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SOME LIPSCHITZ REGULARITY FOR INTEGRAL KERNELS ON SUBVARIETIES OF PSEUDOCONVEX DOMAINS IN $\mathbb{C}^2$

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ABSTRACT. Let D be a smoothly bounded pseudoconvex domain in $\mathbb{C}^2$. Let M be a one-dimensional subvariety of D which has no singularities on bM and intersects bD transversally. If bM consists of the points of finite type, then we can construct an integral kernel $C^M(\zeta, z)$ for M which satisfies the reproducing property of holomorphic functions $f \in \mathcal{O}(M) \cap C(\overline{M})$ from their boundary values. Furthermore, we get a Lipschitz estimate of the operator induced by the integral kernel, which depends on the type of the boundary bM.

1. INTRODUCTION

The Cauchy kernel $C(\zeta, z)$ (see [7], [10], [12]) for a strongly pseudoconvex domain D in $\mathbb{C}^n$ satisfies the reproducing property of holomorphic functions from their boundary values, that is, for all $f \in A(D) = \mathcal{O}(D) \cap C(\overline{D})$ one has

$$f(z) = \int_{bD} f(\zeta) C(\zeta, z) dS(\zeta) \quad \text{for} \quad z \in D, \tag{1.1}$$

where $dS(\zeta)$ is the surface measure on bD. If $\mathbf{C}f(z)$ denotes the holomorphic function obtained by plugging in an arbitrary function $f \in L^1(bD)$ in the integral in (1.1), then for $0 < \alpha < 1$, the operator $\mathbf{C} : \Lambda_\alpha(bD) \to \mathcal{O}(D) \cap \Lambda_\alpha(D)$ is bounded (see [2], [8], [9], [12]). In [11], Range introduced a new method for constructing integral kernels on bounded pseudoconvex domains in $\mathbb{C}^n$. By using the integral kernel, he obtained Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$. In this paper, we consider an integral kernel for a one-dimensional subvariety M of a smoothly bounded pseudoconvex domain D in $\mathbb{C}^2$. With the finite type condition only on bM we construct an integral kernel $C^M(\zeta, z)$ for

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M that represents holomorphic functions on M in terms of its boundary values along the boundary bM. Furthermore, we get a Lipschitz estimate of the operator induced by the integral kernel, that depends on the type of the boundary bM. For the case of a convex domain $D \subset \mathbb{C}^n$ and a subvariety M of dimension one, without any assumption of finite type, we can obtain Lipschitz estimates of the operator induced by the integral kernel C^M [1].

Let D be a bounded pseudoconvex domain in $\mathbb{C}^2$ with a smooth defining function r. Let $\widetilde{M}$ be a subvariety of dimension one in a neighborhood $\widetilde{D}$ of $\overline{D}$ which has no singularities on bM and intersects bD transversally. Suppose that $\widetilde{M}$ is written in the following form $\widetilde{M} = \{ z \in \widetilde{D} \ ; \ h(z) = 0 \}$, where h is a holomorphic function in $\widetilde{D}$ which satisfies $\partial h \wedge \partial r \neq 0$ on $\widetilde{M} \cap bD$. Let $M = \widetilde{M} \cap D$ and $bM = \widetilde{M} \cap bD$. We can state our main result.

Theorem 1.1. *If bM consists of the points of finite type m, then we can construct an integral kernel $C^M(\zeta, z)$ for M such that for all $f \in A(M)$ one has*

$$f(z) = \int_{bM} f(\zeta) C^M(\zeta, z) d\sigma(\zeta) \quad \text{for} \quad z \in M. \tag{1.2}$$

Moreover, for $f \in L^1(bD)$ if we define

$$(\mathbf{C}^M f)(z) = \int_{bM} f(\zeta) C^M(\zeta, z) d\sigma(\zeta) \quad \text{for} \quad z \in M,$$

then the operator $\mathbf{C}^M : \Lambda_\alpha(bM) \to \mathcal{O}(M) \cap \Lambda_{\frac{\alpha}{m} - \eta}(M)$ is bounded for $0 < \alpha < 1$ and $\eta > 0$.

