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HYERS-ULAM STABILITY OF ISOMETRIES

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ABSTRACT. Let X and Y be real Banach spaces. A mapping $\phi : X \longrightarrow Y$ is called an ε-isometry if $| \, \|\phi(x) - \phi(y)\| - \|x - y\| \, | \leq \varepsilon$ holds for all $x, y \in X$. If ϕ is surjective, then its distance to the set of all isometries of X onto Y is at most $\gamma_X \varepsilon$, where γ_X denotes the Jung constant of X.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The first result in what we call now the theory of Hyers-Ulam stability is due to Hyers who solved a stability problem for additive functions posed by Ulam in 1940 [24]. For some recent results on stability behaviour of additive mappings we refer to [1, 9, 10, 11, 15, 18].

In 1945 Hyers and Ulam posed a similar stability problem for isometries [16]. A mapping $\phi : X \to Y$ is called an ε-isometry if

$$| \, \|\phi(x) - \phi(y)\| - \|x - y\| \, | \leq \varepsilon, \quad x, y \in X.$$

The natural question here is, of course, whether there exists an isometry $U : X \to Y$ that is close to ϕ. Note that when studying ε-isometries there is no loss of generality in assuming that $\phi(0) = 0$. Indeed, if a mapping ϕ is an ε-isometry then the same must be true for $\phi - \phi(0)$ and $\phi - \phi(0)$ can be approximated by an isometry U if and only if ϕ is close to isometry $U + \phi(0)$. Hyers and Ulam [16] gave an example of an ε-isometry mapping the real line into the real plane which can not be approximated by an isometry. So, in order to get a stability result one has to assume that ϕ is surjective.

It took almost forty years to get a general stability result for all pairs of Banach spaces X and Y. After several partial results [4, 5, 6, 7, 16, 17] a breakthrough was

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made by Gruber [13] who showed that all surjective asymptotically isometric ε-isometries satisfying $\phi(0) = 0$ can be uniformly approximated by linear isometries. He also showed that in the finite-dimensional case there exists a linear isometry $U : X \to Y$ such that $||\phi(x) - U(x)|| \leq 5\varepsilon$ for every $x \in X$. This result was extended to arbitrary real Banach spaces by Gevirtz [12]. Recently, Omladič and the author [22] showed that the estimate 5ε can be improved to 2ε. In the same paper a simple example was given to show that this estimate is sharp. Consider a surjective function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = -3t$ for $t \in [0, 1/2]$ and $\phi(t) = t - 1$ elsewhere. Clearly, ϕ is a surjective 1-isometry. There are only two linear isometries $U : \mathbb{R} \to \mathbb{R}$, namely, $U(t) = t$ and $U(t) = -t$. Obviously, the second one does not approximate ϕ uniformly. One can easily verify that $\max_{t \in \mathbb{R}} |\phi(t) - t| = 2$, which proves that 2ε is a sharp estimate.

The reason that this simple example works is that the problem was ill-posed in a sense. We should have asked for the distance of a surjective ε-isometry to the set of all (not necessarily linear) surjective isometries. Here, the distance between an approximate isometry ϕ and an isometry U is defined as $\operatorname{dist}(\phi, U) = \sup\{||\phi(x) - U(x)|| : x \in X\} \in [0, \infty]$. Let $\mathcal{U}$ denote the set of all surjective isometries of X onto Y (note that by Mazur-Ulam theorem [21] surjective isometries are linear up to a translation). Then the distance of an approximate isometry ϕ to the set of all surjective isometries is defined by $\operatorname{dist}(\phi, \mathcal{U}) = \inf\{\operatorname{dist}(\phi, U) : U \in \mathcal{U}\}$. In the above example $\operatorname{dist}(\phi, \mathcal{U})$ is the distance between ϕ and $U(t) = t - 1$, which is equal to 1. At this point it would be tempting to conjecture that the distance of a surjective ε-isometry to the set of all surjective isometries is at most ε. Surprisingly, it turns out that the estimate 2ε remains sharp after replacing the set of all linear surjective isometries by the set of all surjective isometries in our stability problem. Namely, let $K \subset \mathbb{R}$, $K = \{1/n : n = 1, 2, \dots\} \cup \{-1/n : n = 1, 2, \dots\} \cup \{-2, 0, 2\}$, and let $C(K)$ be the space of all real-valued continuous functions on K with the usual norm. Then we can prove the following result.

