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THE HOMOGENEOUS APPROXIMATION PROPERTY IN THE BERGMAN SPACE

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ABSTRACT. It is shown that sets of sampling for the Bergman space A^2 have the "homogeneous approximation property" (HAP) and that sets with this property are sampling for $A^{2+\epsilon}$. In addition, previous results concerning the boundary behaviour of sampling sets are improved.

1. INTRODUCTION

For $0 < p < \infty$, the Bergman space A^p is the set of functions f analytic in the unit disk $\mathbf{D} = \{z : |z| < 1\}$ with

$$\|f\|_p^p = \frac{1}{\pi} \int_{\mathbf{D}} |f(z)|^p dA(z) < \infty,$$

where dA denotes Lebesgue area measure. If $p \geq 1$, A^p is a Banach space with norm $\|\cdot\|_p$. If $0 < p < 1$, it is a complete metric space, where the metric is given by $d(f,g) = \|f - g\|_p^p$.

A^2 is a Hilbert space with inner product

$$\langle f, g\rangle = \frac{1}{\pi} \int_{\mathbf{D}} f(z)\overline{g(z)} dA(z)$$

and reproducing kernel $\tilde{k_a}(z) = \dfrac{1}{(1-\bar{a}z)^2}$ at $a \in \mathbf{D}$. That is, $\langle f, \tilde{k_a}\rangle = f(a)$ for all $f \in A^2$. It actually turns out that this holds for $f \in A^p$, $1 \leq p < \infty$. Let now

$$k_a(z) = \frac{\tilde{k_a}(z)}{\|\tilde{k_a}\|_2} = \frac{1-|a|^2}{(1-\bar{a}z)^2}$$

be the normalized kernel at a.

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A space closely related to A^p is $A^{-n}(n>0)$, which consists of functions f analytic in $\mathbf{D}$ with

$$\|f\|_{-n}=\sup_{z\in\mathbf{D}}(1-|z|^2)^n|f(z)|<\infty.$$

A^{-n} is also a Banach space. It is easy to see that for any $\delta>0$, $A^{-(1/p-\delta)}\subseteq A^p$. One can also check that $A^p\subseteq A^{-2/p}$.

A sequence Γ of distinct points in $\mathbf{D}$ is said to be a set of sampling for A^p if there exist positive constants K_1 and K_2 such that

$$K_1\|f\|_p^p\leq\sum_{z\in\Gamma}(1-|z|^2)^2|f(z)|^p\leq K_2\|f\|_p^p \tag{1}$$

for all $f\in A^p$. Likewise, Γ is a set of sampling for A^{-n} if there is a K such that

$$\|f\|_{-n}\leq K\sup_{z\in\Gamma}(1-|z|^2)^n|f(z)|$$

for all $f\in A^{-n}$. (The analogue of the upper inequality in (1) is automatically satisfied here.)

Given a sequence Γ, let T_p be the linear operator which maps an analytic function f to the sequence $\{f(z)(1-|z|^2)^{\frac{2}{p}}\}_{z\in\Gamma}$. Then Γ is said to be a set of interpolation for A^p if $T_p(A^p)\supseteq\ell^p$. Likewise, Γ is a set of interpolation for A^{-n} if $T_{2/n}(A^{-n})\supseteq\ell^\infty$.

Seip [13] completely characterizes sets of sampling and interpolation for A^{-n}, as well as for A^2, using methods which may be extended to A^p for $0<p<\infty$.

Let $\mathbf{B}$ now be A^p or A^{-n}. We say that Γ is a $\mathbf{B}$ zero set if there is a nontrivial function $f\in\mathbf{B}$ which vanishes precisely on Γ. By theorems of Horowitz [4] and Luecking [5], it suffices for f to vanish at least on Γ. We say that Γ is a set of uniqueness for $\mathbf{B}$ if it is not a $\mathbf{B}$ zero set. It is clear from the definition that a $\mathbf{B}$ zero set cannot be a set of sampling for $\mathbf{B}$ and it is not difficult to show that a set of interpolation for $\mathbf{B}$ must be a $\mathbf{B}$ zero set. One may also demonstrate that Γ is a set of uniqueness for A^2 if and only if the span of $\{k_a\}_{a\in\Gamma}$ is dense in A^2.

The main results of this paper will be divided into two sections. In §3 we determine a condition on a sequence, the homogeneous approximation property, which is necessary and almost sufficient for it to be a set of sampling for A^2. In §4, we improve results in [8] about the behaviour of sampling sets near the boundary and we discuss some of the relationships between sampling and zero sets.

2. SEIP'S DESCRIPTION OF SETS OF SAMPLING AND INTERPOLATION

In order to state Seip's theorems, we need a few definitions. The pseudo-hyperbolic metric ρ is defined on $\mathbf{D}$ by $\rho(z,\zeta) = |\phi_\zeta(z)|$, where

$$\phi_\zeta(z) = \frac{\zeta - z}{1 - \overline{\zeta} z}, \qquad z, \zeta \in \mathbf{D}.$$

For $r < 1$, let $B(\zeta, r) = \{z : \rho(\zeta, z) < r\}$. A sequence $\Gamma = \{z_k\}$ is uniformly discrete if there is a $\delta > 0$ such that $\rho(z_i, z_j) \geq \delta$ for all $i \neq j$.