Remark. In [3], Catlin proved that the boundary of a pseudoconvex domain can be pushed out essentially as far as possible near a boundary point of finite type. In [11], to get Hölder estimates for $\overline{\partial}$ on pseudoconvex domains of finite type in $\mathbb{C}^2$ Range constructed a holomorphic generating form with good estimates. First, Range obtained pointwise estimates for holomorphic L^2 functions on the pushed-out domain. However, the pushed-out domain is only defined locally in a fixed neighborhood of the boundary point of finite type. To extend the locally defined pushed-out domain to the globally defined pseudoconvex domain he used the fact that a pseudoconvex domain $D \subset \mathbb{C}^2$ of finite type is regular in the sense of Diederich and Fornaess [5]. If we assume that every boundary point of the boundary bD is finite type, then we can quote Range's result directly. However, in our case, we stress the point that we assume the finite type condition only on bM. For the construction of the globally defined pseudoconvex domain we use Catlin's bumping theorem instead of the theorem of Diederich and Fornaess.

2. Construction of an integral kernel for M

Let $p_0 \in bM$ be a point of finite type m. For each $p \in U_0 \cap bD$, where U_0 is sufficiently small neighborhood of p_0, we introduce a special holomorphic coordinate system $\zeta(z) = \zeta_p(z)$ as in Catlin [3], Proposition 1.1. ζ is defined by a holomorphic map $\phi_p : \mathbb{C}^2 \to \mathbb{C}^2, z = \phi_p(\zeta)$, with $\phi_p(0) = p$. The defining function $\rho = r \circ \phi_p$ for the domain $\Omega_p = \phi_p^{-1}(D)$ has the form

$$\rho(\zeta) = \mathrm{Re}\ \zeta_2 + \sum_{\substack{j+k\leq m \\ j,k>0}} a_{j,k}(p)\zeta_1^j\overline{\zeta}_1^k + \mathcal{O}(|\zeta_1|^{m+1} + |\zeta_2||\zeta|).$$

The function ϕ_p and the coefficients $a_{j,k}(p)$ depend smoothly on $p \in U_0 \cap bD$. For $l = 2, \ldots, m$, and $\delta > 0$, set

$$A_l(p) = \max \{\ |a_{j,k}(p)|\ ;\ j+k = l\ \}$$

and

$$\tau(p,\delta) = \min\left\{ \left(\frac{\delta}{A_l(p)}\right)^{1/l}\ ;\ 2 \leq l \leq m \right\}.$$

For $\delta \geq 0$ we define

$$J_\delta(p,\zeta) = \left[\delta^2 + |\zeta_2|^2 + \sum_{k=2}^{m} A_k(p)^2|\zeta_1|^{2k}\right]^{1/2},$$

and for $a > 0$, we define the nonisotropic polydisc $P_\delta^{(a)}(\zeta')$ centered at ζ' by

$$P_\delta^{(a)}(\zeta') = \{\zeta \in \mathbb{C}^2; |\zeta_2 - \zeta_2'| < aJ_\delta(p,\zeta'), |\zeta_1 - \zeta_1'| < \tau(p, aJ_\delta(p,\zeta'))\}.$$

We set $J(p,\zeta) = J_0(p,\zeta)$ and $P^{(a)}(\zeta') = P_0^{(a)}(\zeta')$.

We will now push out the boundary of Ω_p near the origin maintaining pseudoconvexity. We fix $c > 0$. For all small s and $\delta > 0$ define

$$W_{s,\delta}(p) = \{\zeta \in \mathbb{C}^2; |\zeta| < c \quad \text{and} \quad |\rho(\zeta)| < sJ_\delta(p,\zeta)\}.$$

