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A FOURTH ORDER NONLINEAR ELLIPTIC EQUATION WITH JUMPING NONLINEARITY

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ABSTRACT. We investigate the existence of solutions of the fourth order nonlinear elliptic boundary value problem under Dirichlet boundary condition $\Delta^2 u + c\Delta u = bu^+ + f$ in Ω, where Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary and the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\Delta$. We also investigate a relation between multiplicity of solutions and source terms of the equation with the nonlinearity crossing an eigenvalue.

1. INTRODUCTION

We investigate the existence of solutions of the fourth order nonlinear elliptic boundary value problem

$$\begin{aligned} \Delta^2 u + c\Delta u = bu^+ + f \quad &\text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad &\text{on} \quad \partial\Omega, \end{aligned} \tag{1}$$

where $u^+ = max\{u, 0\}$ and c is not an eigenvalue of $-\Delta$ under Dirichlet boundary condition. Here we assume that Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The operator Δ^2 denotes the biharmonic operator. We assume that b is not an eigenvalue of $\Delta^2 + c\Delta$ under Dirichlet boundary condition.

The nonlinear equation with jumping nonlinearity have been extensively studied by many authors [3,4,6,7,8]. They studied the existence of solutions of the nonlinear equation with jumping nonlinearity for the second order elliptic operator [6], for one dimensional wave operator [3,4], and for the other operators [7,8] when the source term is a multiple of the positive eigenfunction.

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In [10], Tarantello considered the fourth order, nonlinear elliptic problem under the Dirichlet boundary condition

$$\begin{aligned} \Delta^2 u + c\Delta u = b[(u+1)^+ + 1] \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{2}$$

She showed by degree theory that if $b \geq \lambda_1(\lambda_1 - c)$, then 2 has a solution u such that $u(x) < 0$ in Ω.

In this paper we investigate the existence of solutions of the fourth order nonlinear equation 1 when the nonlinearity bu^+ crosses eigenvalues of $\Delta^2 + c\Delta$ under Dirichlet boundary condition.

In section 1, we introduce the Banach space spanned by eigenfunctions of $\Delta^2 + c\Delta$ and investigate the existence of solutions of 1 when the nonlinearity bu^+ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ and when it satisfies $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$.

In section 2, we investigate the multiplicity of solutions of 1 under the following two conditions.

Condition(1) : $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $f = s > 0$.

Condition(2) : $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $s < 0$.

In section 3, we investigate a relation between multiplicity of solutions and source terms of 1 with the nonlinearity crossing an eigenvalue.

2. The Banach space spanned by eigenfunctions

In this section we introduce the Banach space spanned by eigenfunctions of the operator $\Delta^2 + c\Delta$ and we investigate the existence of solutions of the boundary value problem

$$\begin{aligned} \Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3}$$

Here s is real, c is not an eigenvalue of $-\Delta$ under Dirichlet boundary condition and the nonlinearity bu^+ satisfies $\lambda_1 < c$, $b < \lambda_1(\lambda_1 - c)$ or $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$.

Let $\lambda_k (k = 1, 2, \cdots)$ denote the eigenvalues and $\phi_k (k = 1, 2 \cdots)$ the corresponding eigenfunctions, suitably normalized with respect to $L^2(\Omega)$ inner product, of the eigenvalue problem $\Delta u + \lambda u = 0$ in Ω, under Dirichlet boundary condition, where each eigenvalue λ_k is repeated as often as its multiplicity. We recall that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, $\lambda_i \to +\infty$ and that $\phi_1(x) > 0$ for $x \in \Omega$. The eigenvalue problem

$$\begin{aligned} \Delta^2 u + c\Delta u = \mu u \quad \text{in} \quad \Omega, \\ u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega \end{aligned}$$

has infinitely many eigenvalues

$$\mu_k = \lambda_k(\lambda_k - c), \qquad k = 1, 2, \cdots$$

and corresponding eigenfunctions $\phi_k(x)$.
The set of functions $\{\phi_k\}$ is an orthogonal base for $W_0^{1,2}(\Omega)$. Let us denote an element u of $W_0^{1,2}(\Omega)$ as

$$u = \sum h_k \phi_k, \qquad \sum h_k^2 < \infty.$$

Let c be not an eigenvalue of $-\Delta$ and define a subspace H of $W_0^{1,2}(\Omega)$ as follows

$$H = \{u \in W_0^{1,2}(\Omega) : \sum |\lambda_k(\lambda_k - c)| h_k^2 < \infty\}.$$

Then this is a complete normed space with a norm

$$|||u||| = [\sum |\lambda_k(\lambda_k - c)| h_k^2]^{1/2}.$$

Since $\lambda_k \to +\infty$ and c is fixed, we have the following simple properties.

Proposition 2.1. *Let c be not an eigenvalue of $-\Delta$ under Dirichlet boundary condition. Then we have : For $u \in W_0^{1,2}(\Omega)$,*
(i) $\Delta^2 u + c\Delta u \in H$ implies $u \in H$.
(ii) $|||u||| \geq C\|u\|_{L^2(\Omega)}$ for some $C > 0$.
(iii) $\|u\|_{L^2(\Omega)} = 0$ if and only if $|||u||| = 0$.

PROOF. (i) Suppose c is not an eigenvalue of $-\Delta$ and let $u = \sum h_k \phi_k$. Then

$$\Delta^2 u + c\Delta u = \sum \lambda_k(\lambda_k - c) h_k \phi_k.$$

Hence

$$\begin{aligned} \infty > |||\Delta^2 u + c\Delta u|||^2 &= \sum |\lambda_k(\lambda_k - c)|(\lambda_k(\lambda_k - c))^2 h_k^2 \\ &\geq C \sum |\lambda_k(\lambda_k - c)| h_k^2 = |||u|||^2, \end{aligned}$$

where $C = \inf_k \{[\lambda_k(\lambda_k - c)]^2 : k = 1, 2, \cdots\}$. (ii) and (iii) are trivial. □

Lemma 2.2. *Let d be not an eigenvalue of $\Delta^2 + c\Delta$ and $u \in L^2(\Omega)$. Then $(\Delta^2 + c\Delta + d)^{-1} u \in H$.*

PROOF. Suppose that d is not an eigenvalue of $\Delta^2 + c\Delta$ and finite. We know that the number of elements of $\{\lambda_k(\lambda_k - c) : |\lambda_k(\lambda_k - c)| < |d|\}$ is finite, where $\lambda_k(\lambda_k - c)$ is an eigenvalue of $\Delta^2 + c\Delta$. Let $u = \sum h_k \phi_k$. Then

$$(\Delta^2 + c\Delta + d)^{-1} u = \sum \frac{1}{\lambda_k(\lambda_k - c) + d} h_k \phi_k.$$

Hence we have the inequality

$$|||(\Delta^2 + c\Delta + d)^{-1}u|||^2 = \sum |\lambda_k(\lambda_k - c)| \frac{1}{(\lambda_k(\lambda_k - c) + d)^2} h_k^2 \le C \sum h_k^2$$

for some C, which means that

$$|||(\Delta^2 + c\Delta + d)^{-1}u||| \le C_1 \|u\|_{L^2(\Omega)}, \quad C_1 = \sqrt{C}.$$

□

With Lemma 2.2, we can obtain the following lemma.

