N}_G(H)$, then $G = MK$, and so $\chi_M = \lambda \in \mathrm{Irr}(M)$. Since $\ker(\lambda^G) \leq \ker(\lambda) \leq M$, it follows that $K \not\leq \ker(\lambda^G)$. Therefore, there exists $\vartheta \in \mathrm{Irr}(\lambda^G)$ such that $K \not\leq L = \ker(\vartheta)$.

Assume that $L \leq M$. Then, by reciprocity,

$$L = M \cap L = M \cap \ker(\vartheta) = \ker(\vartheta_M) \leq \ker(\lambda) = \ker(\chi_M) = M \cap K < K,$$

contrary to the choice of K as a minimal kernel. Thus $L \not\leq M$, and hence $G = LM$. In that case, $\vartheta_M \in \mathrm{Irr}(M)$. Then $\vartheta_M = \lambda$ since $\lambda \in \mathrm{Irr}(\vartheta_M)$ by reciprocity, and

$$M \cap L = M \cap \ker(\vartheta) = \ker(\vartheta_M) = \ker(\lambda) = K \cap M.$$

Put $\ker(\lambda) = N$. Note that $N = \ker(\chi_M) = K \cap M$.

Since $K \cap L \cap M = K \cap (K \cap M) = K \cap M$ we get $K \cap M \leq L$ and

$$|(K \cap L)M| = \frac{|K \cap L| \cdot |M|}{|K \cap L \cap M|} = \frac{|K \cap L| \cdot |M|}{|K \cap M|} < \frac{|K| \cdot |M|}{|K \cap M|} = |KM| = |G|,$$

and hence $K \cap L < M$. Therefore, $K \cap L \leq K \cap M \leq K \cap L$, and so $K \cap L = K \cap M$. Thus,

$$L \cap M = K \cap M = K \cap L = N. \tag{1}$$

Now, by what has been proved and (1),

$$\begin{aligned} |Q : Q \cap M| &= |QM : M| = |G : M| = |ML : M| = |L : L \cap M| \\ &= |L : K \cap L| = |KL : K|. \end{aligned} \tag{2}$$

Since $H \leq Q \cap M$ and H is a π-Hall subgroup of Q, it follows from (2) that $|KL/K|$ is a π'-number. Since $KH/K = Q/K$ is a π-group, we obtain $KL/K \cap KH/K = \{1\}$, and so KL/K and KH/K commute element by element: $KH = Q$ and KL are normal in G since K and L are normal in G as kernels of characters of G. In particular, $[L, KH] \leq K$. Therefore, $[L, KH] \leq K \cap L = N$, so that $[L, NH] \leq [L, KH] \leq N \leq NH$. Thus $L \leq \mathrm{N}_G(NH)$. Since $N = M \cap K$, $KH = Q$, $MQ = G = MK$ and $H < M$, it follows, by the modular law, that $NH = (M \cap K)H = M \cap KH = M \cap Q$, and hence NH is normal in M since Q is normal in G. Then $\mathrm{N}_G(NH) \geq \langle M, L \rangle = G$ and NH is normal in G. Since H is a π-Hall subgroup of $KH = Q$, it follows that H is a π-Hall subgroup of $NH \leq KH$. Therefore, since NH as a subgroup of Q has a normal π'-Hall subgroup, π-Hall subgroups of NH are conjugate by the Schur-Zassenhaus Theorem, an so, by Frattini's argument, $G = \mathrm{N}_G(H)HN = \mathrm{N}_G(H)N \leq MN = M$, contrary to assumption. Thus H is normal in G, as claimed. □

The theorem is also true if Q is subnormal in G.

Remark. If K is as in the theorem and Q/K is a nilpotent normal subgroup of G, then Q is nilpotent as well. It follows that, if a minimal kernel K coincides with the Fitting subgroup of a nonidentity group G, then G/K is semisimple. If $F/\ker(\chi)$, where $\chi \in \mathrm{Irr}(G)$, is the Fitting subgroup of $G/\ker(\chi)$ and F is not nilpotent, then $\ker(\chi)$ is not a minimal kernel, by the theorem. This result coincides with [I], Theorem 12.24.

