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ON THE STRUCTURE OF SOLUTIONS OF A CLASS OF BOUNDARY VALUE PROBLEMS

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Abstract. Behaviour of continua of the solution set of both operator equations and a class of boundary value problems are obtained, which partially answers an open problem of Ambrosetti [1].

1. Introduction

In a recent paper[1], A. Ambrosetti, H. Brezis and C. Cerami studied the combined effects of concave and convex nonlinearities to elliptic boundary value problems of the following type

$$\begin{cases} -\Delta u = \lambda u^q + u^p, & x \in \Omega \\ u > 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases} \tag{1.1}$$

with $0 < q < 1 < p$. They proved the existence of two positive solutions to (1.1) for λ small by upper and lower solutions and variational techniques when p is subcritical. In that paper, they also indicated several interesting open problems. See Ma [2] for example. One of those is what the structure of the solutions is in the one-dimensional case.

The purpose of the present paper is to study this problem. We will give a different approach and a general setting of the problem. The main feature is the presence of a nonlinearity having a sublinear and superlinear behavior. By applying topological methods on cones we will show the existence of a branch $\mathcal{C}$ of solutions bifurcating from $(0,0)$ that touches back $\{0\} \times (P \backslash \{0\})$.

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As applications, we will discuss in detail a class of boundary value problems of ordinary differential equations. Some further structure theorems are obtained, and a partial answer is given to the question raised in [1].

2. Structure of Solutions of Operator Equations

This section is devoted to the abstract setting of the problem. We will discuss the behaviour of continua of solutions of equations with a parameter when both superlinear and sublinear effects are present. The main results are Theorem 2.1–2.3.

Let E be a Banach space with a cone P, and $I : R^+ \times P \to P$ be a completely continuous operator, where $R^+ = [0, \infty)$. Let $\sum = \{(\lambda, x) \in R^+ \times P : x = I(\lambda, x)\}$. Then clearly $\sum$ is closed and locally compact. Write $B_r = \{x \in P : \|x\| < r\}$ for $r > 0$. First we list the following conditions for this section.

(H1) $\lim\limits_{\|x\|\to 0} \frac{\|I(0,x)\|}{\|x\|} < 1$.

(H2) $\lim\limits_{\|x\|\to 0, \lambda\to\lambda_0} \frac{\|I(\lambda,x)\|}{\|x\|} > 1$, for any $\lambda_0 > 0$.

(H3) $\lim\limits_{\|x\|\to\infty} \frac{\|I(\lambda,x)\|}{\|x\|} > 1$, uniformly for $\lambda \in R^+$.

(H4) $\lim\limits_{\lambda\to+\infty} \|I(\lambda, x)\| = \infty$, uniformly for $x \in P, \varepsilon \le \|x\| \le \frac{1}{\varepsilon}$ where $\varepsilon \in (0, 1)$ is arbitrary.

(H5) $\lim\limits_{\|x\|\to 0, \lambda\to+\infty} \frac{\|I(\lambda,x)\|}{\|x\|} > 1$.

Lemma 2.1. *Suppose that (H1) is satisfied. Then $x = 0$ is the isolated fixed point of $I(0, x)$. Moreover,*

$$i(I(0, \cdot), 0, P) = 1.$$

Proof. By condition (H1) there exists $\delta > 0$ such that $\|I(0, x)\| \le \lambda_1 \|x\|$ for $\|x\| < \delta$, where $\lambda_1 < 1$. Hence $I(0, 0) = 0$ and $i(I(0, \cdot), 0, P) = 1$ by [3]. □

Lemma 2.2. *Suppose that (H2) is satisfied. Then for any $\lambda_1, \lambda_2 > 0$, there exists $\tau > 0$ such that*

$$([\lambda_1, \lambda_2] \times B_\tau) \bigcap \sum \subset [\lambda_1, \lambda_2] \times \{0\}.$$

Proof. Suppose that there exists a sequence $\lambda_n \in [\lambda_1, \lambda_2], x_n \ne 0, x_n \to 0$ such that $(\lambda_n, x_n) \in \sum$. Assume without loss of generality that $\lambda_n \to \lambda_0 \in [\lambda_1, \lambda_2]$. Then $\|x_n\| = \|I(\lambda_n, x_n)\|$ in contradiction with condition (H2). □

Now suppose (H1) is satisfied. Then $(0, 0) \in \sum$. Recall that a continuum is a maximal connected set. Let $\mathcal{C}$ be the continuum of $\sum$ containing $(0, 0)$. Clearly $\mathcal{C}$ is closed.