For $0 < s < 1$, let $n_\Gamma(\zeta, s)$ be the number of points of Γ contained in $B(\zeta, s)$. Define, for Γ uniformly discrete,

$$D_\Gamma(\zeta, r) = \frac{\int_0^r n_\Gamma(\zeta, s) ds}{2 \int_0^r a(B(0,s)) ds},$$

where

$$a(\Omega) = \int_\Omega \frac{1}{(1 - |z|^2)^2} dA(z)$$

is the hyperbolic area of a measurable subset Ω of the disk. Note that $a(B(\zeta, s)) = a(B(0,s))$ for all $\zeta \in \mathbf{D}$.

The lower and upper uniform densities are defined, respectively, to be

$$D^-(\Gamma) = \liminf_{r \to 1} \inf_{\zeta \in \mathbf{D}} D_\Gamma(\zeta, r)$$

and

$$D^+(\Gamma) = \limsup_{r \to 1} \sup_{\zeta \in \mathbf{D}} D_\Gamma(\zeta, r).$$

The following results were proved in [13] for A^{-n} and A^2 and stated for A^p ($0 < p < \infty$) in [3]. (For a fully detailed proof of this last case, see [9] or [10].) See also [11] for a different characterization of sets of interpolation for A^p.

Theorem 2.1 (Seip). *A sequence Γ of distinct points in the disk is a set of sampling for A^p if and only if it is a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > \frac{1}{p}$. Also, Γ is a set of sampling for A^{-n} if and only if it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > n$.*

Theorem 2.2 (Seip). *A sequence Γ of distinct points in the disk is a set of interpolation for A^p if and only if Γ is uniformly discrete and $D^+(\Gamma) < \frac{1}{p}$. Also, Γ is a set of interpolation for A^{-n} if and only if Γ is uniformly discrete and $D^+(\Gamma) < n$.*

Roughly, the theorems indicate that Γ is a set of sampling if there are many points of Γ per unit of hyperbolic area everywhere in the disk. Γ is a set of interpolation if there are *not* too many points per unit of hyberbolic area anywhere in the disk.

3. The homogeneous approximation property

In the proofs of Seip's theorems, the concept of a weak limit, introduced by Beurling in [1], is used extensively. If A is a closed set in $\mathbf{D}$ and $t \geq 0$, then

$$A_t = \{z \in \mathbf{D} : \rho(z, a) \leq t \quad \text{for some } a \in A\}.$$

For A and B closed, the Hausdorff distance is

$$[A, B] = \inf\{t : A \subseteq B_t \quad \text{and} \quad B \subseteq A_t\};$$

and we say that A_n converges weakly to A ($A_n \rightharpoonup A$) if

$$[(A_n \cap K) \cup \partial K, (A \cap K) \cup \partial K] \to 0$$

for every compact set $K \subseteq \mathbf{D}$. We denote by $W(\Gamma)$ the collection of sequences Λ such that $\phi_{\zeta_n}(\Gamma) \rightharpoonup \Lambda$ for some sequence of Möbius transformations $\{\phi_{\zeta_n}\}$.

In [13] it is noted that if Γ is uniformly discrete and $W(\Gamma)$ consists only of sets of uniqueness for A^{-n}, then Γ is a set of sampling for A^{-n}. The converse also holds. In some sense then, a set of sampling can be viewed as a set whose Möbius transforms are "uniformly" far from being zero sets. We make this idea precise in what follows.

Let $\zeta \in \mathbf{D}$ and $g \in A^2$. Define $\zeta * g$ by $\zeta * g(z) = g(\phi_\zeta(z))\phi'_\zeta(z)$.

If $\mathbf{B}$ is a Banach space, $f \in \mathbf{B}$ and $\mathbf{M}$ is a closed subset of $\mathbf{B}$, then the distance between f and $\mathbf{M}$ is defined to be

$$d(f, \mathbf{M}) = \inf\{\|f - h\|_{\mathbf{B}} : h \in \mathbf{M}\}.$$

Suppose now that Γ is a sequence of distinct points in $\mathbf{D}$. We say that Γ has the homogeneous approximation property (HAP) for A^2 if given $\epsilon > 0$ and $g \in A^2$, there is an $R < 1$ such that

$$d(\zeta * g, \mathbf{S}_{B(\zeta,R)\cap\Gamma}) < \epsilon$$

for all $\zeta \in \mathbf{D}$, where $\mathbf{S}_A$ is the closed span of $\{k_a : a \in A\}$. This definition was inspired by a similar concept of the same name in [6]. By earlier remarks and an application of the definition to $\zeta = 0$, we see that a set with the HAP for A^2 is a set of uniqueness for A^2. We are now in a position to state the main result of this section.