Let $H_{p,\delta}$ be the smooth real function on $W_{s,\delta}(p)$ given by Proposition 4.1 in [3]. Set $\rho_{p,\delta}^\epsilon(\zeta) = \rho(\zeta) + \epsilon H_{p,\delta}(\zeta)$, with $\epsilon < 0$. Catlin proved ([3], p.449-453) that c, ϵ, s and δ_0 can be chosen so that for all $0 < \delta \leq \delta_0$ the set

$$S_\delta = \{\zeta \in W_{s,\delta}(p); \rho_{p,\delta}^\epsilon(\zeta) = 0\}$$

is a smooth pseudoconvex hypersurfaces (from the side $\rho_{p,\delta}^\epsilon < 0$), and that the constants can be chosen independently of $p \in U_0 \cap bD$. Thus we fix $\epsilon = \epsilon_0$ and we let $\rho_{p,\delta}(\zeta) = \rho_{p,\delta}^\epsilon(\zeta)$. It follows that

$$\Omega_{p,\delta} = \{|\zeta| < c; \rho(\zeta) < 0\} \cup \{\zeta \in W_{s,\delta}(p); \rho_{p,\delta}(\zeta) < 0\}$$

is a pseudoconvex domain. Catlin ([3], Lemma 3) proved that there exists a constant $a > 0$ (independent of p, ζ', and δ) so that if $\zeta' \in \overline{\Omega}_p$ and $|\zeta'| < a$, then

$$P_\delta^{(a)}(\zeta') \subset \Omega_{p,\delta}.$$

In [3], Catlin proved a bumping theorem near a boundary point of finite type.

Theorem 2.1. *Let p_0 be a point of finite type in the boundary of a pseudoconvex domain D in $\mathbb{C}^2$, defined by $D = \{z; r(z) < 0\}$. Then for any sufficiently small neighborhood V of p_0, there exists a smooth 1-parameter family of pseudoconvex domain $D_t, 0 \leq t < \alpha_0$, each defined by $D_t = \{z; r(z,t) < 0\}$, where $r(z,t)$ has the following properties:*

(i) *$r(z,t)$ is smooth in z for z near bD, and in t for $0 \leq t < \alpha_0$,*
(ii) *$r(z,t) = r(z)$, for $z \notin V$,*
(iii) *$\frac{\partial r}{\partial t}(z,t) \leq 0$,*
(iv) *$r(z,0) = r(z)$, and*
(v) *for z in V, $\frac{\partial r}{\partial t} < 0$.*

Remark. From the construction of ϕ_p and $\rho_{p,\delta}$, for $p_0 \in bM$ we can choose c and a neighborhood $U_0 \Subset V$ of p_0 (independent of p) so that $\rho_{p,\delta}$ is defined in $\{\zeta; |\zeta| < c\}$ and satisfies all the properties in this section for all $p \in U_0 \cap bD$.

Definition 1. Suppose $D, p_0 \in bD$, and V be as in Theorem 2.1. Then we say $\{D_t\}_{0 \leq t < \alpha_0}$ *a bumping family of D at p_0 with front V.*

Let ϕ_p be the map associated with p and set

$$\Omega_t = \{\zeta \in \mathbb{C}^2; \phi_p(\zeta) \in D_t\},$$

where D_t is the family of domains given in Theorem 2.1. If we choose sufficiently small neighborhood U_0 of p_0, then there is a constant $c_1 > 0$ and sufficiently small t_0 with $0 < t_0 < \alpha_0$ so that if $p \in U_0 \cap bD$, then

$$d(\zeta, b\Omega_{p,\delta}) \geq c_1 \quad \text{if} \quad \frac{c}{2} < |\zeta| < c$$

and

$$d(\zeta, b\Omega_{t_0}) < \frac{c_1}{2} \quad \text{if} \quad \frac{c}{2} < |\zeta| < c$$

(see p.456 in [3]).

Now, we will extend the locally defined pushed-out domain $\Omega_{p,\delta}$ to the globally defined pseudoconvex domain which contains Ω_p and which is bumped out near $\phi_p^{-1}(bM)$.