Lemma 2.3. *Suppose that (H1)(H2) are satisfied. Let* $\mathcal{C}^+ = \mathcal{C} \setminus ((0,\infty) \times \{0\})$. *Then* $\mathcal{C}^+$ *is connected and closed.*

Proof. Let $(\lambda_n, x_n) \in \mathcal{C}^+$, $(\lambda_n, x_n) \to (\lambda, x) \in \sum$. Then $(\lambda, x) \in \mathcal{C}$. If $\lambda = 0$, then $(\lambda, x) \in \mathcal{C}^+$. If $\lambda > 0$, then by Lemma 2.2 we know $x \neq 0$. Hence $(\lambda, x) \in \mathcal{C}^+$ and $\mathcal{C}^+$ is closed. Next if there exist closed nonempty sets S, T such that $\mathcal{C}^+ = S \bigcup T$. Let $(0,0) \in S$. Then $\mathcal{C} = ([S \bigcup ([0,\infty) \times \{0\})] \bigcap \mathcal{C}^+) \bigcup T$. Clearly $T \bigcap ([0,\infty) \times \{0\}) = \emptyset$, and $[S \bigcup ([0,\infty) \times \{0\})] \bigcap \mathcal{C}^+$ is closed, which implies that $\mathcal{C}$ is not connected. □

Now we are in a position to give the structure of $\sum$.

Theorem 2.1. *Suppose that (H1)(H2) are satisfied. Then the continuum* $\mathcal{C}$ *of* $\sum$ *containing* $(0,0)$ *has the following properties.*

(i) $\mathcal{C}$ *contains a connected closed subset* $\mathcal{C}^+ \subset [(0,\infty) \times (P \backslash \{0\})] \bigcup (\{0\} \times P)$.

(ii) $\lambda = 0$ *is the bifurcation point of* I *if* $I(\lambda, 0) \equiv 0$.

(iii) *There exists* $\lambda_0 > 0$ *such that* $[\{\lambda\} \times (P \backslash \{0\})] \bigcap \mathcal{C}^+ \neq \emptyset$ *for* $\lambda \in (0, \lambda_0)$.

Proof. Let $\mathcal{C}^+$ be as in Lemma 2.3. Then $\mathcal{C}^+$ is closed and connected by Lemma 2.3. Thus the projection of $\mathcal{C}^+$ onto R^+ is an interval, and we need only to show that there exists $\lambda > 0$ such that $[\{\lambda\} \times (P \backslash \{0\})] \bigcap \mathcal{C} \neq \emptyset$.

In fact, if $[\{\lambda\} \times (P \backslash \{0\})] \bigcap \mathcal{C} = \emptyset$ for any $\lambda > 0$, then $\mathcal{C} \subset ((0,\infty) \times \{0\}) \bigcup (\{0\} \times P)$. Take $\lambda_0 > 0$ and let $Z = [0, \lambda_0] \times P$. Then Z is closed and convex. By Lemma 2.1, 2.2 and condition (H2) there exists $\tau > 0$ such that $[\{\lambda_0\} \times B_\tau] \bigcap \sum \subset (\lambda_0, 0)$, $[\{0\} \times B_\tau] \bigcap \sum \subset (0,0)$, and $\|I(\lambda_0, x)\| > \|x\|$ for $x \in \partial B_\tau$. Write $Q = [0, \lambda_0] \times B_\tau$. Then $\partial Q = [0, \lambda_0] \times (\partial B_\tau \bigcap P)$ in Z. Let $X = \sum \bigcap \overline{Q}$, then X is a compact metric space. Denote $S_1 = \mathcal{C} \bigcap \overline{Q}, S_2 = \sum \bigcap \partial Q$. Thus S_1, S_2 are compact disjoint subsets of X, and no subcontinuum of X can both meet S_1 and S_2. By Lemma 1.1 of [4] there exist compact disjoint subsets K_1, K_2 of X such that $X = K_1 \bigcup K_2, S_1 \subset K_1, S_2 \subset K_2$. Thus $K_1 \bigcap \partial Q = \emptyset$, and we can choose an open set U of Q with $K_1 \subset U, \partial U \bigcap K_1 = \emptyset, \partial U \bigcap K_2 = \emptyset$, hence $\partial U \bigcap \sum = \emptyset$. By the general homotopy invariance of fixed point index (see Amann [5]) we have

$$i(I(\lambda, \cdot), U(\lambda), P) = \mu = \text{const}, \quad \lambda \in [0, \lambda_0]$$

where $U(\lambda) = \{x : (\lambda, x) \in U\}$. By Lemma 2.1 $\mu = 1$ when $\lambda = 0$. Since $\|I(\lambda_0, x)\| > \|x\|$ for $x \in \partial B_\tau$, then by Lemma 2.3.3 of [3] (page 91) we have $i(I(\lambda_0, \cdot), U(\lambda_0), P) = i(I(\lambda_0, \cdot), 0, P) = 0$. □