Lemma 2.3. *Let $f \in L^2(\Omega)$. Let b be not an eigenvalue of $\Delta^2 + c\Delta$. Then all solutions in $W_0^{1,2}(\Omega)$ of*

$$\Delta^2 u + c\Delta u = bu^+ + f(x)$$

belong to H.

With the aid of Lemma 2.3, it is enough to investigate the existence of solutions of 3 in the subspace H of $W_0^{1,2}(\Omega)$, namely,

$$\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad H. \tag{4}$$

Let $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c)$, $\lambda_{k+1}(\lambda_{k+1} - c)$ be successive eigenvalues of $\Delta^2 + c\Delta$ such that there is no eigenvalue between $\lambda_k(\lambda_k - c)$ and $\lambda_{k+1}(\lambda_{k+1} - c)$. Then $\lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c)$ and we have the uniqueness theorem.

Theorem 2.4. *Suppose $\lambda_k < c < \lambda_{k+1}$ and $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$. Then equation 4 has exactly one solution in $L^2(\Omega)$ for all real s. Furthermore equation 4 has a unique solution in H.*

PROOF. We consider the equation

$$-\Delta^2 u - c\Delta u + bu^+ = -s \quad \text{in} \quad L^2(\Omega). \tag{5}$$

Let $\delta = \frac{1}{2}\{\lambda_k(\lambda_k - c) + \lambda_{k+1}(\lambda_{k+1} - c)\}$. Then equation 5 is equivalent to

$$u = (-\Delta^2 - c\Delta + \delta)^{-1}[(\delta - b)u^+ - \delta u^- - s], \tag{6}$$

where $(-\Delta^2 - c\Delta + \delta)^{-1}$ is a compact, self-adjoint, linear map from $L^2(\Omega)$ into $L^2(\Omega)$ with norm $\dfrac{2}{\lambda_{k+1}(\lambda_{k+1} - c) - \lambda_k(\lambda_k - c)}$. We note that

$$\begin{aligned} |(\delta - b)(u_2^+ - u_1^+) - \delta(u_2^- - u_1^-)\| &\le \max\{|\delta - b|, |\delta|\}\|u_2 - u_1\| \\ &< \frac{1}{2}\{\lambda_{k+1}(\lambda_{k+1} - c) - \lambda_k(\lambda_k - c)\}\|u_2 - u_1\|. \end{aligned}$$

It follows that the right hand side of 6 defines a Lipschitz mapping from $L^2(\Omega)$ into $L^2(\Omega)$ with Lipschitz constant $\gamma < 1$. Therefore, by the contraction mapping principle, there exists a unique solution $u \in L^2(\Omega)$ of 6.

On the other hand, by Lemma 2.3, the solution of 6 belongs to H. □

We now examine equation 4 when $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$.

Theorem 2.5. *Assume that $\lambda_1 < c$ and $b < \lambda_1(\lambda_1 - c) < 0$. Then we have :*

(i) If $s < 0$, then equation 4 has no solution.

(ii) If $s = 0$, then equation 4 has only the trivial solution.

PROOF. Assume $s \leq 0$. We rewrite 4 as

$$\{-\Delta^2 - c\Delta + \lambda_1(\lambda_1 - c)\}u + \{-\lambda_1(\lambda_1 - c) + b\}u^+ - \{-\lambda_1(\lambda_1 - c)\}u^- = -s.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{-\Delta^2 - c\Delta + \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega [\{-\lambda_1(\lambda_1 - c) + b\}u^+ - \{-\lambda_1(\lambda_1 - c)\}u^-]\phi_1 = -s\int_\Omega \phi_1. \tag{7}$$

But $\{-\lambda_1(\lambda_1 - c) + b\}u^+ - \{-\lambda_1(\lambda_1 - c)\}u^- \leq 0$ for all real valued function u and $\phi_1(x) > 0$ for $x \in \Omega$. Therefore the left hand side of 7 is always less than or equal to zero. Hence if $s < 0$, then there is no solution of 4 and if $s = 0$, then the only possibility is $u \equiv 0$. □

For the case $s > 0$ in Theorem 2.5, we shall investigate the existence of solutions of 4 in the next section.

If $c < \lambda_1$, $\lambda_1(\lambda_1 - c) < b$ and $s > 0$, then the left hand side of 7 is larger than or equal to zero and the right hand side of it is negative.

Therefore we have the following theorem.

Theorem 2.6. *Assume that $c < \lambda_1$ and $0 < \lambda_1(\lambda_1 - c) < b$, $b \neq \lambda_k(\lambda_k - c)$, $k = 2, 3, \cdots$. Then we have :*

(i) If $s > 0$, then equation 4 has no solution.

(ii) If $s = 0$, then equation 4 has only the trivial solution.

PROOF. Assume $s \geq 0$. We rewrite 4 as

$$\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u + [\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- = s.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega \{[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^-\}\phi_1 = s\int \phi_1. \tag{8}$$

But $[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- \leq 0$ for any real valued function u. Also $\phi_1(x) > 0$ in Ω. Therefore, if $s > 0$, then equation 4 has no solution and if $s = 0$, then the only possibility is that $u = 0$. □

For the case $s < 0$ in Theorem 2.6, we shall investigate the existence of solutions of 3 in the next section.

3. The existence of solutions

In this section we investigate the multiplicity of solutions of the problem

$$\Delta^2 u + c\Delta u = bu^+ + s \quad \text{in} \quad H \tag{9}$$

under the following two conditions.
Condition(1) : $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$.
Condition(2) : $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $s < 0$.

First we investigate the multiplicity of solutions of 9 under the Condition(1).

Theorem 3.1. *Assume that $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the problem 9 has at least two solutions.*

One solution is positive and the existence of the other solution will be proved by critical point theory. For the proof of the theorem, we need several lemmas.

Lemma 3.2. *Let $\lambda_k < c < \lambda_{k+1} (k \geq 1)$ and $b < \lambda_1(\lambda_1 - c)$. Then the problem*

$$\Delta^2 u + c\Delta u = bu^+ \quad \text{in} \quad H \tag{10}$$

has only the trivial solution.

Proof. We rewrite 10 as

$$\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u + [\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- = 0 \quad \text{in} \quad H.$$

Multiply across by ϕ_1 and integrate over Ω. Since $(\{\Delta^2 + c\Delta - \lambda_1(\lambda_1 - c)\}u, \phi_1) = 0$, we have

$$\int_\Omega \{[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^-\}\phi_1 = 0. \tag{11}$$

But $[\lambda_1(\lambda_1 - c) - b]u^+ - \lambda_1(\lambda_1 - c)u^- \geq 0$ for all real valued function u and $\phi_1(x) > 0$ for $x \in \Omega$. Hence the left hand side of 11 is always greater than or equal to zero.

Therefore the only possibility to hold 11 is that $u \equiv 0$. □

Now, we study the existence of the positive solution of 9.

Lemma 3.3. *Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the unique solution u_1 of the problem*

$$\Delta^2 u + c\Delta u = bu + s \quad in \quad L^2(\Omega) \tag{12}$$

is positive.

PROOF. Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_2 - c)$. Then the problem

$$\Delta^2 u + c\Delta u - bu = \mu u \quad \text{in} \quad L^2(\Omega)$$

has eigenvalues $\lambda_k(\lambda_k - c) - b$ and they are positive. Since the inverse $(\Delta^2 + c\Delta - b)^{-1}$ of the operator $\Delta^2 + c\Delta - b$ is positive, the solution $u = (\Delta^2 + c\Delta - b)^{-1}(s)$ of 12 is positive. This proves the lemma. □

An easy consequence of Lemma 3.3 is

Lemma 3.4. *Let $c < \lambda_1$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then the boundary value problem 9 has a positive solution u_1.*

PROOF. The solution u_1 of the linear problem 12 is positive, hence it is also a solution of 9. □

Now, we investigate the existence of the other solution of problem 9 under the condition $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$ by the critical point theory.