Corollary 2. *If $K = \ker(\chi)$ $(\chi \in \mathrm{Irr}(G))$ is a minimal kernel of G, then $\pi(G) = \pi(G/K)$.*

PROOF. Set $\pi = \pi(G/K)$ and let K_1 be a π'-Hall subgroup of the nilpotent subgroup K (see [I], Corollary 12.20); obviously, K_1 is normal in G. Then by the theorem $G = Q \times K_1$, where Q is a π-Hall subgroup of G. Assume that $\pi(G) \neq \pi$. Then $K_1 > \{1\}$. We have $\chi = \tau \times \mu$, where $\tau \in \mathrm{Irr}(Q)$, $\mu \in \mathrm{Irr}(K_1)$. Since

$$K = \ker(\chi) = \ker(\tau) \times \ker(\mu) = (K \cap Q) \times (K \cap K_1) = (K \cap Q) \times K_1,$$

it follows that $\ker(\mu) = K_1$, i.e., $\mu = 1_{K_1}$. Let μ_1 be a nonprincipal character of K_1. Set $\chi_1 = \tau \times \mu_1$. Then $\ker(\chi_1) < \ker(\chi) = K$, contrary to the choice of K. Hence $K_1 = \{1\}$ and $\pi(G) = \pi = \pi(G/K)$. □

We see that minimal kernels behave similarly the Frattini subgroup and, as $\Phi(G)$, they are nilpotent. But, in general, minimal kernels are not contained in $\Phi(G)$ (take $G = \mathrm{E}(p^n)$, the elementary abelian group of order $p^n > p$; see also the example below).

Let $\chi \in \mathrm{Irr}(G)$. The subgroup $\mathrm{Z}(\chi) = \{x \in G \mid |\chi(x)| = \chi(1)\}$ is called the *quasikernel* of χ. It is known that $\mathrm{Z}(\chi)$ is normal in G, $\ker(\chi) \leq \mathrm{Z}(\chi)$ and $\mathrm{Z}(\chi)/\ker(\chi) = \mathrm{Z}(G/\ker(\chi))$ and the last group is cyclic; see [I], Lemma 2.27. A *minimal quasikernel* is defined similarly to a minimal kernel.

Lemma 3. *Let $Z(\chi)$ be a minimal quasikernel of G, where $\chi \in \mathrm{Irr}(G)$. If $\tau \in \mathrm{Irr}(G)$ is such that $\ker(\tau) \leq \ker(\chi)$, then $Z(\tau) = Z(\chi)$.*

PROOF. If $\mathrm{Z}(\tau) \leq \mathrm{Z}(\chi)$, the result holds since $\mathrm{Z}(\chi)$ is minimal. Setting $H = \mathrm{Z}(\tau)\ker(\chi)$, we see that $H/\ker(\chi) \leq \mathrm{Z}(G/\ker(\chi)) = \mathrm{Z}(\chi)/\ker(\chi)$, and so $\mathrm{Z}(\tau) \leq H \leq \mathrm{Z}(\chi)$. □

Corollary 4. *If $Z = Z(\chi)$ ($\chi \in \mathrm{Irr}(G)$) is a minimal quasikernel of G, then Z is nilpotent.*

PROOF. By the lemma, we may assume, without loss of generality, that $\ker(\chi)$ is a minimal kernel of G. In that case, $Z/\ker(\chi) = \mathrm{Z}(G/\ker(\chi))$ is nilpotent, and the result follows from the theorem. □

Corollary 5. *Let $Z = Z(\chi)$ ($\chi \in \mathrm{Irr}(G)$) be a minimal quasikernel of G and let $Q/Z(\chi)$ be normal in $G/Z(\chi)$. Then Q/Z is π-closed if and only if Q is.*

PROOF. In view of the lemma, we may assume that $K = \ker(\chi)$ is a minimal kernel of G. As in the proof of the theorem, we may assume, without loss of generality, that Q/Z is a π-group. In that case, $Q/\ker(\chi)$ as an extension of $Z/\ker(\chi) = \mathrm{Z}(G/\ker(\chi))$ by a π-group Q/Z, is π-closed. By the theorem, Q is π-closed, as desired. □

Let Z be as in Corollary 5. Then G is nilpotent if and only if G/Z is nilpotent. But the analog of Corollary 1 does not hold for minimal quasikernels (let G be a nonidentity abelian group).