Theorem 2.2. *Suppose that (H1)(H2) are satisfied. Let $\mathcal{C}, \mathcal{C}^+$ be as in Theorem 2.1. Then either*

(i) $\mathcal{C}^+$ is unbounded, or

(ii) $\mathcal{C}$ meets $\{0\} \times (P \backslash \{0\})$.

PROOF. Suppose $\mathcal{C}^+$ is bounded and $\mathcal{C} \bigcap [\{0\} \times (P \backslash \{0\})] = \emptyset$. Take $R > 0$ such that $\mathcal{C}^+ \subset [0, R) \times B_R$. Write $Q_R = [0, R] \times B_R, Z = [0, R] \times P, X = (\sum \bigcap \overline{Q_R}) \bigcup (R, 0)$. Then X is compact in Z, and $\partial Q_R = [0, R] \times \partial B_R$ in Z. Let $S_1 = (\mathcal{C} \bigcap \overline{Q_R}) \bigcup (R, 0)$, $S_2 = (\sum \bigcap [\partial Q_R \bigcup (\{0, R\} \times \overline{B_R})]) \backslash \{(R, 0), (0, 0)\}$ which are compact disjoint subsets of X by Lemma 2.1, 2.2. By Lemma 1.1 of Rabinowitz [4] we get compact disjoint subsets K_1, K_2 of X such that $X = K_1 \bigcup K_2, S_1 \subset K_1, S_2 \subset K_2$, and

$$K_1 \bigcap \partial Q_R = \emptyset,\ K_1 \bigcap (\{R\} \times \overline{B_R}) = (R, 0),\ K_1 \bigcap (\{0\} \times \overline{B_R}) = (0, 0).$$

Take open set $U \subset Q_R$ such that $K_1 \subset U, \partial U \bigcap K_1 = \emptyset, \partial U \bigcap \partial Q_R = \emptyset, \partial U \bigcap K_2 = \emptyset, U \bigcap K_2 = \emptyset$. Hence $\partial U \bigcap \sum = \emptyset$, and $U(R) \bigcap P = \{0\}$. Moreover

$$i(I(\lambda, \cdot), U(\lambda), P) = \mu = \text{const}, \quad \lambda \in [0, R].$$

By Lemma 2.1 $\mu = 1$ when $\lambda = 0$ since $U(0) \bigcap \sum = \{0\}$, while

$$i(I(R, \cdot), U(R), P) = i(I(R, \cdot), 0, P) = 0$$

by Lemma 2.3.3 of [3]. □

Theorem 2.3. *Suppose that (H1)–(H5) are satisfied. Then the continuum $\mathcal{C}$ of $\sum$ containing $(0, 0)$ has the following properties.*

(i) $\mathcal{C}$ contains a connected closed subset $\mathcal{C}^+ \subset [(0, \infty) \times (P \backslash \{0\})] \bigcup (\{0\} \times P)$.

(ii) $\lambda = 0$ is the bifurcation point of I if $I(\lambda, 0) \equiv 0$.

(iii) $\mathcal{C}^+$ meets $\{0\} \times (P \backslash \{0\})$.

(iv) There exists $\lambda_0 > 0$ such that $x = I(\lambda, x)$ has at least two nontrivial solutions x'_λ, x''_λ for $\lambda \in (0, \lambda_0)$, and $(\lambda, x'_\lambda), (\lambda, x''_\lambda) \in \mathcal{C}^+$.

PROOF. Let $\mathcal{C}^+$ be as in Theorem 2.1. First we will prove that $\mathcal{C}^+$ is bounded. In fact, by (H3) there exists $R > 0$ such that $\|x\| \leq R$ for $(\lambda, x) \in \sum$. Let $(\lambda_n, x_n) \in \sum, \lambda_n \to \infty$. If there exists $\varepsilon > 0$ with $\|x_n\| > \varepsilon$, then by (H4) we get a contradiction. On the other hand if $x_n \to 0$, then it will contradicts (H5). Thus $\mathcal{C}^+$ is bounded and assertion (iii) is true.