Let us define the functional corresponding to 9 in $H \times R$

$$F_b(u, s) = \int_\Omega \left[\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - su\right] dx \tag{13}$$

For simplicity, we shall write $F = F_b$ when b is fixed. Then F is well-defined. The solutions of 9 coincide with the critical points of $F(u, s)$.

Proposition 3.5. *Let b be fixed and $s \in R$. Then $F(u, s) = F_b(u, s)$ is continuous and Fréchet differentiable in H.*

The proof of Proposition 3.5 is similar to that of Proposition 2.1 of [3].

Let V be the one-dimensional subspace of $L^2(\Omega)$ spanned by ϕ_1 whose eigenvalue is $\lambda_1(\lambda_1 - c)$. Let W be the orthogonal complement of V in H. Let $P : H \to V$ be the orthogonal projection of H onto V and $I - P : H \to W$ denote that of H onto W. Then every element $u \in H$ is expressed by $u = v + z$, where $v = Pu, z = (I - P)u$. Then the problem 9 is equivalent to

$$\Delta^2 v + c\Delta v = P[b(v + z)^+ + s],$$

$$\Delta^2 z + c\Delta z = (I - P)[b(v+z)^+ + s].$$

We look on the above equations as a system of two equations in two unknowns v and w.

Lemma 3.6. *Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then we have :*
(i) There exists a unique solution $z \in W$ of the equation

$$\Delta^2 z + c\Delta z - (I - P)[b(v+z)^+ + s] = 0 \quad in \quad W \tag{14}$$

If for fixed $s \in R$, we put $z = \theta(v, s)$, then θ is continuous on V. In particular, θ satisfies a uniform Lipschitz condition in v with respect to the L^2 norm (also the norm $|||\cdot|||$).
(ii) If $\tilde{F} : V \to R$ is defined by $\tilde{F}(v,s) = F(v+\theta(v,s),s)$, then $\tilde{F}$ has a continuous Fréchet derivative $D\tilde{F}$ with respect to v and

$$D\tilde{F}(v,s)(h) = DF(v + \theta(v,s),s)(h) = 0 \quad for\ all \quad h \in V.$$

If v_0 is a critical point of $\tilde{F}$, then $v_0 + \theta(v_0, s)$ is a solution of the problem 9 and conversely every solution of 9 is of this form.

PROOF. Let $\lambda_1 < c < \lambda_2$, $\alpha < b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Let $\delta = \frac{b}{2} < 0$ and $g(\xi) = b\xi^+$. If $g_1(\xi) = g(\xi) - \delta\xi$, then equation 14 is equivalent to

$$z = (\Delta^2 + c\Delta - \delta)^{-1}(I - P)(g_1(v+z)^+ + s). \tag{15}$$

Since $(\Delta^2 + c\Delta - \delta)^{-1}(I-P)$ is self-adjoint, compact, linear map from $(I-P)L^2(\Omega)$ into itself, the eigenvalues of $(\Delta^2 + c\Delta - \delta)^{-1}(I-P)$ are $(\lambda_l(\lambda_l - c) - \delta)^{-1}$, where $\lambda_l(\lambda_l - c) \geq \lambda_2(\lambda_2 - c)$. Therefore its L^2 norm is $\frac{1}{\lambda_2(\lambda_2 - c) - \delta}$. Since

$$|g_1(\xi_2) - g_1(\xi_1)| \leq \max\{|b - \delta|, |\delta|\}|\xi_2 - \xi_1|,$$

it follows that the right hand side of 15 defines, for fixed $v \in V$, a Lipschitz mapping of $(I-P)L^2(\Omega)$ into itself with Lipschitz constant $\gamma < 1$, where

$$\gamma = \frac{|b|}{2} \cdot \frac{1}{\lambda_2(\lambda_2 - c) - \frac{b}{2}} < 1.$$

Therefore, by the contraction mapping principle, for given $v \in V$, there exists a unique $z \in (I-P)L^2(\Omega)$ which satisfies 15.

Since the constant δ does not depend on v and s, it follows from standard arguments that if $\theta(v, s)$ denotes the unique $z \in (I-P)L^2(\Omega)$ which solves 15, then θ is continuous with respect to v. In fact, if $z_1 = \theta(v_1, s)$ and $z_2 = \theta(v_2, s)$, then we have

$$\begin{aligned} |||z_1 - z_2||| &= ||(\Delta^2 + c\Delta - \delta)^{-1}(I - P)(g_1(v_1 + z_1) - g_1(v_2 + z_2)|| \\ &= \gamma||(v_1 + z_1) - (v_2 + z_2)|| \\ &\leq \gamma(||v_1 - v_2|| + ||z_1 - z_2||). \end{aligned}$$

Hence we have

$$||z_1 - z_2|| \leq C||v_1 - v_2||, \quad C = \frac{\gamma}{1-\gamma},$$

which shows that $\theta(v, s)$ satisfies a uniform Lipschitz condition in v with respect to L^2-norm. With the above inequality we have

$$\begin{aligned} |||z_1 - z_2||| &= |||(\Delta^2 + c\Delta - \delta)^{-1}(I - P)(g_1(v_1 + z_1) - g_1(v_2 + z_2))||| \\ &\leq C_1||(I - P)(g_1(v_1 + z_2) - g_2(v_2 + z_2)|| \\ &\leq C_1\frac{b}{2}||(v_1 + z_1) - (v_2 + z_2)|| \\ &\leq C_1\frac{b}{2}(||v_1 - v_2|| + ||z_1 - z_2||) \\ &\leq C_1\frac{b}{2}(1 + C)||v_1 - v_2|| \end{aligned}$$

for some $C_1 > 0$. Hence we have

$$|||z_1 - z_2||| \leq C_2|||v_1 - v_2||| \tag{16}$$

for some $C_2 > 0$. This shows that $\theta(v, s)$ satisfies a uniform Lipschitz condition in v with respect to the norm $|||\ |||$.

Let $v \in V$ and $z = \theta(v, s)$. If $w \in W$. then from 14 we see that

$$\int_\Omega [\Delta z \cdot \Delta w - c\nabla z \cdot \nabla w - (I - P)[b(v + z)^+ + s]] \cdot w dx = 0. \tag{17}$$

Since

$$\int_\Omega \Delta v \cdot \Delta w = 0 \quad \text{and} \quad \int_\Omega \nabla v \cdot \nabla w = 0,$$

we have

$$DF(v + \theta(v, s), s)(w) = 0 \quad \text{for} \quad w \in W. \tag{18}$$

From Proposition 3.5, $\tilde{F}(v, s)$ has a continuous Fréchet derivative $D\tilde{F}$, and

$$D\tilde{F}(v, s)(h) = DF(v + \theta(v, s), s)(h), \quad h \in V. \tag{19}$$

Suppose that for some fixed $s > 0$, there exists $v_0 \in V$ such that $D\tilde{F}(v_0, s) = 0$. Then it follows 19 that

$$DF(v_0 + \theta(v_0, s), s)(v) = 0 \quad \text{for all} \quad v \in V.$$

Since 18 holds for all $w \in W$ and H is the direct sum of V and W, it follows that

$$DF(v_0 + \theta(v_0, s), s) = 0 \quad \text{in} \quad H.$$

Therefore $u = v_0 + \theta(v_0, s)$ is a solution of 9.