Proposition 1.1. *There exists a surjective* 1*-isometry* $\phi : C(K) \to C(K)$ *such that* $\operatorname{dist}(\phi, \mathcal{U})$ *is* 2.

So, the estimate 2ε is sharp for general Banach spaces. But if we consider surjective ε-isometries between a given pair of Banach spaces X and Y, then in many cases this estimate can be improved. First we need one definition. For a bounded subset $A \subset X$ we define

$$\eta_A = \inf\{r \in \mathbb{R}^+ : \text{there exists } x \in X \text{ such that } A \subset \overline{B}(x, r)\}.$$

Here, $\overline{B}(x,r)$ denotes the closed ball with radius r centered at x. The Jung constant γ_X (see [2, 19]) is defined as

$$\gamma_X = \sup\{\eta_A : A \subset X \text{ and } \operatorname{diam} A \leq 2\}.$$

Clearly, $1 \leq \gamma_X \leq 2$. It is easy to see that $\gamma_{C(K)} = 2$ (in fact, this will follow directly from Proposition 1.1 and the next theorem). An example of a real Banach space X with $\gamma_X = 1$ is a space of all bounded real-valued functions defined on an arbitrary set M equiped with the sup norm.

Theorem 1.2. *Let X and Y be real Banach spaces. Suppose that $\varepsilon > 0$ and that $\phi : X \to Y$ is a surjective ε-isometry. Then* $\operatorname{dist}(\phi, \mathcal{U}) \leq \gamma_X \varepsilon$.

Note that if there exists a surjective ε-isometry $\phi : X \to Y$, then X and Y are isometric, and hence, $\gamma_X = \gamma_Y$.

The Jung constant γ_H of an infinite dimensional real Hilbert space is $\sqrt{2}$ [23]. At the end of this note we will give a much shorter elementary proof of this statement. So, if $\phi : H \to H$ is a surjective ε-isometry then its distance to the set of all surjective isometries on H is at most $\sqrt{2}\varepsilon$. In the finite-dimensional case an analogous result holds without the surjectivity assumption.

Corollary 1.3. *Let E_n be the n-dimensional Euclidean space. Suppose that $\varepsilon > 0$ and that $\phi : E_n \to E_n$ is an ε-isometry. Then there exists an isometry $U : E_n \to E_n$ such that*

$$||\phi(x) - U(x)|| \leq \sqrt{\frac{2n}{n+1}}\varepsilon$$

for every $x \in E_n$.

2. Proofs

Proof of Proposition 1.1. We denote $K_+ = \{1/n : n = 1, 2. \dots\}$ and $K_- = \{-1/n : n = 1, 2. \dots\}$, so that $K = K_+ \cup K_- \cup \{-2, 0, 2\}$. We define two sequences of functions $f_n, g_n \in C(K)$, $n = 1, 2, \dots$, by $f_n(2) = n$ and $f_n(x) = 0$ for all $x \in K \setminus \{2\}$, and $g_n(-2) = -n$ and $g_n(x) = 0$ for all $x \in K \setminus \{-2\}$. We denote $\mathcal{A} = \mathcal{F} \cup \mathcal{G}$ where $\mathcal{F} = \{f_1, f_2, \dots\}$ and $\mathcal{G} = \{g_1, g_2, \dots\}$. We will need two more sequences of functions $h_n, k_n \in C(K)$ defined by $h_n = f_n - (1/2)\chi_{\{(n+3)^{-1}\}}$ and $k_n = g_n + (1/2)\chi_{\{-(n+3)^{-1}\}}$, $n = 1, 2, \dots$ Here, $\chi_{\{x\}}$ denotes the characteristic function of the singleton $\{x\}$.

