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ON THE EQUIVALENCE OF THE FOMIN EXTENSION AND THE BANASCHEWSKI-FOMIN-ŠANIN EXTENSION

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ABSTRACT. In 1947, Katětov asked for necessary and sufficient conditions on a space X so that its Fomin extension σX and its Banaschewski-Fomin-Šanin extension μX are equivalent. This question was raised again by Tikoo in 1985 in his study of the Banaschewski-Fomin-Šanin extension. In this paper we present an answer to this question and look at several examples related to it. Prime open filters and a variant of regularly nowhere dense sets are used as tools in obtaining these results.

1. INTRODUCTION

A fundamental problem in H-closed extension theory is to determine the structure of the semilattice of H-closed extensions of a space or of a space with certain properties. Several extensions have been studied extensively and have been given names. These extensions serve as sign posts within the semilattice of H-closed extensions. One particularly useful enterprise is to determine necessary and sufficient conditions for these named extensions to be equivalent. Applied to a particular space, this gives a crude understanding of the lattice. In some cases, this information is enough to show us that the extension lattices of two spaces are different.

In 1947 Katětov [4] asked for conditions under which the extensions now known as the Katětov extension κX, the Fomin extension σX, the Banaschewski-Fomin-Šanin extension μX, and the Stone-Čech compactification βX are pairwise equivalent. Later that same year, he answered the $\mu X \equiv \beta X$, $\sigma X \equiv \beta X$, and $\kappa X \equiv \beta X$

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questions [5]. Others, including Porter, Thomas, and Votaw, also looked at these equivalences and gave some alternate proofs. Flachsmeyer in 1966 [2] and Porter and Votaw in 1973 [9] found different characterizations for the $\sigma X \equiv \kappa X$ problem. One question which has remained open is when the Fomin extension σX is equivalent to the Banaschewski-Fomin-Šanin extension μX. In 1985 Tikoo investigated μX and gave a more general definition for this extension. He again inquired as to when $\mu X \equiv \sigma X$. The main result of this paper is an answer to this question.

We begin by recalling a few definitions. For us the word "space" will mean "Hausdorff space." A space is said to be H-closed if and only if it is closed in every Hausdorff space containing it as a subspace, thus H-closed is short for Hausdorff closed. Several other useful characterizations are that a space is H-closed iff every open cover has a finite subcollection whose union is dense iff every open filter has an adherent point. An extension of a space X is any space containing X as a dense subspace. The collection of H-closed extensions of a space will be denoted by $H(X)$. The set $H(X)$ can be given a partial order as follows. If Y and Z are H-closed extensions of X, define $Y \geq Z$ if there exists a continuous surjection from Y to Z which keeps the points of X fixed. If $Y \geq Z$ and $Z \geq Y$, we say Y and Z are equivalent extensions of X and denote this by $Y \equiv Z$. The set $H(X)$ with this partial ordering is a complete upper semilattice. For more information on extensions and for undefined terms see [10].

Two named extensions will be of interest to us. The Fomin extension σX has as its point set $X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on X$\}$ and has a base consisting of all sets of the form $o_\sigma(U) = U \cup \{\mathcal{U} \in \sigma X \backslash X : U \in \mathcal{U}\}$ where U is open in X. Recall that an open filter $\mathcal{F}$ is said to be *free* if $\bigcap\{cl(U) : U \in \mathcal{F}\} = \emptyset$, otherwise it is said to be *fixed*. The Banaschewski-Fomin-Šanin extension μX has the same point set as σX, but with a base consisting of all sets of the form $o_\mu(U) = U \cup \{\mathcal{U} \in \mu X \backslash X : U \in \mathcal{U}_s\}$ where U is open in X and $\mathcal{U}_s = \langle\{int(cl(U)) : U \in \mathcal{U}\}\rangle$. The notation $\langle\mathcal{A}\rangle$ means the open filter generated by $\mathcal{A}$. This definition of μX is due to Tikoo [12] and applies to all Hausdorff spaces thus extending earlier definitions that were valid only for semiregular spaces (such as μX is the semiregularization of the Katětov extension).