Conversely, our reasoning shows that if u is a solution of 9 and $v = Pu$, then $D\tilde{F}(v, s) = 0$ in V. □

Let $\lambda_1 < C < \lambda_2$, $b < \lambda_1(\lambda_1 - c), \lambda_k(\lambda_k - c) < 0 < \lambda_{k+1}(\lambda_{k+1} - c)$ and $s > 0$. From lemma 3.4, we see that 9 has a positive solution $u_1(x)$. From lemma 3.6, $u_1(x)$ is of the form $u_1(x) = v_1 + \theta(v_1, s)$.

Lemma 3.7. *Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then there exists a small open neighborhood B of v_1 in V such that $v = v_1$ is a strict local point of minimum of $\tilde{F}$.*

PROOF. Let $s > 0$. Then equation 9 has a positive solution $u_1(x)$ which is of the form $u_1(x) = v_1 + \theta(v_1, s) > 0$, $\theta(v_1, s) \in W$. Since $I + \theta$, where I is an identity map on V, is continuous on V, there exists a small open neighborhood B of v_1 in V such that if $v \in B$, then $v + \theta(v, s) > 0$. Therefore, if $z = \theta(v, s)$, $z_1 = \theta(v_1, s)$ and $v + z = (v_1 + z_1) + (\tilde{v} + \tilde{z})$, then we have

$$\begin{aligned}
\tilde{F}(v, s) &= F(v + z, s) \\
&= \int_\Omega [\frac{1}{2}|\Delta(v + z)|^2 - \frac{c}{2}|\nabla(v + z)|^2 - \frac{b}{2}|v + z|^2 - s(v + z)]dx \\
&= \int_\Omega [\frac{1}{2}|\Delta(v_1 + z_1) + \Delta(\tilde{v} + \tilde{z})|^2 - \frac{c}{2}|\nabla(v_1 + z_1) + \nabla(\tilde{v} + \tilde{z})|^2 \\
&\qquad - \frac{b}{2}|(v_1 + z_1) + (\tilde{v} + \tilde{z})|^2 - s\{(v_1 + z_1) + (\tilde{v} + \tilde{z})\}]dx \\
&= \int_\Omega [\frac{1}{2}|\Delta(v_1 + z_1)|^2 - \frac{c}{2}|\nabla(v_1 + z_1)|^2 - \frac{b}{2}|v_1 + z_1|^2 - s(v_1 + z_1)]dx \\
&\quad + \int_\Omega [\Delta(v_1 + z_1) \cdot \Delta(\tilde{v} + \tilde{z}) - c\nabla(v_1 + z_1) \cdot \nabla(\tilde{v} + \tilde{z}) \\
&\qquad - b(v_1 + z_1) \cdot (\tilde{v} + \tilde{z}) - s(\tilde{v} + \tilde{z})]dx \\
&\quad + \int_\Omega [\frac{1}{2}|\Delta(\tilde{v} + \tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v} + \tilde{z})|^2 - \frac{b}{2}|\tilde{v} + \tilde{z}|^2]dx.
\end{aligned}$$

Here

$$\int_\Omega [\frac{1}{2}|\Delta(v_1+z_1)|^2 - \frac{c}{2}|\nabla(v_1+z_1)|^2 - \frac{b}{2}|v_1+z_1|^2 - s(v_1+z_1)]dx$$
$$= F(v_1+z_1,s) = \tilde{F}(v_1,s)$$

and

$$\int_\Omega [\Delta(v_1+z_1)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_1+z_1)\cdot\nabla(\tilde{v}+\tilde{z})$$
$$-b(v_1+z_1)\cdot(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]dx$$
$$= \int_\Omega [\Delta^2(v_1+z_1) + c\Delta(v_1+z_1) - b(v_1+z_1) - s]\cdot(\tilde{v}+\tilde{z})dx = 0,$$

since v_1+z_1 is a positive solution of 9. Since $\tilde{v}+\tilde{z}$ can be expressed by $\tilde{v}+\tilde{z} = e_1\phi_1 + e_2\phi_2 + \cdots$, we have

$$\begin{aligned}\tilde{F}(v,s) - \tilde{F}(v_1,s) &= \int_\Omega [\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2 - \frac{b}{2}|\tilde{v}+\tilde{z}|^2]dx \\ &= \frac{1}{2}\{[\lambda_1(\lambda_1-c)-b]e_1^2 + [\lambda_2(\lambda_2-c)-b]e_2^2 + \cdots > 0,\end{aligned}$$

since $b < \lambda_1(\lambda_1 - c)$ and $\lambda_1 < c < \lambda_2$. Therefore $v = v_1$ is a strict local point of minimum of $\tilde{F}$. This proves the lemma. □

We now define the functional on H

$$F^*(u) = F(u,0) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2]dx.$$

Then the critical points of $F^*(u)$ coincide with solutions of the equation

$$\Delta^2 u + c\Delta u = bu^+ \quad \text{in} \quad H. \tag{20}$$

If $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$, then 20 has only the trivial solution and hence $F^*(u)$ has only one critical point $u = 0$. Given $v \in V$, let $\theta^*(v) = \theta(v,0) \in W$ be the unique solution of the equation

$$\Delta^2 z + c\Delta z - (I-P)[b(v+z)^+] = 0 \quad \text{in} \quad W.$$

Let us define the reduced functional $\tilde{F}^*(v)$ on V, by $F^*(v + \theta^*(v))$. We note that we can obtain the same result as lemma 3.6 when we replace $\theta(v,s)$ and $\tilde{F}(v,\theta(v,s))$ by $\theta^*(v)$ and $\tilde{F}^*(v)$. We also note that $\tilde{F}^*(v)$ has only one critical point $v = 0$.

Lemma 3.8. *For $d > 0$ and $v \in V$,* $\tilde{F}^*(dv) = d^2\tilde{F}^*(v)$.

PROOF. If $v \in V$ satisfy

$$\Delta^2 z + c\Delta z - (I-P)(b(v+\theta^*(v))^+) = 0 \quad \text{in} \quad W,$$

then for $d > 0$,

$$\Delta^2(dz) + c\Delta(dz) - (I-P)(b(dv + d\theta^*(v))^+) = 0 \quad \text{in} \quad W.$$

Therefore $\theta^*(dv) = d\theta^*(v)$ for $d > 0$. From the definition of $F^*(u)$ we see that

$$F^*(du) = d^2 F^*(u) \quad \text{for} \quad u \in H \quad \text{and} \quad d > 0.$$

Hence, for $v \in V$ and $d > 0$,

$$\tilde{F^*}(dv) = F^*(dv + \theta^*(dv)) = d^2 F^*(v + \theta^*(v)) = d^2 \tilde{F^*}(v).$$

□

Now we remember the notation F_b, which was defined in equation 13. Until now, the notations F, F^* and $\tilde{F^*}$ denote F_b, F_b^* and $\tilde{F_b^*}$ respectively. In the following lemma we use the latter notations.