We define a mapping $\phi_1 : C(K) \to C(K)$ by

$$\phi_1(f)(x) = f(x) + \min\{1, |f(x)| - |f(2) - 1/x| + 3\}$$

if $x \in K_+$ and $|f(x)| \geq |f(2) - 1/x| - 3$,

$$\phi_1(f)(x) = f(x) - \min\{1, |f(x)| - |f(-2) - 1/x| + 3\}$$

if $x \in K_-$ and $|f(x)| \geq |f(-2) - 1/x| - 3$, and

$$\phi_1(f)(x) = f(x)$$

otherwise. Note that for every $f \in C(K)$ we have $\phi_1(f)(x) = f(x)$ for all but finitely many $x \in K$. Hence, $\phi_1(f)$ is continuous, and so, ϕ_1 is well-defined. In order to show that ϕ_1 is surjective we first observe that for every $a \in [-3, \infty)$ the functions $\varphi_a, \psi_a : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi_a(t) = \begin{cases} t + \min\{1, |t| - a\} & \text{if} \quad |t| \geq a \\ t & \text{if} \quad |t| \leq a \end{cases}$$

and

$$\psi_a(t) = \begin{cases} t - \min\{1, |t| - a\} & \text{if} \quad |t| \geq a \\ t & \text{if} \quad |t| \leq a \end{cases}$$

are surjective. Let g be any function in $C(K)$. We define $f : K \to \mathbb{R}$ by $f(x) = g(x)$ if $x \in \{-2, 0, 2\}$. For $x \in K_+$ we denote $a = |g(2) - 1/x| - 3$. Since φ_a is surjective we can find a real t such that $\varphi_a(t) = g(x)$. We choose such a number t and define $f(x) = t$. Similarly, for $x \in K_-$ and $a = |g(-2) - 1/x| - 3$ we define $f(x) = t$ where t was chosen in such a way that $\psi_a(t) = g(x)$. It is easy to see that $f(x) = g(x)$ for all but finitely many $x \in K$. It follows that $f \in C(K)$. Obviously, we have $\phi_1(f) = g$. Hence, ϕ_1 is surjective.

Now, we define $\phi : C(K) \to C(K)$ by

$$\phi(f) = \begin{cases} \phi_1(f) & \text{if} \quad f \notin \mathcal{A} \\ f_n + 2\chi_{\{1/n\}} & \text{if} \quad f = f_n \\ g_n - 2\chi_{\{-1/n\}} & \text{if} \quad f = g_n \end{cases}.$$

First we observe that whenever we have $f(2) = g(2)$, $f(-2) = g(-2)$, and $f(x) = g(x)$, the equation $\phi_1(f)(x) = \phi_1(g)(x)$ must be satisfied. It follows that $\phi_1(f_n)(x) = \phi_1(h_n)(x)$ if $x \neq (n+3)^{-1}$. It is easy to verify that this equation holds true also for $x = (n+3)^{-1}$, and consequently, $\phi_1(f_n) = \phi_1(h_n)$, $n = 1, 2, \ldots$ Similarly, we get $\phi_1(g_n) = \phi_1(k_n)$. This together with the surjectivity of ϕ_1 yield the surjectivity of ϕ.

Next, we will prove that ϕ is a 1-isometry. Assume first that neither f nor g belong to $\mathcal{A}$. For every $x \in K$ we have $\phi(f)(x) = f(x) + A_x$ and $\phi(g)(x) =$

$g(x) + B_x$ for some real numbers A_x, B_x satisfying $|A_x|, |B_x| \leq 1$ and $A_x B_x \geq 0$. It follows that

$$|f(x) - g(x)| - 1 \leq |\phi(f)(x) - \phi(g)(x)| \leq |f(x) - g(x)| + 1.$$

Since this is true for every $x \in K$ we have the desired inequalities

$$||f - g|| - 1 \leq ||\phi(f) - \phi(g)|| \leq ||f - g|| + 1.$$

It is trivial to see that $|\,||\phi(f) - \phi(g)|| - ||f - g||\,| \leq 1$ whenever both f and g belong to $\mathcal{A}$. So, it remains to consider the case that $f \notin \mathcal{A}$ while $g \in \mathcal{A}$. Once again we have two possibilities. We will consider only the first one that $g \in \mathcal{F}$ since almost the same argument can be applied when $g \in \mathcal{G}$. So, let $g = f_n$ for some positive integer n. We will first show that $||\phi(f) - \phi(f_n)|| \geq ||f - f_n|| - 1$. Let us first assume that $||f - f_n|| = |f(x) - f_n(x)|$ for some $x \neq 1/n$. Then we have $\phi(f)(x) = f(x) + A$ for some real A with $|A| \leq 1$ and $\phi(f_n)(x) = f_n(x)$. Consequently, $||\phi(f) - \phi(f_n)|| \geq |\phi(f)(x) - \phi(f_n)(x)| \geq |f(x) - f_n(x)| - |A| \geq ||f - f_n|| - 1$, as desired. If $||f - f_n|| = |f(1/n) - f_n(1/n)| = |f(1/n)| \geq |f(2) - n|$ then $\phi(f)(1/n) = f(1/n) + 1$. This yields $|\phi(f)(1/n) - \phi(f_n)(1/n)| = |f(1/n) - 1| \geq ||f - f_n|| - 1$.