Let $p_1 \in bM \setminus V \subset bD_{t_0}$. There exists a bumping family $\{D_{t_0 t}\}_{0 \le t < \alpha_1}$ of D_{t_0} at p_1 with front $B(p_1, \epsilon_1)$ for small $\epsilon_1 > 0$. Choose t_1 with $0 < t_1 < \alpha_1$. Since $bM \setminus V$ is compact, by induction, we can choose $p_1, \ldots, p_N \in bM \setminus V$, $\epsilon_1, \ldots, \epsilon_N > 0$, and $t_1, \ldots, t_{N-1} > 0$ such that

(i) $bM \setminus V \subset \cup_{i=1}^N B(p_i, \epsilon_i)$,
(ii) $p_i \notin B(p_j, \epsilon_j)$ for $i \neq j$, and
(iii) $\{D_{t_0 t_1 \ldots t_{i-1} t}\}_{0 \le t < \alpha_i}$ is a bumping family of $D_{t_0 \ldots t_{i-1}}$ at p_i with front $B(p_i, \epsilon_i)$ for $i = 1, \ldots, N$.

We choose t_N with $0 < t_N < \alpha_N$ and set $D_{t_*} = D_{t_0 \ldots t_N}$. If $t_0, \ldots, t_N$ are sufficiently small, then $D_{t_*} \cap \widetilde{M} \Subset \widetilde{M}$ and $\widetilde{M}$ intersects bD_{t_*} transversally. We define Ω_{t_*} by $\Omega_{t_*} = \{\zeta \in \mathbb{C}^2; \phi_p(\zeta) \in D_{t_*}\}$, and a domain $\widetilde{\Omega}_{p,\delta}$ by

$$\widetilde{\Omega}_{p,\delta} = \{\zeta \in \Omega_{t_*}; |\zeta| \ge c\} \cup \{\Omega_{t_*} \cap \Omega_{p,\delta}\}.$$

Since pseudoconvexity is a local condition, $\widetilde{\Omega}_{p,\delta}$ is a pseudoconvex domain. By combing the properties of $\Omega_{p,\delta}$, Ω_{t_0}, and Ω_{t_*}, we obtain the following results as in ([3], Lemma 2.8).

Lemma 2.2. *For all $p \in U_0 \cap bD$ and all $\delta, 0 < \delta < \delta_0$, the domain $\widetilde{\Omega}_{p,\delta}$ has the following properties:*

(i) *$\widetilde{\Omega}_{p,\delta}$ is a bounded pseudoconvex domain that contains Ω_p,*
(ii) *$\phi_p^{-1}(\overline{M}) \subset \phi_p^{-1}(\widetilde{M}) \cap \widetilde{\Omega}_{p,\delta}$, and*
(iii) *there is a constant $a_1 > 0$ so that for all $\zeta' \in \overline{\Omega}_p$ with $|\zeta'| < c$,*

$$P_\delta^{(a_1)}(\zeta') \subset \widetilde{\Omega}_{p,\delta}.$$

Now we define

$$\Omega_p^* = \operatorname{Int}\left[\bigcap_{0<\delta<\delta_0} \widetilde{\Omega}_{p,\delta}\right].$$

Proposition 2.3. *Ω_p^* is a bounded pseudoconvex domain such that*

(i) *$0 \in b\Omega_p^*$,*
(ii) *$\Omega_p \subset \Omega_p^*$,*
(iii) *$\phi_p^{-1}(\overline{M}) \setminus \{0\} \subset \phi_p^{-1}(\widetilde{M}) \cap \Omega_p^*$, if $p \in bM$ and $\phi_p^{-1}(\overline{M}) \subset \phi_p^{-1}(\widetilde{M}) \cap \Omega_p^*$, if $p \in bD \setminus bM$, and*
(iv) *$P^{(a_1)}(\zeta') \subset \Omega_p^*$ for $\zeta' \in \Omega_p$ and $|\zeta'| < c$.*

Let $\delta^*(\zeta) = \operatorname{dist}(\zeta, b\Omega_p^*)$ for $\zeta \in \Omega_p^*$. Suppose $h \in \mathcal{O}(\Omega_p^*)$ satisfies

$$\int_{\Omega_p^*} \frac{|h(\zeta)|^2 \delta^{*2\eta}(\zeta)}{|\zeta|^2} dV(\zeta) = (T_\eta(h))^2 < \infty \quad \text{for some} \quad \eta > 0.$$