Lemma 3.9. *Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$. Then there exist v_1 and v_2 in V such that $\tilde{F_b^*}(v_1) > 0$ and $\tilde{F_b^*}(v_2) < 0$.*

PROOF. First, we choose $v_1 \in V$ such that $v_1 + \theta(v_1, 0) > 0$. In this case $z = \theta(v_1, 0) = 0$. Hence $v_1 + z = d_1\phi$, and we have

$$\begin{aligned}
\tilde{F_b^*}(v_1) &= \int_\Omega [\frac{1}{2}|\Delta(v_1+z)|^2 - \frac{c}{2}|\nabla(v_1+z)|^2 - \frac{b}{2}|(v_1+z)^+|^2]dx \\
&= \int_\Omega [\frac{1}{2}|\Delta(v_1+z)|^2 - \frac{c}{2}|\nabla(v_1+z)|^2 - \frac{b}{2}|v_1+z|^2]dx \\
&= \int_\Omega [\frac{1}{2}(\Delta^2 + c\Delta)(v_1+z)\cdot(v_1+z) - \frac{b}{2}(v_1+z)\cdot(v_1+z)]dx \\
&= \frac{1}{2}[\{\lambda_1(\lambda_1 - c) - b\}d_1^2] > 0.
\end{aligned}$$

Next, we choose $v_2 \in V$ such that $v_2 + \theta(v_2, 0) < 0$. In this case $z = \theta(v_2, 0) = 0$. Hence if we write $v_2 + z = e_1\phi_1$, then we have

$$\begin{aligned}
\tilde{F_b^*}(v_2) &= \int_\Omega \left[\frac{1}{2}|\Delta(v_2+z)|^2 - \frac{c}{2}|\nabla(v_2+z)|^2\right] dx \\
&= \int_\Omega \left[\frac{1}{2}(\Delta^2 + c\Delta)(v_2+z)\cdot(v_2+z)\right] dx \\
&= \frac{1}{2}[\lambda_1(\lambda_1 - c)e_1^2] < 0,
\end{aligned}$$

since $b < \lambda_1(\lambda_1 - c) < 0 < \lambda_2(\lambda_2 - c)$. □

Lemma 3.10. *Let $\lambda_1 < c < \lambda_2$ and $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. Then $\tilde{F}_b(v,s)$ is neither bounded above nor below on V.*

PROOF. From lemma 3.9, $\tilde{F}_b^*(v)$ has negative (positive) value. Suppose that $\tilde{F}_b^*(v)$ assumes negative values and that $\tilde{F}_b(v,s)$ is bounded below. Let v_0 denote a fixed point in V with $\|v_0\| = 1$. Let $z_n = nv_0 + \theta(nv_0, s)$ and let $z_n^* = v_0 + \frac{\theta(nv_0,s)}{n} = v_0 + w_n^*$. Since θ is Lipschitzian, the sequence $\{z_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$. We have $DF(z_n, s)(y) = 0$ for all n and arbitrary $y \in W$. Dividing this equation by n gives

$$\int_\Omega [\Delta z_n^* \cdot \Delta y - c\nabla z_n^* \cdot \nabla y - bz_n^{*+} y - \frac{s}{n} y]dx = 0. \tag{21}$$

Setting $y = z_n$ we know that $\{z_n^*\}_{n=1}^\infty$ is bounded in $L^2(\Omega)$. Hence $\{w_n^*\}_1^\infty$ is bounded in $L^2(\Omega)$ so we may assume that it converges weakly to an element $w^* \in W$. If $z^* = w^* + v_0$ and we let $n \to \infty$, in 21 we obtain

$$\int_\Omega [\Delta z^* \cdot \Delta y - c\nabla z^* \cdot \nabla y - bz^{*+} y]dx = 0 \tag{22}$$

for arbitrary $y \in W$. Hence $w^* = \theta(v_0, 0)$. If we set $y = w_n$ in (21) and dividing by n, then we have

$$\int_\Omega [|\Delta w_n^*|^2 - c|\nabla w_n^*|^2 - (b|z_n^{*^+}| + \frac{s}{n})w_n^*]dx = 0. \tag{23}$$

Letting $n \to \infty$ in 23, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_\Omega [|\Delta w_n^*|^2 - c|\nabla w_n^*|^2]dx &= \lim_{n\to\infty} \int_\Omega [b(|z_n^{*^+}| + \frac{s}{n})w_n^*]dx \\
&= \int_\Omega b|z^{*^+}|w^* dx \\
&= \int_\Omega [\Delta z^* \cdot \Delta w^* - c\nabla z^* \cdot \nabla w^*]dx \\
&= \int_\Omega [|\Delta w^*|^2 - c|\nabla w^*|^2]dx,
\end{aligned}$$

where we have used 22. Hence

$$\lim_{n\to\infty} \int_\Omega [|\Delta z_n^*|^2 - |\nabla z_n^*|^2]dx = \int_\Omega [|\Delta z^*|^2 - c|\nabla z^*|^2]dx.$$

The assumption that $\tilde{F}(v,s)$ is bounded below implies the existence of a constant M such that

$$\tilde{F}_b(nv_0, s)/n^2 \geq M/n^2.$$

Letting $n \to \infty$, our previous reasoning shows that

$$\tilde{F_b}^*(v_0) = \tilde{F}_b(v_0, 0) = \lim_{n\to\infty} \tilde{F}_b(nv_0, s)/n^2 \geq 0.$$

Since v_0 was an arbitrary member of V with $\|v_0\| = 1$ and $\tilde{F}_b(kv, 0) = k^2\tilde{F}_b(v, 0)$, this contradicts the assumption $\tilde{F}_b^*(v)$ is negative for some value of $v \in V$. Hence $\tilde{F}_b(v, s)$ cannot be bounded below. The proof that $\tilde{F}_b(v, s)$ cannot be bounded above if $\tilde{F}_b^*(v)$ assumes positive values is essentially the same. □

PROOF OF THEOREM 2.1. Let $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$ and $s > 0$. By Lemma 3.4, 9 has a positive solution $u_1(x) = v_1 + \theta(v_1, s)$. By Lemma 3.7, there exists a small open neighborhood B of v_1 in V such that $v = v_1$ is a strict local point of minimum of $\tilde{F}_b$. Since $\tilde{F}_b(v, s)$ is not bounded below, there exits a point $v_2 \in V$ with $v_1 \neq V_2$ and $\tilde{F}_b(v_1, s) = \tilde{F}_b(v_2, s)$. The Rolle's theorem and the fact that $\tilde{F}_b(v, s)$ has a continuous Fréchet derivative imply that there exists a strict local point of maximum $\tilde{F}_b$. Thus $\tilde{F}_b$ has at least two critical points. Therefore 9 has at least two solutions. □

Next, we investigate the multiplicity of solutions of 9 under the Condition (2), Condition(2) : $c < \lambda_1$ (in this case $0 < \lambda_1(\lambda_1 - c)$), $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$ $(k = 1, 2, \cdots)$ and $s < 0$.

Theorem 3.11. *Assume that $c < \lambda_1$, $0 < \lambda_1(\lambda_1 - c)$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $(k \geq 0)$ and $s < 0$. Then the problem 9 has at least two solutions.*

One solution is a negative solution and the existence of another solution will be shown by critical point theory.
To prove Theorem 3.11, we need several lemmas.

Lemma 3.12. *Let $c < \lambda_1$, $b \geq 0$ and $b \neq \lambda_1(\lambda_1 - c)$. Then the problem*

$$\Delta^2 u + c\Delta u = bu^+ \quad \text{in} \quad H \tag{24}$$

has only the trivial solution.