Next we will show that if there exists $\lambda > 0, x \in P$ such that $\mathcal{C}^+ \bigcap (\{\lambda\} \times P) = \{x\}$, then $\mathcal{C}^+ \bigcap ([0, \lambda] \times P)$ is connected.

In fact, if there exist nonempty closed disjoint subsets S_1, S_2 with $\mathcal{C}^+ \bigcap ([0, \lambda] \times P) = S_1 \bigcup S_2$ and $(\lambda, x) \in S_2$, then $\mathcal{C}^+ = S_1 \bigcup S_3$, where $S_3 = S_2 \bigcup (\mathcal{C}^+ \bigcap [\lambda, \infty) \times$

P). Evidently S_3 and S_1 are disjoint. This contradicts with the fact that $\mathcal{C}^+$ is connected.

Now suppose that there exist $\lambda_n > 0, \lambda_n \to 0$ such that the set $\mathcal{C}^+ \bigcap(\{\lambda_n\} \times P)$ is single-pointed for $n > 1$. Let $\mathcal{C}_n = \mathcal{C}^+ \bigcap([0, \lambda_n] \times P)$. Then $\mathcal{C}_n$ is connected and closed. By (iii) there exists $x_0 > 0, (0, x_0) \in \mathcal{C}^+$. Let $\mathcal{C}_0 = \overline{\lim_{n\to\infty}} \mathcal{C}_n = \{z :$ there exist a subsequence $n_k \to \infty$ with $z_{n_k} \in \mathcal{C}_{n_k}, z_{n_k} \to z\}$. Hence $(0, x_0) \in \mathcal{C}_0$. By Liu [6] we know that $\mathcal{C}_0$ is connected and closed. Moreover $\mathcal{C}_0 \subset \sum$, and by definition $\mathcal{C}_0 \subset \{0\} \times P$. Hence $x = 0$ could not be an isolated fixed point of $I(\lambda, \cdot)$. $\square$

3. Autonomous and Non-autonomous Boundary Value Problems

In this section, we will use the results obtained in section 2 to study a class of autonomous and non-autonomous boundary value problems of ordinary differential equations. First we consider the following non-autonomous problem

$$\begin{cases} -(Lx)(t) = f(\lambda, t, x(t)), \quad t \in (0,1) \\ \alpha x(0) - \beta \lim_{t\to 0} p(t)x'(t) = \gamma x(1) + \delta \lim_{t\to 1} p(t)x'(t) = 0 \end{cases} \tag{3.1}$$

where $(Lx)(t) = \frac{1}{p(t)}(p(t)x'(t))', p \in C[0,1] \bigcap C^1(0,1), p(t) > 0$ for $t \in (0,1)$, $\alpha, \beta, \gamma, \delta \geq 0, \beta\gamma + \alpha\delta + \alpha\gamma > 0$, and $f \in C[R^+ \times (0,1) \times R^+, R^+]$. We will assume $\int_0^1 \frac{1}{p(t)} dt < \infty$ throughout this section. Denote $\tau_0(t) = \int_0^t \frac{1}{p(t)} dt, \tau_1(t) = \int_t^1 \frac{1}{p(t)} dt$, $\rho^2 = \beta\gamma + \alpha\delta + \alpha\gamma \int_0^1 \frac{1}{p(t)} dt$, and $\rho > 0$. Define

$$u(t) = \frac{1}{\rho}[\delta + \gamma\tau_1(t)], \quad v(t) = \frac{1}{\rho}[\beta + \alpha\tau_0(t)], \tag{3.2}$$

Then $\gamma v + \alpha u \equiv \rho$. Let $E = C[0,1]$ and

$$k(t,s) = \begin{cases} u(t)v(s)p(s), & 0 \leq s \leq t \leq 1 \\ v(t)u(s)p(s), & 0 \leq t \leq s \leq 1 \end{cases} \tag{3.3}$$

Then problem (3.1) is equivalent to the operator equation $x = I(\lambda, x)$, $x \in P^{[7]}$, where

$$I(\lambda, x) = \int_0^1 k(t,s) f(\lambda, s, x(s)) ds \tag{3.4}$$

and $P = P(a,b) = \{x \in E : \min_{t\in[a,b]} x(t) \geq m(a,b)\|x\|\}$, where $m(a,b)$ is determined by the next lemma, and $a, b \in (0,1)$ be fixed ($a = \frac{1}{4}, b = \frac{3}{4}$ for example).