By Proposition 2.3 and Cauchy estimates on $P^{(a_1)}(\zeta) \subset \Omega_p^*$ as in [11], for $\zeta \in \Omega_p$ with $|\zeta| < c$ it follows that

$$|h(\zeta)| \lesssim \frac{T_\eta(h)}{(|\rho(\zeta)| + |\zeta_2| + |\zeta|^m)^{1+\eta}}, \tag{2.1}$$

$$\left|\frac{\partial h}{\partial \zeta_1}(\zeta)\right| \lesssim \frac{T_\eta(h)}{|\zeta|(|\rho(\zeta)| + |\zeta_2| + |\zeta|^m)^{1+\eta}}, \quad \text{and} \tag{2.2}$$

$$\left|\frac{\partial h}{\partial \zeta_2}(\zeta)\right| \lesssim \frac{T_\eta(h)}{(|\rho(\zeta)| + |\zeta_2| + |\zeta|^m)^{2+\eta}}. \tag{2.3}$$

We now must transport these estimates back to the domain D. Define the domain D_p by $D_p = \phi_p(\Omega_p^*)$.

Proposition 2.4. *D_p is a bounded pseudoconvex domain such that*

(i) $p \in bD_p$,
(ii) $D \subset D_p$, *and*
(iii) $\overline{M} \setminus \{p\} \subset \widetilde{M} \cap D_p$, *if* $p \in bM$ *and* $\overline{M} \subset \widetilde{M} \cap D_p$, *if* $p \in bD \setminus bM$.

For $p \in U_0 \cap bD$ we set $\delta_p(z) = \operatorname{dist}(z, bD_p)$, and given $\eta > 0$, we define the weighted L^2 norm $I_{p,\eta}$ on D_p by

$$I_{p,\eta}(h) = \left[\int_{D_p} \frac{|h(z)|^2}{|z-p|^2} \delta_p^{2\eta}(z) dV(z)\right]^{1/2}.$$

Furthermore, let $g(p,\cdot)$ denote the second component of the inverse of the biholomorphic map $\phi_p : \mathbb{C}^2 \to \mathbb{C}^2$. After perhaps shrinking U_0, we may choose a fixed orthonormal frame $\{L_1, L_2\}$ for $T^{0,1}$ on a neighborhood of $\overline{U}_0$ which satisfies $L_1 r = 0$.

Proposition 2.5. *There are positive constants c and C, such that for all $p \in U_0 \cap bD$ the following holds. If $h \in \mathcal{O}(D_p)$ and $I_{p,\eta}(h) < \infty$ for some $\eta > 0$, then*

(i)

$$|h(z)| + |d_z h(z)| \le C I_{p,\eta}(h) \quad \textit{for} \quad z \in M \quad \textit{with} \quad |z-p| \ge c,$$

and if $z \in D$ with $|z-p| < c$, then

(ii)

$$|h(z)| \le C \frac{I_{p,\eta}(h)}{[|r(z)| + |g(p,z)| + |z-p|^m]^{1+\eta}},$$

(iii)

$$|L_1 h(z)| \le C \frac{I_{p,\eta}(h)}{|z-p|[|r(z)|+|g(p,z)|+|z-p|^m]^{2+\eta}}, \quad \text{and}$$

(iv)

$$|L_2 h(z)| \le C \frac{I_{p,\eta}(h)}{[|r(z)|+|g(p,z)|+|z-p|^m]^{2+\eta}}.$$

PROOF. Since $D_{t_*} \cap \{|z-p| \ge c\} = D_p \cap \{|z-p| \ge c\}$, $\widetilde{M}$ intersects bD_p transversally on $\{|z-p| \ge c\}$, and hence $\operatorname{dist}(\overline{M} \setminus \{|z-p| < c\}, \widetilde{M} \cap bD_p) \approx \operatorname{dist}(\overline{M} \setminus \{|z-p| < c\}, bD_p)$. From (iii) in Proposition 2.4 it follows that $\operatorname{dist}(\overline{M} \setminus \{|z-p| < c\}, \widetilde{M} \cap bD_p) > 0$. Hence we get $\delta_p(z) > 0$ for $z \in M$ with $|z-p| \ge c$ and (i) is an immediate consequence of Cauchy estimates.