Theorem 3.1. *Let Γ be a uniformly discrete sequence of points in $\mathbf{D}$ and let $\epsilon > 0$. If Γ is a set of sampling for A^2, then Γ has the HAP for A^2. On the other hand, if Γ has the HAP for A^2, then Γ is a set of sampling for $A^{2+\epsilon}$.*

This will follow from

Lemma 3.2. *Let Γ be a uniformly discrete sequence in $\mathbf{D}$. Γ has the HAP for A^2 if and only if $W(\Gamma)$ consists only of sets of uniqueness for A^2.*

PROOF OF THEOREM 3.1. The first statement follows from Lemma 3.2 and the fact that if Γ is a set of sampling for A^2, then every $\Lambda \in W(\Gamma)$ is a set of uniqueness for A^2. Consider now the second statement and suppose Γ has the HAP for A^2. Choose $\delta > 0$ such that $1/2 - \delta > \frac{1}{2+\epsilon}$. Since $A^{-(1/2-\delta)} \subseteq A^2$ and by Lemma 3.2, $W(\Gamma)$ contains only sets of uniqueness for $A^{-(1/2-\delta)}$. By the previously noted fact in [13] then, Γ is a set of sampling for $A^{-(1/2-\delta)}$ and so

$$D^-(\Gamma) > 1/2 - \delta > \frac{1}{2+\epsilon}.$$

This, in turn, implies that Γ is a set of sampling for $A^{2+\epsilon}$. □

We start by listing some technical facts concerning the above definitions. By a change of variables argument, one sees that

$$\|\zeta * g\|_2 = \|g\|_2.$$

Elementary calculations give us

$$\phi_\zeta(\phi_w(z)) = \theta(\zeta, w)\phi_{\phi_w(\zeta)}(z), \tag{2}$$

where $\theta(\zeta, w) = \dfrac{\phi_\zeta(w)}{\phi_w(\zeta)}$ is a complex number of modulus one. By differentiating (2), we obtain

$$\phi'_\zeta(\phi_w(z))\phi_w{}'(z) = \theta(\zeta, w)\phi'_{\phi_w(\zeta)}(z).$$

$$\phi'_\zeta(\phi_\zeta(z))\phi'_\zeta(z) = 1$$

is arrived at by differentiating the equation $\phi_\zeta(\phi_\zeta(z)) = z$. From this follows the fact that

$$\zeta * \zeta * g = g.$$

Given $\theta \in \mathbf{T} = \{z : |z| = 1\}$, define g_θ by $g_\theta(z) = g(\theta z)$. A calculation shows that

$$(\phi_w(\zeta) * g_{\theta(\zeta,w)})(z) = (w * \zeta * g)(z)/\theta(\zeta, w). \tag{3}$$

Using the identity

$$(1-\overline{\phi_\zeta(z)}\phi_\zeta(w))^{-2}\phi_\zeta'(w)\overline{\phi_\zeta'(z)} = (1-\overline{z}w)^{-2},$$

we obtain

$$k_{\phi_w(a)}(z) = \frac{\overline{\phi_w'(\phi_w(a))}}{|\phi_w'(\phi_w(a))|}(w * k_a)(z),$$

from which it follows that

$$w * \mathbf{S}_A = \mathbf{S}_{\phi_w(A)}. \tag{4}$$

We now proceed with the proof of Lemma 3.2. We need to ensure that any Möbius transform of a set with the HAP also has the HAP. The constant $\theta(\zeta, w)$ on the righthand side of (3) presents a problem that we have to deal with first.

Lemma 3.3. *Suppose uniformly discrete* Γ *has the* HAP *for* A^2 *and let* $g \in A^2$. *Consider* $\{g_\theta\}_{\theta\in\mathbf{T}}$, *where* $g_\theta(z) = g(\theta z)$. *If* $\epsilon > 0$, *then there exists* $R < 1$ *such that*

$$d(\zeta * g_\theta, \mathbf{S}_{B(\zeta,R)\cap\Gamma}) < \epsilon$$

for all $\zeta \in \mathbf{D}$ *and all* $\theta \in \mathbf{T}$.

PROOF. Let $h \in H^\infty$, the set of bounded analytic functions, and $\theta, \theta_0 \in \mathbf{T}$. Suppose that $|h(z)| \le C$ for all $z \in \mathbf{D}$.

$$|h_\theta(z) - h_{\theta_0}(z)|^2 \le 2(|h(\theta z)|^2 + |h(\theta_0 z)|^2) \le 4C^2$$

so by the dominated convergence theorem, we see that $\lim_{\theta\to\theta_0} \|h_\theta - h_{\theta_0}\|_2 = 0$.

For every $\theta_0 \in \mathbf{T}$ then, there is a neighbourhood N_0 of θ_0 such that

$$\theta \in N_0 \Rightarrow \|h_\theta - h_{\theta_0}\|_2 < \epsilon/6.$$

Let $g \in A^2$. Since H^∞ is dense in A^2, there is an $h \in H^\infty$ such that $\|h-g\|_2 < \epsilon/6$. Therefore, $\theta \in N_0$ implies that

$$\begin{aligned}\|g_\theta - g_{\theta_0}\|_2 &\le \|g_\theta - h_\theta\|_2 + \|h_\theta - h_{\theta_0}\|_2 + \|g_{\theta_0} - h_{\theta_0}\|_2 \\ &= 2\|g-h\|_2 + \|h_\theta - h_{\theta_0}\|_2 \\ &< \epsilon/2.\end{aligned}$$

Since N_0 is compact, we obtain finite sets $\{\theta_1, \ldots, \theta_s\}$, $\{N_1, \ldots, N_s\}$ for which the above holds. For $i = 1, \ldots, s$, there exists R_i such that

$$d(\zeta * g_{\theta_i}, \mathbf{S}_{B(\zeta,R_i)\cap\Gamma}) < \epsilon/2 \tag{5}$$

for all $\zeta \in \mathbf{D}$, since Γ has the HAP for A^2.