2. Prime open filters

In studying H-closed extensions one often works with open filters. Prime open filters offer some additional structure that can be useful when looking at extensions. Recall that an open filter is prime, if whenever the union of two open sets is in the filter, one of the two sets is in the filter. For more information on prime open filters see the papers [1] and [7]. Proofs of the statements in this section can be found in [7].

A useful technique is borrowed from commutative ring theory [3]. An ideal in a commutative ring which is maximal with respect to the exclusion of a multiplicative set is prime. Translating into the language of open filters we have the following lemma:

Lemma 2.1. *An open filter on X which is maximal with respect to the exclusion of some collection of open sets of X which is closed under finite unions is prime.*

Working with open sets allows one to strengthen this lemma and provides a characterization of prime open filters involving dense open sets.

Lemma 2.2. *An open filter on X is prime iff it is maximal with respect to the exclusion of some collection of dense open sets of X which is closed under finite unions.*

Several useful facts follow from these lemmas.

Proposition 2.3. [1] *Every open filter on X is the intersection of the prime open filters containing it.*

Proposition 2.4. *A prime open filter on X which contains all of the dense open sets of X (this is said to be* saturated*) is an open ultrafilter.*

Note that if a space is not discrete then it will have prime open filters which are not open ultrafilters.

3. Preliminaries and terminology

Porter and Woods [10] define a nowhere dense set A to be *regularly nowhere dense* if there are open sets U and V such that $A \subseteq cl(U) \cap cl(V)$ and $U \cap V = \emptyset$. The following definition generalizes the notion of a regularly nowhere dense set in a natural way.

Definition 1. A set A is said to be *n-regularly nowhere dense* (for n an integer greater than 1) if there exist open sets $U_1, \cdots, U_n$ such that $A \subseteq \bigcap\{cl(U_i) : i \in n\}$ and $\bigcap\{U_i : i \in n\} = \emptyset$.

If X is a space, it will be convenient to let $\mathcal{R} = \{X \backslash A : A$ is n-regularly nowhere dense for some $n > 1\}$. Notice that $\mathcal{R}$ is an open filter. The following definition will simplify the discussion which follows.

Definition 2. Let X be a space. A set $A \subseteq X$ is *$\mathcal{R}$-nowhere dense* if A is n regularly nowhere dense for some $n > 1$.

Since we will be working with complements, we point out the obvious fact that A is $\mathcal{R}$-nowhere dense if and only if $X \setminus A \in \mathcal{R}$. This is also the motivation for the name "$\mathcal{R}$-nowhere dense."

The $\mathcal{R}$-nowhere dense property is related to the notion of π-sets introduced by Zaĭcev [13]. A π-set is the intersection of finitely many regular closed sets. Thus immediately from the definitions we have:

Proposition 3.1. *A subset of a space is $\mathcal{R}$-nowhere dense if and only if it is a subset of a nowhere dense π-set.*

Recall the definition of $\mathcal{U}_s$ from Section 1. The following basic facts about the filter $\mathcal{U}_s$ associated with an ultrafilter $\mathcal{U}$ will be useful.

Proposition 3.2. *Let X be a space. If $\mathcal{U}$ is an open ultrafilter on X, then*
1) $\mathcal{U}$ is the unique open ultrafilter containing $\mathcal{U}_s$;
2) $\mathcal{U}$ is saturated (contains all of the dense open sets of X);
3) $\mathcal{R} \subseteq \mathcal{U}_s$ (is $\mathcal{R}$-saturated);
4) If $\mathcal{U}_s$ is saturated then $\mathcal{U}_s = \mathcal{U}$.

PROOF. 1) and 2) are both well known and easy. See [10] Problem 7V for details.

3) Suppose $U \in \mathcal{R}$ then $U = X \setminus A$ where A is n-regularly nowhere dense. Thus A is contained in a nowhere dense set which is a finite intersection of regularly closed sets. Thus $U = X \setminus A \supseteq V$ where V is dense and $V = \bigcup\{V_i : i \in n\}$ with V_i regular open. Since $\mathcal{U}$ is saturated, $V \in \mathcal{U}$. The open filter $\mathcal{U}$ is prime so $V_i \in \mathcal{U}$ for some i. The set V_i is regular open so $int(cl(V_i)) = V_i \in \mathcal{U}$, thus $V_i \in \mathcal{U}_s$. We have then that $U \supseteq V_i \in \mathcal{U}_s$, so $U \in \mathcal{U}_s$.