Corollary 6. *Let π be a set of primes. Suppose that $\chi \in \mathrm{Irr}(G)$ is such that $|\ker(\chi)|_\pi$ is minimal with respect to divisibility. Then $\ker(\chi) = A \times B$, where A is a nilpotent π-Hall subgroup of $\ker(\chi)$.*

PROOF. Set $K = \ker(\chi)$. Let $\tau \in \mathrm{Irr}(G)$ be such that $L = \ker(\tau)$ is a minimal kernel with $L \leq K$. Since $|L|_\pi$ divides $|K|_\pi$, it follows that $|K|_\pi = |L|_\pi$, by the choice of K. Since L is nilpotent ([I], Corollary 12.20), it follows that A, a π-Hall subgroup of L (and of K), is normal in G and so in K. Obviously, K/L is a π'-group. Therefore, by the theorem, $K = A \times B$, where B is a π'-Hall subgroup of K. Since L is nilpotent and $A \leq L$, the result follows. □

We will show that the nilpotency is the only restriction on the structure of minimal kernels. Indeed, let K be a nilpotent group and $\pi(K) = \{p_1, \ldots, p_n\}$. Let $H = P_1 \times \cdots \times P_n$, where P_i is a nonabelian group of order p_i^3, $i = 1, \ldots, n$. Let ϕ_i be an irreducible character of P_i of degree p_i, $i = 1, \ldots, n$. Set $G = H \times K$ and $\chi = \phi_1 \times \cdots \times \phi_n \times 1_K$. Then $\chi \in \mathrm{Irr}(G)$ and $\ker(\chi) = K$. We claim that K is a minimal kernel of G. Indeed, let N be a proper normal divisor of K; then N is normal in G. By the construction of G, $Z(G/N)$ is not cyclic. Therefore, there is no $\tau \in \mathrm{Irr}(G)$ such that $\ker(\tau) = N$ ([I], Lemma 2.27(d), (f)). This means that K is a minimal kernel of G. Similarly, let $G = C \times K$, where C is cyclic of order $p_1 \ldots p_n$ with the same p_i as above. Then K, C are minimal kernels of G. Moreover, in this case, K is the kernel of a linear character of G.

Question 1. *Let G be a nonabelian p-group such that the kernel of some linear character λ of G is minimal. Study the embedding of* $\ker(\lambda)$ *in G.*

Question 2. *Let G be a nonabelian p-group such that some minimal kernel $K =$* $\ker(\chi)$ *($\chi \in \mathrm{Irr}(G)$ is nonlinear) has index p^3 in G. Study the embedding of K in G.*

Question 3. *Classify minimal kernels of: (a) a standard wreath product of two elementary abelian p-groups, (b) $A \times B$ in terms of A and B.*

Proposition 7. *Let*

$$m = \min\{|\ker(\chi)| \mid \chi \in \mathrm{Irr}(G)\},\ X = \{\chi \mid \chi \in \mathrm{Irr}(G),\ |\ker(\chi)| = m\}.$$

Then $D = \bigcap_{\chi \in X} \ker(\chi) \leq \Phi(G)$.

PROOF. Let $\chi \in X$, $K = \ker(\chi)$. Suppose that $K \not\leq \Phi(G)$. Then there exists $H < G$ such that $G = KH$. We assume that $|H|$ is as small as possible. Then $K \cap H \leq \Phi(H)$. Obviously, $\chi_H = \theta \in \mathrm{Irr}(H)$. Let $\tau \in \mathrm{Irr}(\theta^G)$. Then, by reciprocity,

$$H \cap \ker(\tau) = \ker(\tau_H) \leq \ker(\theta) = \ker(\chi_H) = K \cap H. \tag{3}$$