Let $R = \max\{R_1, \ldots, R_s\}$, $\theta \in \mathbf{T}$ and $\zeta \in \mathbf{D}$. There is an i such that $\theta \in N_i$ and so by (5), there is a function $f \in \mathbf{S}_{B(\zeta,R_i)\cap\Gamma}$ such that $\|\zeta * g_{\theta_i} - f\|_2 < \epsilon/2$. Therefore,

$$\begin{aligned}\|\zeta * g_\theta - f\|_2 &\leq \|\zeta * g_\theta - \zeta * g_{\theta_i}\|_2 + \|\zeta * g_{\theta_i} - f\|_2 = \|g_\theta - g_{\theta_i}\|_2 + \|\zeta * g_{\theta_i} - f\|_2 \\ &< \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

Thus, $d(\zeta * g_\theta, \mathbf{S}_{B(\zeta,R)\cap\Gamma}) < \epsilon$. $\square$

Lemma 3.4. *Let $\epsilon > 0$ and suppose Γ is a uniformly discrete sequence in $\mathbf{D}$ with the HAP for A^2. If $g \in A^2$, then there is an $R < 1$ such that*

$$d(\zeta * g, \mathbf{S}_{B(\zeta,R)\cap\phi_w(\Gamma)}) < \epsilon$$

for all $\zeta, w \in \mathbf{D}$.

PROOF. By Lemma 3.3, there is an $R < 1$ such that $d(\phi_w(\zeta) * g_\theta, \mathbf{S}_{B(\phi_w(\zeta),R)\cap\Gamma}) < \epsilon$ for all $\zeta, w \in \mathbf{D}$ and all $\theta \in \mathbf{T}$. Now fix $w, \zeta \in \mathbf{D}$. By (3), there is a $\theta_0 \in \mathbf{T}$ such that $\phi_w(\zeta) * g_{\theta_0} = (w * \zeta * g)/\theta_0$. Then

$$\begin{aligned}d(\zeta * g, \mathbf{S}_{B(\zeta,R)\cap\phi_w(\Gamma)}) &= d(w * \zeta * g, w * \mathbf{S}_{B(\zeta,R)\cap\phi_w(\Gamma)}) \\ &= d(w * \zeta * g, \mathbf{S}_{B(\phi_w(\zeta),R)\cap\Gamma}) \\ &= d((w * \zeta * g)/\theta_0, \mathbf{S}_{B(\phi_w(\zeta),R)\cap\Gamma}) \\ &= d(\phi_w(\zeta) * g_{\theta_0}, \mathbf{S}_{B(\phi_w(\zeta),R)\cap\Gamma}) < \epsilon.\end{aligned}$$

$\square$

Before we can complete the proof of the necessity part of Lemma 3.2, we need some results concerning the behaviour of $\mathbf{S}_A$ when A is shifted slightly.

Lemma 3.5. *Let $\mathbf{B}$ be a Banach space. If $f, g \in \mathbf{B}$ with $C_1 \leq \|f\|_{\mathbf{B}}, \|g\|_{\mathbf{B}} \leq C_2$, then $\|\frac{f}{\|f\|_{\mathbf{B}}} - \frac{g}{\|g\|_{\mathbf{B}}}\|_{\mathbf{B}} \leq C\|f - g\|_{\mathbf{B}}$ for some other constant C.*

PROOF. This follows from a straightforward application of the properties of a norm. $\square$

Lemma 3.6. *Let $0 < s < 1$ and suppose that $a, b \in B(0,s)$. There is a constant C, depending only on s, such that $\|k_a - k_b\|_2 \leq C\rho(a,b)$.*

PROOF.

$$\begin{aligned}
\|\tilde{k}_a - \tilde{k}_b\|_2^2 &= \frac{1}{\pi}\int_{\mathbf{D}} \left|\frac{1}{(1-\bar{a}z)^2} - \frac{1}{(1-\bar{b}z)^2}\right|^2 dA(z) \\
&= \frac{1}{\pi}\int_{\mathbf{D}} \left|\frac{(1-\bar{b}z)^2 - (1-\bar{a}z)^2}{(1-\bar{a}z)^2(1-\bar{b}z)^2}\right|^2 dA(z) \\
&\le C\int_{\mathbf{D}} |(1-\bar{b}z)^2 - (1-\bar{a}z)^2|^2 dA(z) \\
&= C\int_{\mathbf{D}} |z(\bar{b}-\bar{a})(2-(\bar{a}+\bar{b})z)|^2 dA(z) \\
&\le C|a-b|^2 \le 4C(\rho(a,b))^2.
\end{aligned}$$

An application of Lemma 3.5 yields the desired result. □

If $\{x_i\}_{i=1}^m$ is a linearly independent set of elements of unit norm in a Hilbert space, then we say that $\{w_i\}_{i=1}^m$ is its orthogonalization if $w_1 = x_1$ and

$$w_i = x_i - \sum_{j=1}^{i-1} \langle x_i, w_j\rangle w_j$$

for $i = 2, \ldots, m$. This process is similar to the Gram-Schmidt process except that we don't normalize at each stage.