For (ii), (iii), and (iv), one pulls back the estimates given by (2.1), (2.2), and (2.3) via the map ϕ_p^{-1}, using the fact that the Jacobian determinants of ϕ_p and ϕ_p^{-1} are bounded by constants which are independent of p. □

Let $D_\epsilon = \{z \in D; r(z) < -\epsilon\}$ and $M_\epsilon = M \cap D_\epsilon$ for $\epsilon > 0$. Let us introduce $\Gamma_\epsilon(\zeta, z) = \operatorname{dist}(z, bD_\epsilon) + |g(\zeta,z)| + |\zeta - z|^m$. By Skoda's theorem ([13], Theorem 1) and a partition of unity, we can obtain the following results as in ([11], p.70-71).

Theorem 2.6. *Let $p_0 \in bM$ be a point of finite type m and let U_0 be a sufficiently small neighborhood of p_0. Let $\eta > 0$. If $\epsilon > 0$ is sufficiently small, then there are $c > 0, C_\eta < \infty$ and C^∞ functions $h_j^{(\epsilon)}$ on $(U_0 \cap bD) \times D_\epsilon$, $j = 1, 2$, with the following properties:*

(i) $h_1^{(\epsilon)}(\zeta,z)(z_1 - \zeta_2) + h_2^{(\epsilon)}(\zeta,z)(z_2 - \zeta_2) = 1$;

(ii) $h_j^{(\epsilon)}(\zeta, \cdot) \in \mathcal{O}(D_\epsilon)$ *for* $\zeta \in U_0 \cap bD$;

(iii) $|h_j^{(\epsilon)}(\zeta,z)| + |d_z h_j^{(\epsilon)}(\zeta,z)| \le C_\eta$ *for* $z \in M_\epsilon$ *with* $|z-\zeta| \ge c$;

(iv) $|h_j^{(\epsilon)}(\zeta,z)| \le \frac{C_\eta}{\Gamma_\epsilon(\zeta,z)^{1+\eta}}$,

(v) $|L_1 h_j^{(\epsilon)}(\zeta,z)| \le \frac{C_\eta}{|z-\zeta|\Gamma_\epsilon(\zeta,z)^{1+\eta}}$, *and*

(vi) $|L_2 h_j^{(\epsilon)}(\zeta,z)| \le \frac{C_\eta}{\Gamma_\epsilon(\zeta,z)^{2+\eta}}$,

for $z \in D_\epsilon$ with $|z-\zeta| < c$.

The functions $h_j^{(\epsilon)}$ depend also on η, but the constants C_η are independent of $\epsilon > 0$.

For $p_0 \in bD \setminus bM$ also, we can apply Skoda's theorem to the domain D. In this case, there are C^∞ functions $h_j^{(\epsilon)}$ on $(U_0 \cap bD) \times D_\epsilon, j = 1, 2$, with the following properties:

(i) $h_1^{(\epsilon)}(\zeta,z)(z_1-\zeta_2)+h_2^{(\epsilon)}(\zeta,z)(z_2-\zeta_2)=1$ and
(ii) $h_j^{(\epsilon)}(\zeta,\cdot)\in\mathcal{O}(D_\epsilon)$ for $\zeta\in U_0\cap bD$.

However, in this case we cannot obtain the pointwise estimates for $h_j^{(\epsilon)}$ and their derivatives. By Theorem 2.6 and (i), (ii) above, for any point $p_0\in bD$ and a sufficiently small neighborhood U_0 of p_0 we can construct the functions $h_j^{(\epsilon)}$ on $(U_0\cap bD)\times D_\epsilon$ which satisfy the properties (i) and (ii) above. By compactness of bD we can use a partition of unity in ζ to patch together the locally defined functions $h_j^{(\epsilon)}$ to obtain smooth functions $w_j^{(\epsilon)}$ on $bD\times D_\epsilon, j=1,\dots,n$ which are holomorphic in z and satisfy

$$w_1^{(\epsilon)}(\zeta,z)(z_1-\zeta_1)+w_2^{(\epsilon)}(\zeta,z)(z_2-\zeta_2)=1 \quad \text{for} \quad (\zeta,z)\in bD\times D_\epsilon.$$