4) Suppose $\mathcal{U}_s$ is saturated. Proposition 2.3 states that $\mathcal{U}_s$ is the intersection of the prime open filters containing it. Each prime containing it must also be saturated and hence an ultrafilter. But $\mathcal{U}$ is the only ultrafilter containing $\mathcal{U}_s$. Therefore $\mathcal{U}_s = \mathcal{U}$. □

Also note that:

Proposition 3.3. *If $\mathcal{P}$ is a prime open filter on a space X and $\mathcal{P} \supseteq \mathcal{R}$ then $\mathcal{P} \supseteq \mathcal{U}_s$ for some open ultrafilter $\mathcal{U}$.*

PROOF. The open filter $\mathcal{P}$ is contained in some open ultrafilter $\mathcal{U}$. Let $V \in \mathcal{U}_s$. Without loss of generality, assume that $V = int(cl(U))$ for some $U \in \mathcal{U}$. Then

$V \cup (X \setminus cl(V))$ is dense in X and is the union of regular open sets. Thus $V \cup X \setminus cl(V) \in \mathcal{R} \subseteq \mathcal{P}$. By primeness, either V or $X \setminus cl(V)$ is in $\mathcal{P}$. But $V \in \mathcal{U}$ implies $V \in \mathcal{P}$. Therefore $\mathcal{P}$ contains $\{int(cl(U)) : U \in \mathcal{U}\}$ which generates $\mathcal{U}_s$. □

As a preliminary to the $\mu X \equiv \sigma X$ problem, Tikoo noticed:

Lemma 3.4. *$\mu X \equiv \sigma X$ if and only if $\mathcal{U} = \mathcal{U}_s$ for all free open ultrafilters.*

Using the notion of $\mathcal{R}$-nowhere dense we easily have the following proposition.

Proposition 3.5. *For a space X the following are equivalent:*
1) Every closed nowhere dense set is $\mathcal{R}$-nowhere dense;
2) $\mathcal{U} = \mathcal{U}_s$ for all open ultrafilters.

PROOF. 1) $\Longrightarrow$ 2) If D is any dense open set, then by hypothesis, $D \in \mathcal{R}$. Thus $D \in \mathcal{U}_s$, hence $\mathcal{U}_s$ is saturated, and it follows from Proposition 3.2 that $\mathcal{U} = \mathcal{U}_s$.
2) $\Longrightarrow$ 1) Suppose $D \notin \mathcal{R}$ then there is a prime open filter $\mathcal{P}$ maximal with respect to containing $\mathcal{R}$ and excluding D. Hence $\mathcal{P} \supseteq \mathcal{U}_s$ for some open ultrafilter $\mathcal{U}$. Thus $\mathcal{U} \supset \mathcal{P} \supseteq \mathcal{U}_s$. □

A careful look at the proof of Proposition 3.3 reveals that only the complements of regularly nowhere dense sets are used and not the complements of other $\mathcal{R}$-nowhere dense sets. This might suggest that Proposition 3.5 might be strengthened to use only regularly nowhere dense sets. The proof of the 2) $\Longrightarrow$ 1) implication, however, uses Lemma 2.1 and requires that $\mathcal{R}$ be closed under finite unions. Further, the example presented in Section 5 demonstrates that the concept of regularly nowhere dense alone is not sufficient.

Of course this is not the $\mu X \equiv \sigma X$ characterization which has $\mathcal{U} = \mathcal{U}_s$ for all free open ultrafilters.

4. THE THEOREM AND EXAMPLES

Theorem 4.1. *The following are equivalent for a space X:*
1) $\mu X \equiv \sigma X$;
2) $\mathcal{U} = \mathcal{U}_s$ for all free open ultrafilters;
3) For every closed nowhere dense set A in X, every free open filter on X contains a set whose intersection with A is $\mathcal{R}$-nowhere dense;
4) For every closed nowhere dense set A in X, every open cover $\mathcal{C}$ of X has a finite subcollection whose closures cover all but an $\mathcal{R}$-nowhere dense subset of A.