Lemma 3.7. *Let $m \in \mathbf{N}$. There is a constant C such that if $\{x_i\}_{i=1}^m$ and $\{y_i\}_{i=1}^m$ are two linearly independent sets of elements of unit norm in a Hilbert space and $\{w_i\}_{i=1}^m$, $\{z_i\}_{i=1}^m$ are their orthogonalizations, respectively, then*

$$\|w_i - z_i\| \le C \max_{j=1,\ldots,m} \|x_j - y_j\|$$

for $i = 1, \ldots, m$.

PROOF. Because of the inequality

$$\|w_i\| \le 1 + \sum_{j=1}^{i-1} \|w_j\|^2$$

there is a constant C such that $\|w_i\|, \|z_i\| \le C$ for $i = 1, \ldots, m$. Using this fact, as well as the triangle and Cauchy-Schwarz inequalities several times, we obtain

$$\|w_i - z_i\| \le (Cm - C + 1)\|x_i - y_i\| + 2C\sum_{j=1}^{i-1} \|w_j - z_j\|.$$

This last inequality leads directly to the desired result. □

We turn now to the proof of Lemma 3.2.

PROOF OF THE NECESSITY. We prove that every $\Lambda \in W(\Gamma)$ has the HAP for A^2. Let $\epsilon > 0$ and $g \in A^2$. Without loss of generality, we may assume that $\|g\|_2 = 1$. Since $\Lambda \in W(\Gamma)$, there is a sequence $\{\Lambda_n\}$ of images of Möbius transformations of Γ such that $\Lambda_n \rightharpoonup \Lambda$. By Lemma 3.4, there is an $R < 1$ such that

$$d(\zeta * g, \mathbf{S}_{B(\zeta,R)\cap\Lambda_n}) < \epsilon/3 \tag{6}$$

for all ζ in $\mathbf{D}$ and all n.

Let $\zeta \in \mathbf{D}$ and choose $s < 1$ such that $B(\zeta, R) \subseteq B(0, s)$. We note that the uniform discreteness of Γ implies that Λ is also uniformly discrete. Thus we write $B(\zeta, R) \cap \Lambda = \{a_1, \ldots, a_m\}$. By the definition of weak convergence, there is an N_1 such that $n \geq N_1$ implies that there are precisely m points in $B(\zeta, R) \cap \Lambda_n$.

$\{k_{a_j}\}_{j=1}^m$ is a linearly independent set, so we let $\{l_{a_j}\}_{j=1}^m$ be its orthogonalization. Let $K = \min_{j=1,\ldots,m}\{\|l_{a_j}\|_2\}$. Lemmas 3.6 and 3.7 tell us that there is a constant C such that if $\{b_1^n, \ldots, b_m^n\}$ is another set of distinct points, then for $i = 1, \ldots, m$,

$$\|l_{a_i} - l_{b_i^n}\|_2 \leq C \max_{j=1,\ldots,m}\{\rho(a_j, b_j^n)\},$$

where $\{l_{b_j^n}\}_{j=1}^m$ is the orthogonalization of $\{k_{b_j^n}\}_{j=1}^m$. There is a δ_1 such that

$$\max_{j=1,\ldots,m}\{\rho(a_j, b_j^n)\} < \delta_1 \Rightarrow \|l_{b_i^n}\|_2 \geq K/2$$

for $i = 1, \ldots, m$ and we choose N_2 such that

$$n \geq N_2 \Rightarrow [B(\zeta, R) \cap \Lambda_n, B(\zeta, R) \cap \Lambda] < \min\{\delta_1, \frac{K\epsilon}{C2^{(m+5)/2}}\}.$$

Now, let $n \geq \max\{N_1, N_2\}$. By (6), there is an $f \in \mathbf{S}_{B(\zeta,R)\cap\Lambda_n}$ such that $\|\zeta * g - f\|_2 < \epsilon/2$ and $\|f\|_2 \leq 2$. Let $\{b_j\}_{j=1}^m = B(\zeta, R) \cap \Lambda_n$. We suppress the superscript as n is now fixed.