So, it remains to show that $|\phi(f)(x) - \phi(f_n)(x)| \leq ||f - f_n|| + 1$ for every $x \in K$. If $x \neq 1/n$ then $|\phi(f)(x) - \phi(f_n)(x)| \leq |f(x) - f_n(x)| + 1 \leq ||f - f_n|| + 1$. In the case that $x = 1/n$ and $|f(1/n)| \geq |f(2) - n| - 2$ we have $|\phi(f)(1/n) - \phi(f_n)(1/n)| \leq |f(1/n)| + 1 \leq ||f - f_n|| + 1$. Finally, if $x = 1/n$ and $|f(1/n)| \leq |f(2) - n| - 2$ then $|\phi(f)(1/n) - 2| \leq |f(1/n)| + 2 \leq |f(2) - n| \leq ||f - f_n||$. This completes the proof of the fact that ϕ is a 1-isometry.

Assume now that there exist a surjective isometry $U : C(K) \to C(K)$ and a positive real number $M < 2$ such that

$$||\phi(f) - U(f)|| \leq M$$

for every $f \in C(K)$. Then, by Mazur-Ulam theorem [21], $U(f) = Af + g$ for some linear isometry $A : C(K) \to C(K)$ and some $g \in C(K)$. Dividing the inequality $||\phi(nf) - A(nf) - g|| \leq M$ by n and sending n to infinity we get $Af = \lim_{n\to\infty} n^{-1}\phi(nf)$ which further yields that $Af = f$ for every $f \in C(K)$. Hence, $||\phi(f) - f - g|| \leq M$, $f \in C(K)$. Replacing f by f_n we get $|\phi(f_n)(1/n) - f_n(1/n) - g(1/n)| \leq M$ which implies $g(1/n) \geq 2 - M$. Replacing f once again, this time by g_n, we get $g(-1/n) \leq -2 + M$. As $g \in C(K)$ is continuous at zero we have $g(0) \geq 2 - M$ and $g(0) \leq -2 + M$. This contradiction completes the proof of Proposition 1.1. □

PROOF OF THEOREM 1.2. As we have already mentioned in the introduction there is no loss of generality in assuming that $\phi(0) = 0$. For every $x \in X$ we define $\phi_x : X \to Y$ by $\phi_x(z) = \phi(z + x) - \phi(x)$. Clearly, ϕ_x is a surjective ε-isometry satisfying $\phi_x(0) = 0$. It follows from [22, Main Theorem] that for every $x \in X$ there exists a unique surjective linear isometry $U_x : X \to Y$ such that $||\phi_x(z) - U_x(z)|| \leq 2\varepsilon$ for every $z \in X$. Choose any x in X and denote $U_0 = U$. For an arbitrary $z \in X$ we have

$$||U(z) - U_x(z)|| \leq$$
$$||U(z) - \phi(z)|| + ||\phi((z - x) + x) - \phi(x) - U_x(z - x)|| + ||\phi(x)|| + ||U_x(x)|| \leq$$
$$4\varepsilon + ||\phi(x)|| + ||U_x(x)||.$$

The right hand side of this inequality is independent of z. As $U - U_x$ is linear, it must be zero. Therefore, $U_x \equiv U$ is independent of x. It follows that for every $x, y \in X$ we have

$$||(\phi(x) - U(x)) - (\phi(y) - U(y))|| = ||\phi_y(x - y) - U(x - y)|| \leq 2\varepsilon.$$