PROOF. The equivalence of 1) and 2) is Tikoo's result, Lemma 3.4. The equivalence of 3) and 4) is a standard conversion between filters and covers. We show the equivalence of 2) and 3).

2) $\Longrightarrow$ 3) Suppose A is closed nowhere dense. Let $\mathcal{F}$ be any open filter on X. If $\mathcal{F}$ does not meet A, i.e., if there is an $F \in \mathcal{F}$ with $F \cap A = \emptyset$, then 3) follows since the empty set is trivially 2-regularly nowhere dense. So suppose $\mathcal{F}$ meets A, hence $X \backslash A \notin \mathcal{F}$. Notice that if $\langle \mathcal{F}, \mathcal{R} \rangle$ excludes $X \backslash A$, there would be an open filter $\mathcal{G}$ maximal with respect to containing $\langle \mathcal{F}, \mathcal{R} \rangle$ and excluding $X \backslash A$. The open filter $\mathcal{G}$ contains $\mathcal{R}$ and is prime, so by Proposition 3.3 for some free open ultrafilter $\mathcal{U}$, we have $\mathcal{U} \supset \mathcal{G} \supseteq \mathcal{U}_s$ which contradicts the hypothesis. Thus $X \backslash A \in \langle \mathcal{F}, \mathcal{R} \rangle$ so there is a $F \in \mathcal{F}$ and a $R \in \mathcal{R}$ such that $F \cap R \subseteq X \backslash A$. Therefore $F \cap A \subseteq X \backslash R$ hence $F \cap A$ is $\mathcal{R}$-nowhere dense.

3) $\Longrightarrow$ 2) Let $\mathcal{U}$ be any free open ultrafilter on X, and let D be any dense open set. Then $X \setminus D$ is closed nowhere dense. Since $\mathcal{U}_s$ is free, there exists a $U \in \mathcal{U}_s$ and an $R \in \mathcal{R}$ such that $U \cap X \setminus D \subseteq X \setminus R$ or $R \cap U \cap X \setminus D = \emptyset$. Since $R \in \mathcal{U}_s$, $D \supseteq R \cap U \in \mathcal{U}_s$. Thus $D \in \mathcal{U}_s$, so $\mathcal{U}_s$ is saturated, and hence $\mathcal{U} = \mathcal{U}_s$. □

We notice that if A is $\mathcal{R}$-nowhere dense, then condition 3) in Theorem 4.1 is satisfied. Condition 3) is also satisfied if no free open filters meet A. This latter condition is a property similar to, but weaker than, compactness.

Definition 3. [6] Let X be a space. A set A is *H-bounded* iff every open filter on X which meets A is fixed.

An equivalent characterization is that a subset A of a space X is H-bounded iff every open cover of X has a finite subcollection whose closures in X cover A. It should be noted that if a set is compact, then it is H-closed, which implies that it is an H-set, which implies it is H-bounded. In fact, any subset of a compact set, an H-closed set, or an H-set is H-bounded. Also in a regular space, a closed H-bounded set is compact. For detailed information on H-bounded sets, see [6] or [8].

The notions of H-bounded and $\mathcal{R}$-nowhere dense play off each other to give the following corollary.

Corollary 4.2. *Let X be a space. If each closed nowhere dense set of X is either H-bounded or $\mathcal{R}$-nowhere dense then $\mu X \equiv \sigma X$.*

This is useful for testing spaces and finding examples. In particular, notice its combination with the following proposition.

Proposition 4.3. [14] *In a metric space every closed set is the intersection of two regularly closed sets. In particular, every closed nowhere dense set is regularly nowhere dense.*

Proposition 4.3 shows that $\mu X \equiv \sigma X$ for an important class of spaces. Also note that all the nowhere dense sets of an ordinal space are 2-regularly nowhere dense (split the isolated points into evens and odds), so $\mu X \equiv \sigma X$ also holds for ordinal spaces.

On the other side of Corollary 4.2, extremally disconnected spaces have no nonempty n-regularly nowhere dense sets, since the equality $cl(U \cap V) = cl(U) \cap cl(V)$ holds in extremally disconnected spaces. The next two corollaries follow immediately.