Write $f = \sum_{j=1}^m \lambda_j l_{b_j}$. Since $\{l_{b_j}\}_{j=1}^m$ is orthogonal, $f = \displaystyle\sum_{j=1}^m \frac{\langle f, l_{b_j}\rangle l_{b_j}}{\|l_{b_j}\|^2}$. Therefore, $\lambda_j = \dfrac{\langle f, l_{b_j}\rangle}{\|l_{b_j}\|_2^2}$ and so Bessel's inequality implies that

$$\sum_{j=1}^m \frac{|\langle f, l_{b_j}\rangle|^2}{\|l_{b_j}\|_2^2} \leq \|f\|_2^2.$$

Therefore,

$$\sum_{j=1}^{m} |\lambda_j|^2 = \sum_{j=1}^{m} \frac{|\langle f, l_{b_j}\rangle|^2}{\|l_{b_j}\|_2^2} \frac{1}{\|l_{b_j}\|_2^2} \leq 4\|f\|_2^2/K^2 \leq 16/K^2.$$

Define $h = \sum_{j=1}^{m} \lambda_j l_{a_j}$. Then $h \in \mathbf{S}_{B(\zeta,R)\cap\Lambda}$ and

$$\begin{aligned}\|f - h\|_2^2 = \|\sum_{j=1}^{m} \lambda_j (l_{a_j} - l_{b_j})\|_2^2 &\leq 2^{m-1} \sum_{j=1}^{m} |\lambda_j|^2 \|(l_{a_j} - l_{b_j})\|_2^2 \\ &\leq 2^{m-1} \sum_{j=1}^{m} |\lambda_j|^2 \frac{K^2\epsilon^2}{2^{m+5}} \leq \epsilon/2.\end{aligned}$$

Therefore

$$\|\zeta * g - h\|_2 \leq \|\zeta * g - f\|_2 + \|f - h\|_2 < \epsilon$$

and so $d(\zeta * g, \mathbf{S}_{B(\zeta,R)\cap\Lambda}) < \epsilon$. Hence Λ has the *HAP* for A^2. □

The idea for the following is based on the proof of Theorem 3 in [6].

PROOF OF THE SUFFICIENCY. If Γ doesn't have the *HAP*, then there is an $\epsilon > 0$ and a $g \in A^2$ that defeat it. Let $\{\zeta_n\} \in \mathbf{D}$ and $R_n \to 1$ such that by (4),

$$d(g, \mathbf{S}_{B(0,R_n)\cap\Gamma_n}) = d(\zeta_n * g, \mathbf{S}_{B(\zeta_n,R_n)\cap\Gamma}) \geq \epsilon$$

for all n, where $\Gamma_n = \phi_{\zeta_n}(\Gamma)$. By passing to a subsequence if necessary, we may assume that $\Gamma_n \rightharpoonup \Lambda$, for some Λ. Let $R < 1$ and $B(0,R) \cap \Lambda = \{a_1, \dots, a_m\}$. Let $f \in \mathbf{S}_{B(0,R)\cap\Lambda}$ and write $f = \sum_{j=1}^{m} \lambda_j k_{a_j}$. Choose n so large that $R_n > R$ and the number of points in $B(0,R) \cap \Gamma_n$ is precisely m. By Lemma 3.6, choose N such that

$$n \geq N \Rightarrow \|k_{a_i} - k_{b_i^n}\|_2 < \frac{\epsilon}{2^{(m+1)/2}(\sum |\lambda_j|^2)^{1/2}},$$

where $\{b_1^n, \dots, b_m^n\} = B(0,R) \cap \Gamma_n$.

Let $n \geq N$ and now suppress the superscript n as it is fixed. If we define $h = \sum_{j=1}^{m} \lambda_j k_{b_j}$, then $h \in \mathbf{S}_{B(0,R)\cap\Gamma_n} \subseteq \mathbf{S}_{B(0,R_n)\cap\Gamma_n}$ and

$$\begin{aligned}\|h - f\|_2^2 = \|\sum_{j=1}^{m} \lambda_j (k_{a_j} - k_{b_j})\|_2^2 &\leq 2^{m-1} \sum_{j=1}^{m} |\lambda_j|^2 \|(k_{a_j} - k_{b_j})\|_2^2 \\ &\leq 2^{m-1} \sum_{j=1}^{m} |\lambda_j|^2 \frac{\epsilon^2}{2^{m+1}\sum |\lambda_j|^2} = \epsilon^2/4\end{aligned}$$

so $\|h - f\|_2 < \epsilon/2$. Therefore

$$\|g - f\|_2 \geq \|g - h\|_2 - \|h - f\|_2 > \epsilon - \epsilon/2 = \epsilon/2$$

and so

$$d(g, \mathbf{S}_{B(0,R)\cap\Lambda}) > \epsilon/2. \tag{7}$$

Since $R < 1$ was arbitrary, (7) implies that Λ is an A^2 zero set, contradicting the hypotheses of the lemma. Thus Γ has the HAP for A^2. □

A natural question is whether the $\epsilon > 0$ is necessary in the statement of Theorem 3.1. In other words, is there a set with the HAP for A^2 which is not a set of sampling for A^2? Unfortunately, the answer is yes. Before we can discuss an example of such a situation, we require a technical lemma.

Lemma 3.8. *Let Γ be a uniformly discrete sequence in $\mathbf{D}$. Suppose Γ has the property that there is an analytic function g which vanishes precisely on Γ with*

$$|g(z)| \approx \rho(z, \Gamma)(1 - |z|^2)^{-1/2} \tag{8}$$

for all $z \in \mathbf{D}$. Then every member of $W(\Gamma)$ also has this property. We say that $f_1(z) \approx f_2(z)$ if the ratio of $f_1(z)$ and $f_2(z)$ is bounded above and below by positive constants.