Furthermore, at the boundary point $\zeta\in bM$, $w_j^{(\epsilon)}$ satisfy the pointwise estimates stated in Theorem 2.6. By the theorem of Hatziafratis [6], for $f\in A(M)=\mathcal{O}(M)\cap C(\overline{M})$, and $z\in M_\epsilon$, it follows that

$$f(z)=\int_{bM}f(\zeta)C_\epsilon(\zeta,z)d\sigma(\zeta) \tag{2.4}$$

where $C_\epsilon(\zeta,z)=\sum_{j=1}^2 w_j^{(\epsilon)}(\zeta,z)\varphi_j(\zeta,z)$ and $\varphi_j(\zeta,z)$ are C^∞ functions in $\overline{D}\times\overline{D}$ depending holomorphically on z.

Lemma 2.7. *For $\zeta\in bM$ we can choose a subsequence $C_k(\zeta,\cdot)$ of $C_\epsilon(\zeta,\cdot)$ which converges uniformly on each compact subset of D.*

PROOF. Let $\{\epsilon_j\}$ be a decreasing sequence of positive numbers, with $\epsilon_j\to 0$ as $j\to\infty$. Let $C_j(\zeta,z)=C_{\epsilon_j}(\zeta,z)$ and $D_j=D_{\epsilon_j}$. For $\zeta\in bM, z\in\overline{D}_k$, and $j>k+1$, it follows that $\operatorname{dist}(z,bD_j)>0$ and $|\zeta-z|\gtrsim\epsilon_k$. Thus for $\eta>0$,

$$\begin{aligned}|C_j(\zeta,z)|&\le\frac{C_\eta}{[\Gamma_{\epsilon_j}(\zeta,z)]^{1+\eta}}\\&\le\frac{C_\eta}{\epsilon_k^{m(1+\eta)}}.\end{aligned}$$

Thus $\{C_j(\zeta,\cdot);j>k+1\}$ is uniformly bounded in $\overline{D}_k$. Therefore we can choose a subsequence $C_{j_k}(\zeta,\cdot)$ of $C_j(\zeta,\cdot)$ which converges uniformly in $\overline{D}_{k-1}$. Denote $k:=k_k$. Then for $\zeta\in bM$, $C_k(\zeta,\cdot)$ converges uniformly on each compact subset of D. □

Define $C^M(\zeta, z) = \lim_{k\to\infty} C_k(\zeta, z)$ for $(\zeta, z) \in bM \times D$. Then $C^M(\zeta, z)$ depends holomorphically on z and, by (2.4),

$$f(z) = \int_{bM} f(\zeta) C^M(\zeta, z) d\sigma(\zeta)$$

for $f \in A(M)$ and $z \in M$. Thus we proved (1.2) in Theorem 1.1. For $f \in L^1(bM)$, define

$$\mathbf{C}_k f(z) = \int_{bM} f(\zeta) C_k(\zeta, z) d\sigma(\zeta) \quad \text{for} \quad z \in M_k$$

and

$$\mathbf{C}^M f(z) = \int_{bM} f(\zeta) C^M(\zeta, z) d\sigma(\zeta) \quad \text{for} \quad z \in M.$$

Then

$$\lim_{k\to\infty} \mathbf{C}_k f(z) = \mathbf{C}^M f(z) \quad \text{for} \quad z \in M \tag{2.5}$$

and the convergence is uniform on each compact subset of M.

3. Lipschitz estimates for the integral kernel

By (2.5), it is enough to show that for sufficiently small ϵ,

$$\mathbf{C}_\epsilon : \Lambda_\alpha(bM) \to \mathcal{O}(M_\epsilon) \cap \Lambda_{\frac{\alpha}{m}-\eta}(M_\epsilon) \quad \text{is bounded.}$$

Thus we shall prove that there is a constant $C_\eta < \infty$ such that

$$\left| d_z \int_{bM} f(\zeta) C_\epsilon(\zeta, z) d\sigma(\zeta) \right| \le C_\eta |f|_{\Lambda_\alpha(bM)} \mathrm{dist}(z, bM_\epsilon)^{-1+(\frac{\alpha}{m}-\eta)} \quad \text{for} \quad z \in M_\epsilon.$$