Corollary 4.4. *If X is extremally disconnected, then $\mu X \equiv \sigma X$ iff every closed nowhere dense set is H-bounded.*

Corollary 4.5. [12] *If X is extremally disconnected and semiregular (and hence regular) then $\mu X \equiv \sigma X$ iff every closed nowhere dense set is compact.*

5. A 3-REGULARLY NOWHERE DENSE SET WHICH IS NOT REGULARLY NOWHERE DENSE

A natural question, especially in light of Proposition 4.3, is whether the concept of n-regularly nowhere dense is different from regularly nowhere dense. We conclude by showing that these ideas are distinct.

Example 5.1. An example of a 3-regularly nowhere dense set which is not 2-regularly nowhere dense.

Partition the real line $\mathbf{R}$ into six disjoint dense subsets, call them P_0, P_1, P_2, Q_{01}, Q_{12}, Q_{20}. Let X be the topological space consisting of the points of $\mathbf{R}$ with a base for its topology consisting of all sets of the form $B(a,b,i) = (a,b) \cap P_i$ or $B(a,b,i,j) = \{x : a < x < b,\ x \in P_i \cup P_j \cup Q_{ij}\}$ where $a, b \in R$ with $a < b$ and $i < j \in \{0,1\}$. Thus the set P_i has the topology inherited as a subspace of $\mathbf{R}$. The points in Q_{ij} have neighborhoods that reach back to both P_i and P_j. It may be helpful to visualize this space as six spokes of a wheel with the P spokes alternating with Q spokes, and with Q_{ij} lying between P_i and P_j. Let $H = Q_{01} \cup Q_{12} \cup Q_{20}$.

Proposition 5.2. *The set H is 3-regularly nowhere dense, but not regularly nowhere dense.*

PROOF. Let $U_0 = P_0 \cup P_1$, $U_1 = P_1 \cup P_2$, and $U_2 = P_0 \cup P_2$. Then it is easy to see that $\bigcap\{U_i : i < 3\} = \emptyset$, but $\bigcap\{cl(U_i) : i < 3\} = H$. Thus H is 3-regularly nowhere dense. We claim that H is not regularly nowhere dense. By way of contradiction, suppose that H is regularly nowhere dense. Then there exist disjoint open sets U and V such that $H \subseteq cl(U) \cap cl(V)$. Now U must meet at least one of P_0, P_1, and P_2, say P_0. The set U meets P_0 in an interval. That is, there exist $a, b \in R$ such that $(a, b) \cap P_0 \subseteq U \cap P_0$. Then $[a, b] \cap Q_{01} \subseteq cl(U)$. In order for these points to be in $cl(V)$, we have that $(a, b) \cap P_1 \subseteq V$. Thus $[a, b] \cap Q_{12} \subseteq cl(V)$. In order for these points to be in $cl(U)$, we have that $(a, b) \cap P_2 \subseteq U$. Thus $[a, b] \cap Q_{20} \subset cl(U)$. In order for these points to be in $cl(V)$, $(a, b) \cap P_0 \subseteq V$. Therefore $(a, b) \cap P_0 \subseteq U \cap V$, contradicting the assumption that U and V are disjoint. □

Proposition 5.3. *Every closed nowhere dense subset of the space X defined in Example 5.1 is $\mathcal{R}$-nowhere dense.*

PROOF. Every closed nowhere dense set is either a subset of $P_0 \cup P_1 \cup P_2$, hence 2-regularly nowhere dense; a subset of H, hence 3-regularly nowhere dense; or a combination of both, hence 6-regularly nowhere dense. □

Finally, note that this example shows that the concept of regularly nowhere dense will not suffice.

Proposition 5.4. *For space X defined in Example 5.1, $\mu(X) \equiv \sigma(X)$, but not every closed nowhere dense set is regularly nowhere dense.*

Notice that this construction may be generalized to larger values of n.

We have seen that $\mathcal{R}$-regularly nowhere dense sets are an important component in understanding the semilattice of H-closed extensions of a Hausdorff space. Further study of these sets is in order. The example of this section is the only example known to the author of an $\mathcal{R}$-regularly nowhere dense set which is not regularly nowhere dense. Substantially different methods of constructing such examples would be of interest as would characterizing those spaces which have $\mathcal{R}$-regularly nowhere dense sets which are not regularly nowhere dense.

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