Hence, if we define $\alpha : X \to Y$ by $\alpha(x) = \phi(x) - U(x)$, $x \in X$, then the diameter of the range of α is at most 2ε. Consequently, for every positive δ there exists $u_\delta \in Y$ such that the range of α is contained in the closed ball with diameter $\varepsilon\gamma_Y + \delta$ centered at u_δ. Equivalently, we have

$$||\phi(z) - (U(z) + u_\delta)|| \leq \varepsilon\gamma_Y + \delta$$

for every $z \in X$. This completes the proof. □

In the proof of Theorem 1.2 we have shown that if $\phi : X \to Y$ is a surjective ε-isometry then there exists a bijective linear isometry $U : X \to Y$ such that the range of $\alpha = \phi - U$ has diameter at most 2ε. The surjectivity assumption on ε-isometries was used only once in the proof of Theorem 1.2. Namely, for general Banach spaces X and Y it is impossible to obtain the Main Theorem in [22] without this assumption. However, in the special case that $X = Y$ is an Euclidean space the conclusion of the Main Theorem in [22] holds without the surjectivity assumption [3, 8]. Jung [19] proved that the Jung constant of E_n is $\gamma_{E_n} = \sqrt{\frac{2n}{n+1}}$. This gives us Corollary 1.3.

We will conclude this note by giving a short proof of the Routledge's result [23] on the Jung constant for Hilbert spaces. Let H be an infinite dimensional real Hilbert space. Then $\gamma_H = \sqrt{2}$. Routledge considered the special case that H is separable. In the proof he used the fact that $\gamma_{E_n} = (2n/(n+1))^{1/2}$. We will show that the infinite dimensional case can be proved directly without using the finite dimensional result. In fact, the proof in the infinite dimensional case is even simpler than the proof of the finite dimensional case.

We first observe that if $A \subset H$ satisfy $A \subset \overline{B}(x,r) \cap \overline{B}(y,r)$ for some $x, y \in H$ and nonnegative real number r, then

$$A \subset \overline{B}\left(\frac{x+y}{2}, \sqrt{r^2 - \frac{||x-y||^2}{4}}\right).$$

It follows easily that for every nonempty bounded subset $A \subset H$ there exists $x_0 \in H$ such that $A \subset \overline{B}(x_0, \eta_A)$. Moreover, if $A \subset \overline{B}(x,r)$ for some $x \in H \setminus \{x_0\}$ and some positive r, then $r > \eta_A$.

We choose an arbitrary subset $A \subset H$ with $\operatorname{diam} A = 2$ and find x_0 such that $A \subset \overline{B}(x_0, \eta_A)$. With no loss of generality we can assume that $x_0 = 0$. We will first consider the possibility that for every $\varepsilon > 0$ there exist $u, v \in A$ such that $||u||, ||v|| \geq \eta_A - \varepsilon$ and $\langle u, v\rangle \leq \varepsilon$. Then $4 \geq ||u-v||^2 = ||u||^2 + ||v||^2 - 2\langle u, v\rangle \geq 2(\eta_A - \varepsilon)^2 - 2\varepsilon$. As this is true for every positive ε we have necessarily $\eta_A^2 \leq 2$.

So, assume that there exists $\varepsilon > 0$ such that $||u||, ||v|| \geq \eta_A - \varepsilon$ implies $\langle u, v\rangle > \varepsilon$. We choose $u_0 \in A$ with $||u_0|| > \eta_A - \varepsilon$ and $\delta > 0$ such that $\delta\eta_A < \varepsilon/2$ and $\eta_A^2 - 2\delta\varepsilon + \delta^2\eta_A^2 = a^2 < \eta_A^2$. If $v \in A$ satisfies $||v|| \leq \eta_A - \varepsilon$ then $||\delta u_0 - v|| \leq \delta||u_0|| + ||v|| < \eta_A - \varepsilon/2$. If $||v|| \geq \eta_A - \varepsilon$ then $||\delta u_0 - v||^2 = \delta^2||u_0||^2 - 2\delta\langle u_0, v\rangle + ||v||^2 \leq a^2$. Hence, A is contained in the closed ball centered at δu_0 with the radius $\max\{\eta_A - \varepsilon/2, a\} < \eta_A$. This contradiction shows that the second possibility can not occur. It follows that $\gamma_H \leq \sqrt{2}$.

Let $\{e_i : i = 1, 2, \dots\}$ be an orthonormal set of vectors in H, and let $A = \{\sqrt{2}e_i : i = 1, 2, \dots\}$. Then obviously, $\operatorname{diam} A = 2$. Choose x_0 such that $A \subset \overline{B}(x_0, \eta_A)$. As $\lim_{i\to\infty}\langle x_0, e_i\rangle = 0$ we have $\eta_A \geq \sqrt{2}$, and consequently, $\gamma_H \geq \sqrt{2}$. This completes the proof.

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