Assume that $\ker(\tau) \leq H$. Then, by (3), $\ker(\tau) \leq K$, and so $\ker(\tau) = K$ by the minimal choice of K. Since $K \not\leq H$, we obtain a contradiction. Hence it follows that $HL = G = HK$, where $L = \ker(\tau)$. Then by (3), $|L| = |K| \cdot \dfrac{|L \cap H|}{|K \cap H|} \leq |K|$, so that $|L| = |K|$ by the choice of K and $\tau \in X$, $|H \cap L| = |H \cap K|$. Hence by (3), $H \cap \ker(\tau) = H \cap K$ for every $\tau \in \mathrm{Irr}(\theta^G)$. Since $\ker(\theta^G) \leq H$, it follows that $\ker(\theta^G) = H \cap K$. Since $\ker(\theta^G)$ is normal in G and $\ker(\theta^G) = H \cap K \leq \Phi(H)$, it follows that $\ker(\theta^G) \leq \Phi(G)$ by (Φ4). Since $\tau \in \mathrm{Irr}(\theta^G)$ is arbitrary, we obtain that $\mathrm{Irr}(\theta^G) \subseteq X$. Therefore, $D = \bigcap_{\psi \in X} \ker(\psi) \subseteq \ker(\theta^G) \leq \Phi(G)$, proving the proposition. □

It is easy to prove that the intersection of the kernels of characters $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = \max\{\tau(1) \mid \tau \in \mathrm{Irr}(G)\}$ is also contained in $\Phi(G)$.

Let p be a prime. Set

$$\mathrm{Irr}_1(G, p') = \{\chi \in \mathrm{Irr}_1(G) \mid p \nmid \chi(1)\},\ G(p') = \bigcap_{\chi \in \mathrm{Irr}(G,p')} \ker(\chi),$$

$$\mathrm{Irr}(G, p) = \{\chi \in \mathrm{Irr}(G) \mid p \mid \chi(1)\},\ G(p) = \bigcap_{\chi \in \mathrm{Irr}(G,p)} \ker(\chi).$$

Remark. We claim that $G(p)$ is p-closed and its Sylow p-subgroup is abelian. By the Michler-Ito Theorem (see [I], Corollary 12.34 and [M]), it suffices to show that all characters in $\mathrm{Irr}_1(G(p))$ have p'-degrees. Assume that $\mu \in \mathrm{Irr}_1(G(p))$ has degree divisible by p. Take $\chi \in \mathrm{Irr}(\mu^G)$; then $G(p) \not\leq \ker(\chi)$. Then, by Clifford's Theorem, $\mu(1)$ divides $\chi(1)$, and so $G(p) \leq \ker(\chi)$, a contradiction. The subgroup $G(p)$ is characteristic in G. It is known that $G(p')$ has a normal p-complement and solvable; see [B], Proposition 9 and Remark 1 following it. Next, we will prove that $G(p') \leq G'$ unless p divides the degrees of all nonlinear irreducible characters of G (and then G is p-nilpotent by [I], Corollary 12.2). Indeed, let χ be a nonlinear irreducible character of G of p'-degree, λ a linear character of G, $x \in G(p')$. Then $\lambda\chi \in \mathrm{Irr}(G, p')$, and so $\chi(1) = \chi(x) = (\lambda\chi))(x) = \lambda(x)\chi(x) = \lambda(x)\chi(1)$. It follows that $\lambda(x) = 1$ so $x \in G'$ since λ is an arbitrary linear character of G. Similarly, $G(p) \leq G'$ unless all characters in $\mathrm{Irr}(G)$ have p'-degrees (and then G is p-closed with abelian Sylow p-subgroup by [M]; note that the proof in [M] depends on the classification on finite simple groups).

Let N be a normal subgroup of G. Then $\Phi(N) \leq \Phi(G)$, by (Φ2). But, in general, $N(p') \not\leq G(p')$ (take $G = \mathrm{S}_4$, $N = \mathrm{A}_4$ and $p = 3$: in that case, $N = N(3')$ is of order 12 and $|G(3')| = 4$), and $N(p) \not\leq G(p)$ (take the same G, N and $p = 2$). However, the following result holds:

Proposition 8. *If N is a normal subgroup of G, then $N(p') \cap N' \leq G(p')$. In particular, if $N(p') \nleq G(p')$, then N is p-nilpotent.*