PROOF. For $c < \lambda_1$, $0 < \lambda_1(\lambda_1 - c) < b$, the result follows from Theorem 2.6 (ii). We prove the lemma for the case $0 \leq b < \lambda_1(\lambda_1 - c)$. From 9 we have

$$\lambda_1(\lambda_1 - c)\|u\|^2 \leq \int_\Omega |\Delta u|^2 - c|\nabla u|^2 = b\int_\Omega u^+ \cdot u \leq b\|u\|^2, \tag{25}$$

where $\| \ \|$ is the L^2 norm is Ω. It follows from 25 that $b\|u\|^2 \geq \lambda_1(\lambda_1 - c)\|u\|^2$, which yields $u = 0$. □

Now, we investigate the existence of the negative solution of 9 under Condition(2).

Lemma 3.13. *Assume that $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)(k \geq 1)$ and $s < 0$. Then the problem 9 has a negative solution $u_2(x)$.*

PROOF. If u is a smooth function satisfying

$$\Delta^2 u + c\Delta u \geq 0 \quad \text{in} \quad \Omega,$$

$$u = 0, \quad \Delta u = 0 \quad \text{on} \quad \partial\Omega$$

and $c < \lambda_1$, then $u > 0$ in Ω or $u = 0$. This immediately follows by first applying standard (strong) maximum principle to $w = \Delta u$ and consequently to u. Subsequently, for $c < \lambda_1$ and $s < 0$, it follows that if u_2 is the unique solution for

$$\Delta^2 u_2 + c\Delta u_2 = s \quad \text{in} \quad \Omega, \tag{26}$$

$$u_2 = 0, \quad \Delta u_2 = 0 \quad \text{on} \quad \partial\Omega,$$

then $u_2 < 0$ in Ω. The unique negative solution u_2 solution of 26 is also a negative solution of 9. □

Now, we investigate the existence of the other solution of the problem 9 under the condition $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)(k \geq 1)$ and $s < 0$ will be shown by critical point theory. Now we consider the functional

$$F_b(u, s) = \int_\Omega [\frac{1}{2}|\Delta u|^2 - \frac{c}{2}|\nabla u|^2 - \frac{b}{2}|u^+|^2 - su]dx,$$

which is well defined in $H \times R$, continuous and Fréchet differentiable in H (by Proposition 3.5).

Let V be the k-dimensional subspace of H spanned by eigenfunctions $\phi_1, \phi_2, \cdots, \phi_k$. Let W be the orthogonal compliment of V in H. We note that Lemma 3.6 holds under Condition (2). From Lemma 3.13, we see that 9 has a negative solution $u_2(x)$. By Lemma 3.6, u_2 is of the form $u_2 = v_2 + \theta(v_2, s)$.

Lemma 3.14. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)(k \geq 1)$ and $s < 0$. Then there exists a small open neighborhood D of v_2 in V such that $v = v_2$ is a strict local point of minimum of $\tilde{F}_b$.*

PROOF. Let $s < 0$. Then the problem 9 has a negative solution $u_2(x)$ which is of the form $u_2(x) = v_2 + \theta(v_2, s) < 0$. Since $I + \theta$, where I is an identity map on V, is continuous, there exists a small open neighborhood D of v_2 in V

such that if $v \in D$, $v + \theta(v, s) < 0$. Therefore if $z = \theta(v, s)$, $z_2 = \theta(v_2, s)$ and $v + z = (v_2 + z_2) + (\tilde{v} + \tilde{z})$, then we have

$$\begin{aligned}
\tilde{F}_b(v,s) &= F_b(v+z,s) \\
&= \int_\Omega [\frac{1}{2}|\Delta(v+z)|^2 - \frac{c}{2}|\nabla(v+z)|^2 - s(v+z)]dx \\
&= \int_\Omega [\frac{1}{2}|\Delta(v_2+z_2) + \Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(v_2+z_2) + \nabla(\tilde{v}+\tilde{z})|^2 \\
&\qquad -s\{(v_2+z_2) + (\tilde{v}+\tilde{z})\}]dx \\
&= \int_\Omega [\frac{1}{2}|\Delta(v_2+z_2)|^2 - \frac{c}{2}|\nabla(v_2+z_2)|^2 - s(v_2+z_2)]dx \\
&\quad + \int_\Omega [\Delta(v_2+z_2)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_2+z_2)\cdot\nabla(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]dx \\
&\quad + \int_\Omega [\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2]dx.
\end{aligned}$$

Here

$$\begin{aligned}
&\int_\Omega [\frac{1}{2}|\Delta(v_2+z_2)|^2 - \frac{c}{2}|\nabla(v_2+z_2)|^2 - s(v_2+z_2)]dx \\
&= F_b(v_2+z_2,s) = \tilde{F}_b(v_2,s)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega [\Delta(v_2+z_2)\cdot\Delta(\tilde{v}+\tilde{z}) - c\nabla(v_2+z_2)\cdot\nabla(\tilde{v}+\tilde{z}) - s(\tilde{v}+\tilde{z})]dx \\
&= \int_\Omega [\Delta^2(v_2+z_2) + c\Delta(v_2+z_2) - s]\cdot(\tilde{v}+\tilde{z})dx = 0,
\end{aligned}$$

since $v_2 + z_2$ is a negative solution of 9. Since, $\tilde{v} + \tilde{z}$ can be expressed by $\tilde{v} + \tilde{z} = \sum_{i=1}^{\infty} e_i\phi_i$, we have

$$\begin{aligned}
\tilde{F}_b(v,s) - \tilde{F}_b(v_2,s) &= \int_\Omega [\frac{1}{2}|\Delta(\tilde{v}+\tilde{z})|^2 - \frac{c}{2}|\nabla(\tilde{v}+\tilde{z})|^2]dx \\
&= \frac{1}{2}\{\lambda_1(\lambda_1 - c)e_1^2 + \lambda_2(\lambda_2 - c)e_2^2 + \cdots\} > 0,
\end{aligned}$$

since $0 < \lambda_1(\lambda_1 - c)$. Therefore $\tilde{F}_b(v,s)$ has a strict local minimum at $v = v_2$. This proves the lemma. □

Lemma 3.15. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)(k \geq 1)$. Then there exist v_p and v_q in V such that $\tilde{F}_b^*(v_p) < 0$ and $\tilde{F}_b^*(v_q) > 0$.*

PROOF. First, we choose $v_p \in V$ such that $v_p + \theta(v_p, 0) > 0$ and $\theta(v_p, 0) = 0$. If $v_p + z = \sum_{i=1}^{k} f_i\phi_i$, where $\theta(v_p, 0) = 0$, then we have

$$\begin{aligned}\tilde{F_b}^*(v_p) &= \int_\Omega [\frac{1}{2}|\Delta(v_p+z)|^2 - \frac{c}{2}|\nabla(v_p+z)|^2 - \frac{b}{2}|(v_p+z)^+|^2]dx \\ &= \int_\Omega [\frac{1}{2}|\Delta v_p|^2 - \frac{c}{2}|\nabla v_p|^2 - \frac{b}{2}|v_p|^2]dx \\ &= \int_\Omega [\frac{1}{2}(\Delta^2 + c\Delta)v_p \cdot v_p - \frac{b}{2}|v_p|^2]dx \\ &= \frac{1}{2}\{[\lambda_1(\lambda_1 - c) - b]f_1^2 + \cdots + [\lambda_k(\lambda_k - c) - b]f_k^2\} < 0.\end{aligned}$$