Proof. Because of (8) and the identity

$$|\phi'_\zeta(z)| = \frac{1 - |\phi_\zeta(z)|^2}{1 - |z|^2},$$

we have

$$|g(\phi_\zeta(z))(\phi'_\zeta(z))^{1/2}| \approx \rho(z, \phi_\zeta(\Gamma))(1 - |z|^2)^{-1/2}.$$

If $\Lambda \in W(\Gamma)$, then there exists $\{\zeta_n\} \subseteq \mathbf{D}$ such that $\phi_{\zeta_n}(\Gamma) \rightharpoonup \Lambda$. If we define $g_n(z) = g(\phi_{\zeta_n}(z))(\phi'_{\zeta_n}(z))^{1/2}$, we see by the above that there is a constant C such that $\|g_n\|_{-1/2} \leq C$ for all n. Because of the compactness property of $A^{-1/2}$, there is a function f and a subsequence $\{g_{n_k}\}$ such that $g_{n_k} \to f$ uniformly on compact subsets of $\mathbf{D}$. It is easy to see that f vanishes precisely on Λ and satisfies the condition (8). □

In [12], Seip shows that if Γ is a uniformly discrete set in $\mathbf{D}$ with the above property, then

$$D^-(\Gamma) = D^+(\Gamma) = \frac{1}{2}$$

and Γ is a set of uniqueness for A^2. Therefore, by Lemmas 3.2 and 3.8, the sets $\Gamma(a,b)$ with $\frac{2\pi}{b\log a} = \frac{1}{2}$, constructed in [12] and defined later in the present article for the reader, provide examples of sets with the HAP for A^2 which are not sets of sampling for A^2. Notice that $\Gamma_{1/2}$, as defined in [8], is also an example of this phenomenon.

We remark that all of the analysis in the previous section can be applied to a certain class of weighted Bergman spaces, as well as the Bargmann-Fock space.

4. Boundary behaviour of sets of sampling

Sets of sampling, as can be seen from the definition, must be fairly dense near the boundary.

An open set W in $\mathbf{C}$ will be called an N-type neighbourhood if it is internally tangent to $\mathbf{T}$ at one point or if it contains an arc of $\mathbf{T}$. We say that a sequence of points Γ in $\mathbf{D}$ accumulates strongly if for all N-type neighbourhoods W, $D^+(\Gamma\cap W) > 0$.

In [8], it is shown that if a sequence Γ does not intersect every N-type neighbourhood, then $D^-(\Gamma) = 0$. As a corollary, we see that sets of sampling intersect every N-type neighbourhood. The following lemma allows us to strengthen this result.

Lemma 4.1. *Let Γ be a sequence of distinct points in $\mathbf{D}$. If Γ does not accumulate strongly, then $D^-(\Gamma) = 0$.*

Proof. If Γ doesn't accumulate strongly, then there is an N-type neighbourhood W such that $D^+(\Gamma\cap W) = 0$. Note that $D^-(\Gamma\setminus W) = 0$, since $\Gamma\setminus W$ does not intersect W. Lemma 1 of [8] states that

$$D^-(A\cup B) \le D^-(A) + D^+(B) \le D^+(A\cup B), \tag{9}$$

if A and B are disjoint. Therefore,

$$D^-(\Gamma) \le D^+(\Gamma\cap W) + D^-(\Gamma\setminus W) = 0.$$

□

It follows from the work in [2] and [7] on the spectrum of the multiplication operator acting on the invariant subspace of A^p generated by a Bergman space zero set, that if A and B are A^p zero sets whose union is a set of sampling for A^p, then both A and B accumulate everywhere on $\mathbf{T}$. (See [10] for details.) If A and B are actually sets of interpolation, we can say a little bit more. We first need a result about the density of the part of a sequence in an N-type neighbourhood.

Lemma 4.2. *Let Γ be a sequence of points in $\mathbf{D}$ with $D^-(\Gamma) = D^+(\Gamma)$. For any N-type neighbourhood W, $D^+(\Gamma) = D^+(\Gamma \cap W)$.*

PROOF. By (9),

$$D^-(\Gamma) \leq D^-(\Gamma \setminus W) + D^+(\Gamma \cap W) \leq D^+(\Gamma).$$

By the hypotheses of the lemma and since $D^-(\Gamma \setminus W) = 0$, we obtain the result. □

Proposition 4.3. *Let $\Gamma = A \cup B$ be a uniformly discrete set of sampling for A^p. If one of A or B is a set of interpolation for A^p, then the other will accumulate strongly. The result is sharp in the sense that if $q > p$ and Γ is sampling only for A^q, then neither A nor B need accumulate everywhere. It is also sharp in the sense that if $r < p$ and A and B are interpolating for A^r, then neither A nor B need accumulate everywhere.*

PROOF. If A does not accumulate strongly, then $D^-(A) = 0$. Therefore,

$$D^-(\Gamma) = D^-(A \cup B) \leq D^-(A) + D^+(B) < 1/p,$$

which contradicts the fact that Γ is a set of sampling for A^p. To prove the sharpness of the result, we recall the example provided by Seip in [12].