PROOF. Let $\chi \in \mathrm{Irr}_1(G,p')$ and let $\chi_N = e(\mu_1 + \cdots + \mu_t)$ be the Clifford decomposition. If $\mu_1(1) = 1$, then $N(p') \cap N' \leq N' \leq \ker(\mu_1 + \cdots + \mu_t) = \ker(\chi_N) \leq \ker(\chi)$. If $\mu_1(1) > 1$, then p does not divide $\mu_i(1)$ for all i since $\mu_i(1)$ divides $\chi(1)$ by Clifford's Theorem, and so $N(p') \cap N' \leq N(p') \leq \ker(\mu_1 + \cdots + \mu_t) = \ker(\chi_N) \leq \ker(\chi)$. Therefore, $N(p') \cap N' \leq \bigcap_{\chi \in \mathrm{Irr}_1(G,p')} \ker(\chi) = G(p')$. Suppose that $N(p') \nleq G(p')$. Then, by what has just been proved, $N(p') \nleq N'$, and so $\mathrm{Irr}_1(N,p')$ is empty by the Remark; in that case, N is p-nilpotent by [I], Corollary 12.2. □

For a normal subgroup N of G, set $\mathrm{Irr}_1(G \mid N) = \mathrm{Irr}_1(G) - \mathrm{Irr}(G/N)$.

The following proposition was inspired by [IK], Theorem D.

Proposition 9. *Let $K \leq N$ be normal subgroups of G.*

(a) If $NG(p)/G(p)$ is p-closed, then N is. In particular, if N/K is p-closed and all characters in $\mathrm{Irr}_1(G \mid K)$ have p′-degrees, then N is p-closed.

(b) If $NG(p')/G(p')$ is p-nilpotent, then N is. In particular, if N/K is p-nilpotent and all characters in $\mathrm{Irr}_1(G \mid K)$ have degrees divisible by p, then N is p-nilpotent (moreover, if, in addition, N/K is solvable, then N is also solvable; compare to [IK], Theorem D).

PROOF. Let G be a counterexample of minimal order.

(a) Let us prove the first assertion. Without loss of generality, we may assume that $G(p) \leq N$ and, by induction, applied to $N/\mathrm{O}_p(G(p)) \leq G/\mathrm{O}_p(G(p))$, we get $\mathrm{O}_p(G(p)) = \{1\}$, i.e., $G(p)$ is a p'-subgroup by the Remark. Let $P/G(p)$ be a (normal) Sylow p-subgroup of $N/G(p)$. Obviously, P is p-closed if and only if N is. Therefore, we may assume that $P = N$, i.e., $N/G(p)$ is a p-group. Since N is not p-closed, $\{1\} < G(p) < N$. Assume that $\mathrm{O}_p(N) > \{1\}$. Set $\overline{G} = G/\mathrm{O}_p(N)$. Take $\chi \in \mathrm{Irr}(\overline{G}, p)$ (if such a χ does not exist, $\overline{G}$ is p-closed, by Ito-Michler Theorem; in that case, G and so N are p-closed). We see that $\mathrm{O}_p(N)G(p) \leq \ker(\chi)$ (here we consider χ as a character of G). It follows that $\overline{G(p)} \leq \overline{G}(p) = \overline{H}$. Then NH/H, as an epimorphic image of $N/G(p)$, is p-closed. Since $NH/H \cong \overline{NH}/\overline{H}$, it follows by induction that $\overline{NH}$ is p-closed. Since $\mathrm{O}_p(N)$ is a p-group, NH is p-closed so N is, and G is not a counterexample. Thus $\mathrm{O}_p(N) = \{1\}$. Let $L/G(p)$ be a minimal normal subgroup of $G/G(p)$ contained in $N/G(p)$. Then $L/G(p)$ is a (nonidentity) elementary abelian p-group. By what has been just proved, L is not p-closed. Therefore, by Ito's Theorem (see [I], Corollary 12.34

for solvable groups, but the same proof works for p-solvable groups as well), L has an irreducible character λ of degree divisible by p. Let $\chi \in \mathrm{Irr}(\lambda^G)$. By Clifford's Theorem, p divides $\chi(1)$ so $\chi \in \mathrm{Irr}(G,p)$. Then $G(p) \leq \ker(\chi)$, and so $G(p) \leq L \cap \ker(\chi) = \ker(\chi_L) \leq \ker(\lambda)$, which is not the case since $L/G(p)$ is abelian and λ is nonlinear. The first assertion in (a) is proved.