Applying (2.4) to the function $f \equiv 1$ gives $d_z \int_{bM} C_\epsilon(\zeta, z) d\sigma(\zeta) \equiv 0$. For $z \in M_\epsilon$ fixed, we choose $z' \in bM$ with $|z - z'| = \mathrm{dist}(z, bM)$, so that $|\zeta - z'| \le 2|\zeta - z|$ for $\zeta \in bM$. Since $\int_{bM} f(z') d_z C_\epsilon(\zeta, z) d\sigma(\zeta) \equiv 0$, it follows that

$$\int_{bM} f(\zeta) d_z C_\epsilon(\zeta, z) d\sigma(\zeta) = \int_{bM} (f(\zeta) - f(z')) d_z C_\epsilon(\zeta, z) d\sigma(\zeta),$$

and hence

$$\left| \int_{bM} f(\zeta) d_z C_\epsilon(\zeta, z) d\sigma(\zeta) \right| \lesssim |f|_{\Lambda_\alpha} \int_{bM} |\zeta - z|^\alpha |d_z C_\epsilon(\zeta, z)| d\sigma(\zeta). \tag{3.1}$$

Because of (iii) in Theorem 2.6, the nontrivial case occurs for $z \in M_\epsilon$ and $|\zeta - z| < c$. Since M meets bD transversally, if ϵ is sufficiently small, then M_ϵ also meets bD_ϵ transversally. Thus $\mathrm{dist}(z, bM_\epsilon) \approx \mathrm{dist}(z, bD_\epsilon)$ for $z \in M_\epsilon$. Also, note that $\mathrm{dist}(z, bM_\epsilon) \lesssim |\zeta - z|$ for $z \in M_\epsilon$ and $\zeta \in bM$, and hence it follows that

$\Gamma_\epsilon(\zeta, z) \lesssim |\zeta - z|$ for $z \in M_\epsilon$ and $\zeta \in bM$. Thus, it follows from $\Gamma_\epsilon(\zeta, z) \gtrsim |\zeta - z|^m$ that the integral in (3.1) over $\{\zeta \in bM; |\zeta - z| < c\}$ is bounded uniformly by

$$C_\eta |f|_{\Lambda_\alpha} \int_{bM \cap \{|\zeta - z| < c\}} \frac{d\sigma(\zeta)}{\Gamma_\epsilon(\zeta, z)^{2+\eta-\frac{\alpha}{m}}} \tag{3.2}$$
$$\leq C_\eta |f|_{\Lambda_\alpha(bM)} \mathrm{dist}(z, bM_\epsilon)^{\frac{\alpha}{m}-\eta} \int_{bM \cap \{|\zeta - z| < c\}} \frac{d\sigma(\zeta)}{\Gamma_\epsilon(\zeta, z)^2}.$$

Let $t_1 + it_2 = h(\zeta), t_3 = r(\zeta)$, and $t_4 = \mathrm{Im}\ g(\zeta, z)$. In [11], Range proved that $dr(p_0) \wedge d_\zeta \mathrm{Im}\ g(p_0, p_0) \neq 0$. Thus for $z \in bM$ fixed, if $0 < \gamma \leq c$ is sufficiently small, t_1, t_2, t_3, and t_4 form a local coordinate system on $B(z, \gamma)$ in such a way that $t_4 = \mathrm{Im}\ g(\zeta, z)$ is the local coordinate of $bM \cap B(z, \gamma)$. Of course, for this we need the transversal assumption on the intersections of $\widetilde{M}$ and bD. Given that $t_4 = \mathrm{Im}\ g(\zeta, z)$ is a local coordinate on $bM \cap B(z, \gamma)$, it follows that the integral on the right in (3.2) is estimated by $\mathrm{dist}(z, bM_\epsilon)^{-1}$. Thus, altogether, one obtains the required result.

Remark. Regularity properties of the integral kernel $C^M(\zeta, z)$ for M on the Hardy classes H^p were studied in [4] by the author.

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