Let $a > 1, b > 0$ and let

$$\Lambda(a,b) = \{a^m(bn+i)\}_{m,n \in \mathbf{Z}},$$

where $\mathbf{Z}$ is the set of integers. $\Lambda(a,b)$ is a sequence of points in $\mathbf{H}^+$, the upper half-plane. An analytic isomorphism from $\mathbf{D}$ to $\mathbf{H}^+$ is given by

$$\psi(z) = i(\frac{1+z}{1-z})$$

and we define

$$\Gamma(a,b) = \psi^{-1}(\Lambda(a,b)),$$

so $\Gamma(a,b)$ is a sequence of distinct points in $\mathbf{D}$. In fact, $\Gamma(a,b)$ is uniformly discrete and Seip shows that

$$D^-(\Gamma(a,b)) = D^+(\Gamma(a,b)) = \frac{2\pi}{b \log a}.$$

In [8], we introduce a special class of subsequences of $\Gamma(a,b)$.

Let $\mathbf{N}$ be the set of natural numbers. For $u, v, k \in \mathbf{N} \cup \{0\}$ with $u \leq v \leq k-1$, define

$${}_u^v\Lambda_k(a,b) = \{a^m(bn+i)\}_{m \in \mathbf{Z}, n \equiv u \pmod k, n \equiv u+1 \pmod k, \ldots, n \equiv v \pmod k}$$

and

$$ {}_u^v\Gamma_k(a,b) = \psi^{-1}({}_u^v\Lambda_k(a,b)). $$

Lemma 2 in [8] states that

$$ D^-({}_u^v\Gamma_k(a,b)) = D^+({}_u^v\Gamma_k(a,b)) = \frac{2\pi}{b\log a}\left(\frac{v-u+1}{k}\right). $$

We use these subsequences to demonstrate the sharpness of our result. Choose a,b,k,l such that

$$ \frac{2\pi}{b\log a} < \frac{1}{p} \qquad \text{and} \qquad \frac{2\pi}{b\log a}\frac{l}{k} > \frac{1}{q}. $$

Let $A = {}_0^{l-1}\Gamma_k(a,b) \cap \mathbf{H}^+$ and $B = \Gamma(a,b) \cap \overline{\mathbf{H}^-}$, where $\mathbf{H}^-$ is the lower half-plane. Then, by Lemma 4.2,

$$ D^+(A) = \frac{2\pi}{b\log a}\frac{l}{k} < \frac{1}{p} $$

and

$$ D^+(B) = \frac{2\pi}{b\log a} < \frac{1}{p}, $$

so both A and B are A^p interpolating. Also,

$$ A \cup B = {}_0^{l-1}\Gamma_k(a,b) \cup ({}_l^{k-1}\Gamma_k(a,b) \cap \overline{\mathbf{H}^-}), $$

where the union is disjoint. Therefore,

$$ D^-(A \cup B) \geq D^-({}_0^{l-1}\Gamma_k(a,b)) = \frac{2\pi}{b\log a}\frac{l}{k} > \frac{1}{q}. $$

The second sharpness is proved using the same construction. This time choose a,b,k,l such that

$$ \frac{2\pi}{b\log a} < \frac{1}{r} \qquad \text{and} \qquad \frac{2\pi}{b\log a}\frac{l}{k} > \frac{1}{p}. $$

□

We conclude with some remarks about the relationship between Seip's work and zero sets. By the observations made in the introduction, we see that for uniformly discrete Γ,

$$ \Gamma \text{ is an } A^p \text{ zero set } \Rightarrow D^-(\Gamma) \leq \frac{1}{p} \tag{10} $$

and

$$ D^+(\Gamma) < \frac{1}{p} \Rightarrow \Gamma \text{ is an } A^p \text{ zero set.} \tag{11} $$

We show that the necessary condition (10) is far from sufficient and the sufficient condition (11) is far from necessary for Γ to be an A^p zero set, even for Γ uniformly discrete.

Proposition 4.4. *Let $p, M > 0$. There is a uniformly discrete sequence Γ such that $D^-(\Gamma) = 0$ but Γ is not an A^p zero set. There is a uniformly discrete Blaschke sequence Λ such that $D^+(\Lambda) > M$.*

PROOF. Choose a, b such that

$$\frac{2\pi}{b \log a} > \frac{2}{p} \tag{12}$$

and write

$$\Gamma(a,b) = (\Gamma(a,b) \cap \mathbf{H}^+) \cup (\Gamma(a,b) \cap \overline{\mathbf{H}^-}) = A \cup B.$$

Note that $D^-(A) = D^-(B) = 0$. If A and B were both A^p zero sets, then their union would be an $A^{\frac{p}{2}}$ zero set, contradicting (12).

Consider now the second statement of the proposition. Choose any uniformly discrete set Γ with $D^-(\Gamma) = D^+(\Gamma) > M$ and let W be an open disk tangent to $\mathbf{T}$ at 1. Let $\Lambda = \Gamma \cap W$. By Lemma 4.2, $D^+(\Lambda) = D^+(\Gamma) > M$. Since $D^+(\Lambda) < \frac{1}{q}$ for some q, it is an A^q zero set. A well known-result states that an A^q zero set contained in a circle tangent to $\mathbf{T}$ is a Blaschke sequence. This completes the proof of the proposition. $\square$

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