Suppose that N/K is p-closed and all characters in $\mathrm{Irr}_1(G \mid K)$ have p'-degrees (here we do not assume that $G(p)$ is contained in N). Then $K \leq G(p)$. In that case, $NG(p)/G(p) \cong N/N \cap G(p)$ as an epimorphic image of the p-closed group N/K is also p-closed. Therefore, by what has been proved in the previous paragraph, $NG(p)$ is p-closed so is N. The proof of (a) is completed.

(b) Let us prove the first assertion. Without loss of generality, we may assume that $G(p') \leq N$ and, by induction, $\mathrm{O}_{p'}(G(p')) = \{1\}$, i.e., $G(p')$ is a p-subgroup by the Remark. Let $H/G(p')$ be a (normal) p'-Hall subgroup of $N/G(p')$. Obviously, N is p-nilpotent if and only if H is. Therefore, we may assume, without loss of generality, that $H = N$; in that case $N/G(p')$ is a p'-subgroup and $G(p')$ is a p-subgroup. Since G is a counterexample, $N > G(p') > \{1\}$. Let $P \in \mathrm{Syl}_p(G)$; then $G(p') \leq P$. We claim that $G(p') \not\leq \Phi(P)$. Assume that this is false. Then $G(p') \leq \Phi(G)$ by ($\Phi4$), and so N is p-nilpotent since $N/G(p')$ is, which is a contradiction. Since $P' \leq \Phi(P)$, it follows that $G(p') \not\leq P'$, and so P possesses a linear character λ such that $G(p') \not\leq \ker(\lambda)$. Since $\lambda^G(1) = |G : P| \not\equiv 0 \pmod{p}$, there exists $\chi \in \mathrm{Irr}(\lambda^G)$ whose degree is not divisible by p. By reciprocity, $G(p') \not\leq \ker(\chi)$. By the Remark, $G(p') \leq G'$ (otherwise, G is p-nilpotent, and so N is). In that case, $G' \not\leq \ker(\chi)$, i.e., χ is nonlinear and $\chi \in \mathrm{Irr}_1(G,p')$. Then $G(p') \leq \ker(\chi)$, contrary to what has just been proved.

Now suppose that N/K is p-nilpotent and the degrees of all characters in $\mathrm{Irr}_1(G \mid K)$ are divisible by p (here we do not assume that $G(p')$ is contained in N). In that case, $K \leq G(p')$. Then $NG(p')/G(p') \cong N/N \cap G(p')$ is p-nilpotent as an epimorphic image of the p-nilpotent group N/K. By what has been proved in the previous paragraph, $NG(p')$ is p-nilpotent so is N. Since K is solvable, the last assertion of (b) is obvious. □

In general, $G(p')$, $G(p)$ are not necessarily subgroups of $\Phi(G)$. If $G = \mathrm{S}_4$, then $G(2) = G(3') \not\leq \Phi(G) = \{1\}$ since $|G(2)| = 4$. If $W = \mathrm{ASL}(3,2) = \mathrm{SL}(3,2)\cdot\mathrm{E}(2^3)$, where $\mathrm{E}(2^3)$ is elementary abelian of order 8, and $G/\mathrm{E}(2^3)$ is a subgroup of order 21 in $W/\mathrm{E}(2^3)$, then $G(3) = G(7') = \mathrm{E}(2^3) \not\leq \Phi(G) = \{1\}$.

Let N be subnormal in G (i.e., N is a member of some composition series of G), let $NG(p)/G(p)$ be p-closed and $K/G(p)$ its normal p-Sylow subgroup. Then $K/G(p)$ is a subnormal p-subgroup of $G/G(p)$. It is easy to prove by induction that $K/G(p) \leq H/G(p) = \mathrm{O}_p(G/G(p))$. By Proposition 7(a), H is p-closed. Therefore, K is p-closed and subnormal in N. Since $|N : K|$ is a p'-number, a Sylow p-subgroup P of K is a Sylow subgroup of N. Since P is subnormal in G, it is subnormal in N. Then P is normal in N, and hence N is p-closed. Similarly, a subnormal subgroup N is p-nilpotent if and only if $NG(p')/G(p')$ is.

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