Next, we choose $v_q \in V$ such that $v_q + \theta(v_q, 0) < 0$. Let $z = \theta(v_q, 0)$. If $v_q + z = \sum_{i=1}^{\infty} g_i\phi_i$, then we have

$$\begin{aligned}\tilde{F_b}^*(v_q) &= \int_\Omega [\frac{1}{2}|\Delta(v_q+z)|^2 - \frac{c}{2}|\nabla(v_q+z)|^2]dx \\ &= \int_\Omega [\frac{1}{2}(\Delta^2 + c\Delta)(v_q+z) \cdot (v_q+z)]dx \\ &= \frac{1}{2}[\lambda_1(\lambda_1 - c)g_1^2 + \cdots + \lambda_k(\lambda_k - c)g_k^2] > 0,\end{aligned}$$

since $0 < \lambda_l(\lambda_l - c)$. □

Lemma 3.16. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)(k \geq 1)$ and $s < 0$. Then $\tilde{F}_b(v, s)$ is neither bounded above nor below on V.*

The proof of the lemma is the same as that of Lemma 3.10.

Lemma 3.17. *Let $c < \lambda_1$, $\lambda_k(\lambda_k - c) < b < \lambda_{k+1}(\lambda_{k+1} - c)$, $k = 1, 2, \cdots$ and $s < 0$. Then the functional $\tilde{F}_b(v, s)$, defined on V, satisfies the Palais-Smale condition : Any sequence $\{v_n\} \subset V$ for which $\tilde{F}_b(v_n, s)$ is bounded and $D\tilde{F}_b(v_n, s) \to 0$ possesses a convergent subsequence.*

PROOF. Suppose that $\tilde{F}_b(v_n, s)$ is bounded and $D\tilde{F}_b(v_n, s) \to 0$ in V, where $\{v_n\}$ is a sequence in V. Since V is k-dimensional subspace spanned by $\phi_1, \cdots, \phi_k$, we have, with $u_n = v_n + \theta(v_n, s)$

$$\Delta^2 u_n + c\Delta u_n - bu_n^+ = s + DF_b(u_n, s).$$

Assuming [P.S.] condition does not hold, that is $\|v_n\| \to \infty$ ($\||v_n\|| \to \infty$), we see that $\|u_n\| \to \infty$. Dividing by $\|u_n\|$ and taking $w_n = \|u_n\|^{-1}u_n$ we have

$$\Delta_2 w_n + c\Delta w_n - bw_n^+ = \|u_n\|^{-1}(s + DF_b(u_n, s)). \tag{27}$$

Since $DF_b(u_n, s) \to 0$ as $n \to \infty$ and $\|u_n\| \to \infty$. Moreover 27 shows that $\|\Delta^2 w_n + c\Delta w_n\|$ is bounded. Since $(\Delta^2 + c\Delta)^{-1}$ is a compact operator, passing to a subsequence we get that $w_n \to w_0$. Since $\|w_n\| = 1$ for all $n = 1, 2, \cdots$ it follows that $\|w_0\| = 1$. Taking the limit of both sides of 27, we find

$$\Delta^2 w_0 + c\Delta w_0 - bw_0^+ = 0$$

with $\|w_0\| \neq 0$. This contradicts to the fact that the equation

$$\Delta^2 u + c\Delta u = bu^+$$

has only the trivial solution. □

Proof of Theorem 2.2. By Lemma 3.13, 9 has a negative solution $u_2(x) = v_2 + \theta(v_2, s)$. By Lemma 3.14, there exists a small open neighborhood D of v_2 in V such that $v = v_2$ is a strict local point of minimum of $\tilde{F}_b$. Also $\tilde{F}_b \in C^1(V, R)$ satisfies the Palais-Smale condition. Since $\tilde{F}_b(v, s)$ is neither bounded above nor below on V (Lemma 3.16), we can choose $v_3 \in V \setminus D$ such that

$$\tilde{F}_b(v_3, s) < \tilde{F}_b(v_2, s).$$

Let Γ be the set of all paths in V joining v_3 and v_2. The Mountain Pass Theorem implies that

$$c = \inf_{\gamma \in \Gamma} \sup_{\gamma} \tilde{F}_b(v, s)$$

is a critical value of $\tilde{F}_b$. Thus $\tilde{F}_b$ has at least two critical values. Thus 9 has at least two solutions. □

4. Multiplicity of solutions and source terms

We let $Lu = \Delta^2 u + c\Delta u$. We investigate relations between multiplicity of solutions and source terms $f(x)$ of the fourth order nonlinear elliptic boundary value problem, under the condition : $\lambda_1 < c < \lambda_2$, $b < \lambda_1(\lambda_1 - c)$

$$Lu - bu^+ = f \qquad \text{in } H, \tag{28}$$

where we assume that $f = c_1\phi_1 + c_2\phi_2$ $(c_1, c_2 \in R)$.

Theorem 4.1. *If $c_1 < 0$, then 28 has no solution.*

PROOF. We rewrite 28 as

$$(L-\mu_1)u+(-b+\mu_1)u^+-\mu_1u^-=c_1\phi_1+c_2\phi_2 \qquad \text{in } H.$$

Multiply across by ϕ_1 and integrate over Ω. Since L is self-adjoint and $(L-\mu_1)\phi_1=0$, $((L-\mu_1)u,\phi_m u_1)=0$. Thus we have

$$\int_\Omega\{(-b+\mu_1)u^+-\mu_1u^-\}\phi_1=(c_1\phi_1,\phi_1)=c_1.$$

We know that $(-b+\mu_1)u^+-\mu_1u^-\geq 0$ for all real valued function u. Also $\phi_1>0$ in Ω. Therefore $\int_\Omega\{(-b+\mu_1)u^+-\mu_1u^-\}\phi_1\geq 0$. Hence there is no solution of 28 if $c_1<0$. □

Let V be the subspace of H spanned by $\{\phi_1,\phi_2\}$ and W be the orthogonal complement of V in H. Let P be the orthogonal projection of H onto V. Then every $u\in H$ can be written as $u=v+w$, where $v=Pu$ and $w=(I-P)u$. Hence equation 28 is equivalent to a system

$$Lw+(I-P)(-b(v+w)^+)=0, \tag{29}$$

$$Lv+P(-b(v+w)^+)=c_1\phi_1+c_2\phi_2. \tag{30}$$

Now we have a uniqueness theorem, which proof is similar to that of (i) of Lemma 3.6.

Lemma 4.2. *For a fixed $v\in V$, 29 has a unique solution $w=\theta(v)$. Furthermore, $\theta(v)$ is Lipschitz continuous in v.*

By Lemma 4.2, the study of the multiplicity of solutions of 28 is reduced to that of an equivalent problem

$$Lv+P(-b(v+\theta(v))^+)=c_1\phi_1+c_2\phi_2 \tag{31}$$

defined on V.

Proposition 4.3. *If $v\geq 0$ or $v\leq 0$, then $\theta(v)=0$.*

PROOF. Let $v\geq 0$. Then $\theta(v)=0$ and equation 29 is reduced to

$$L0+(I-P)(-bv^+)=0$$

because $v^+=v, v^-=0$ and $(I-P)v=0$. Similarly if $v\leq 0$, then $\theta(v)=0$. □

Since $V = span\{\phi_1, \phi_2\}$ and ϕ_1 is a positive eigenfunction, there exists a cone C_1 defined by

$$C_1 = \{v = c_1\phi_1 + c_2\phi_2 \,|\, c_1 \geq 0, |c_2| \leq kc_1\}$$

for some $k > 0$ so that $v \geq 0$ for all $v \in C_1$, and a cone C_3 defined by

$$C_3 = \{v = c_1\phi_1 + c_2\phi_2 \,|\, c_1 \leq 0, |c_2| \leq k|c_1|\}$$

so that $v \leq 0$ for all $v \in C_3$. Thus $\theta(v) \equiv 0$ for $v \in C_1 \cup C_3$.

Now we set

$$\begin{aligned} C_2 &= \{v = c_1\phi_1 + c_2\phi_2 \,|\, c_2 \geq 0, k|c_1| \leq c_2\} \\ C_4 &= \{v = c_1\phi_1 + c_2\phi_2 \,|\, c_2 \leq 0, k|c_1| \leq |c_2|\}. \end{aligned}$$

Then the union of C_1, C_2, C_3, and C_4 is the space V.

We define a map $\Phi : V \longrightarrow V$ by

$$\Phi(v) = Lv + P(-b(v + \theta(v))^+), \qquad v \in V.$$

Then Φ is continuous on V and we have the following lemma.

Lemma 4.4. $\Phi(cv) = c\Phi(v)$ *for* $c \geq 0$ *and* $v \in V$.

PROOF. Let $c \geq 0$. If v satisfies $L\theta(v) + (I - P)(-b(v + \theta(v))^+) = 0$, then

$$L(c\theta(v)) + (I - P)(-b(cv + c\theta(v))^+) = 0$$

and hence $\theta(cv) = c\theta(v)$. Therefore

$$\begin{aligned} \Phi(cv) &= L(cv) + P(b(cv + \theta(cv))^+) \\ &= L(cv) + P(b(cv + c\theta(v))^+) \\ &= c\Phi(v) \end{aligned}$$

We investigate the image of the cones C_1, C_3 under Φ. First, we consider the image of C_1. If $v = c_1\phi_1 + c_2\phi_2 \geq 0$,

$$\begin{aligned} \Phi(v) &= Lv + P(-b(v + \theta(v))^+) \\ &= c_1\mu_1\phi_1 + c_2\mu_2\phi_2 - b(c_1\phi_1 + c_2\phi_2) \\ &= (-b + \mu_1)c_1\phi_1 + (-b + \mu_2)c_2\phi_2. \end{aligned}$$

Thus the images of the rays $c_1\phi_1 \pm kc_1\phi_2 (c_1 \geq 0)$ are

$$(-b + \mu_1)c_1\phi_1 \pm (-b + \mu_2)kc_1\phi_2 \qquad (c_1 \geq 0).$$

Therefore Φ maps C_1 onto the cone

$$R_1 = \left\{ d_1\phi_1 + d_2\phi_2 \,\middle|\, d_1 \geq 0, |d_2| \leq \frac{-b + \mu_2}{-b + \mu_1} kd_1 \right\}.$$

Second, we consider the image of C_3. If $v = -c_1\phi_1 + c_2\phi_2 \le 0\,(c_1 \ge 0, |c_2| \le kc_1)$,

$$\begin{aligned}\Phi(v) &= Lv + P(-b(v+\theta(v))^+)\\ &= -c_1\mu_1\phi_1 + c_2\mu_2\phi_2.\end{aligned}$$

Thus the images of the rays $-c_1\phi_1 \pm c_1 k\phi_2 (c_1 \ge 0)$ are

$$-c_1\mu_1\phi_1 \pm c_1 k\mu_2\phi_2 \qquad (c_1 \ge 0).$$

Therefore Φ maps C_3 onto the cone

$$R_3 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \ge 0, |d_2| \le \frac{\mu_2}{|\mu_1|} kd_1 \right\}.$$

We have three possibilities that R_1 is a proper subset of R_3, or R_3 is a proper subset of R_1, or $R_1 = R_3$. R_1 is a proper subset of R_3 if and only if the nonlinearity $-bu^+$ satisfies $\frac{\mu_2}{|\mu_1|} > \frac{-b+\mu_2}{-b+\mu_1}$. R_3 is a proper subset of R_1 if and only if the nonlinearity $-bu^+$ satisfies $\frac{\mu_2}{|\mu_1|} < \frac{-b+\mu_2}{-b+\mu_1}$. The relation $R_1 = R_3$ holds if and only if the nonlinearity $-bu^+$ satisfies $\frac{\mu_2}{|\mu_1|} = \frac{-b+\mu_2}{-b+\mu_1}$.

We investigate the multiplicity of solutions of 28 under the condition that R_1 is a proper subset of R_3, that is, $\frac{\mu_2}{|\mu_1|} > \frac{-b+\mu_2}{-b+\mu_1}$.

We consider the restrictions $\Phi|_{C_i} (1 \le i \le 4)$ of Φ to the cones C_i. Let $\Phi_i = \Phi|_{C_i}$, *i.e.*, $\Phi_i : C_i \longrightarrow V$. Then it follows from Lemma 4.4 and the above calculations that $\Phi_1 : C_1 \longrightarrow R_1$ and $\Phi_3 : C_3 \longrightarrow R_3$ are bijective.

Now we investigate the images of the cones C_2, C_4 under Φ. By Theorem 4.1 and Lemma 4.2, the image of C_2 under Φ is a cone containing

$$R_2 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \ge 0, \frac{-b+\mu_2}{-b+\mu_1} kd_1 \le d_2 \le \frac{\mu_2}{|\mu_1|} kd_1 \right\}$$

and the image of C_4 under Φ is a cone containing

$$R_4 = \left\{ d_1\phi_1 + d_2\phi_2 \;\middle|\; d_1 \ge 0, -\frac{\mu_2}{|\mu_1|} kd_1 \le d_2 \le -\frac{-b+\mu_2}{-b+\mu_1} kd_1 \right\}.$$

We note that $\Phi_i(C_i)$ contains R_i, for $i = 2, 4$, respectively. □

Lemma 4.5. *For $i = 2, 4$, let γ be any simple path in R_i with end points on ∂R_i, where each ray in R_i (starting from the origin) intersects only one point of γ. Then the inverse image $\Phi_i^{-1}(\gamma)$ of γ is also a simple path in C_i with end points on ∂C_i, where any ray in C_i (starting from the origin) intersects only one point of this path.*

The proof of Lemma 4.5 is similar to that of Lemma 3.2 of [4]. From Lemma 4.5 we have Theorem 4.6 which implies our last and main result of this section.

Theorem 4.6. *For $1 \leq i \leq 4$, the restriction Φ_i maps C_i onto R_i. Therefore, Φ maps V onto R_3. In particular, Φ_1 and Φ_3 are bijective.*

Theorem 4.7. *Suppose $b < \mu_1 < 0 < \mu_2$ and $\frac{\mu_2}{|\mu_1|} > \frac{-b+\mu_2}{-b+\mu_1}$. Let $f = c_1\phi_1 + c_2\phi_2 \in V$. Then we have :*
(1) If $f \in IntR$, then 28 has exactly two solutions, one of which is positive and the other is negative.
(2) If $f \in IntR_2 \bigcup IntR_4$, then 28 has a negative solution and at least one sign changing solution.
(3) If $f \in \partial R_3$, then 28 has a negative solution.
(4) If $f \in R_3^c$, then 28 has no solution.

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