Lemma 3.1. *The following estimates hold.*

$$\min_{t\in[a,b]} k(t,s) \geq m(a,b) \max_{t\in[0,1]} k(t,s)$$

$$\max_{t\in[0,1]}\int_a^b k(t,s)ds \geq \max\{v(a)\int_a^b up, u(b)\int_a^b vp\}$$

where $m(a,b)=\min\{\frac{u(b)}{u(0)},\frac{v(a)}{v(1)}\}$, *and the operator* I *maps* $R^+\times P(a,b)$ *into* $P(a,b)$ *and is completely continuous.*

Proof. It is straight forward. □

Now we will list the conditions used in this section.

(F1): $\lim\limits_{x\to 0}\frac{f(0,t,x)}{x}\leq \lambda_1$, uniformly for $t\in(0,1)$ and $\lambda_1 u(0)v(1)\max\limits_{t\in[0,1]} p(t)<1$.

F(2): $\lim\limits_{x\to 0,\lambda\to\lambda_0}\frac{f(\lambda,t,x)}{x}\geq \lambda_2(\lambda_0)$, uniformly for $t\in(0,1)$, where $\lambda_2(\lambda_0)C(a,b) > 1$, $C(a,b)=m(a,b)\max\{v(a)\int_a^b up, u(b)\int_a^b vp\}$, and $\lambda_0>0$ is arbitrary.

(F3): $\lim\limits_{x\to\infty}\frac{f(\lambda,t,x)}{x}\geq\lambda_3$, uniformly for $\lambda\in R^+, t\in(0,1)$ where $\lambda_3 C(a,b)>1$.

(F4): $\lim\limits_{\lambda\to+\infty} f(\lambda,t,x)=+\infty$, uniformly for $x\in[x_1,x_2], t\in(0,1)$, and $x_1,x_2>0$.

(F5): $\lim\limits_{x\to 0,\lambda\to+\infty}\frac{f(\lambda,t,x)}{x}\geq\lambda_5$, uniformly for $t\in(0,1)$, where $\lambda_5 C(a,b)>1$.

Lemma 3.2. *Let (F1)(F2) be satisfied. Then conditions (H1)(H2) are valid.*

Proof. Choose $r>0$ such that $f(0,t,x)\leq(\lambda_1+\varepsilon)x$ for $x<r$. Then for $\|x\|<r$ we have

$$\|I(0,x)\|\leq\int_0^1 uvpf(0,s,x)ds\leq(\lambda_1+\varepsilon)\|x\|\int_0^1 uvp$$

Thus condition (H1) is true. Similarly choose $r>0$ such that $f(\lambda,t,x)\geq(\lambda_2(\lambda_0)-\varepsilon)x$ for $|\lambda-\lambda_0|<r, |x|<r$. Then for $|\lambda-\lambda_0|<r, \|x\|<r, x\in P(a,b)$ we have

$$I(\lambda,x)(t)=\int_0^1 k(t,s)f(\lambda,s,x)ds$$

$$\geq(\lambda_2(\lambda_0)-\varepsilon)\int_a^b k(t,s)x(s)ds\geq(\lambda_2(\lambda_0)-\varepsilon)m(a,b)\|x\|\int_a^b k(t,s)ds$$

□

Lemma 3.3. *Let (F1)–(F5) be satisfied. Then conditions (H1)–(H5) are valid.*

Proof. (1) Let $R>0$ such that $f(\lambda,t,x)\geq(\lambda_3-\varepsilon)x$ for $x\geq R,\lambda\geq 0$. Then for $x\in P(a,b), \|x\|>\frac{R}{m(a.b)}$ we have

$$I(\lambda,x)(t)\geq\int_a^b k(t,s)f(\lambda,s,x)ds$$

$$\geq (\lambda_3 - \varepsilon)\int_a^b k(t,s)x(s)ds \geq (\lambda_3 - \varepsilon)m(a,b)\|x\|\int_a^b k(t,s)ds$$

(2) Let $x \in P(a,b), \varepsilon \leq \|x\| \leq \frac{1}{\varepsilon}$. Then for $t \in (a,b)$ we have $\varepsilon m(a,b) \leq x(t) \leq \frac{1}{\varepsilon}$. Let $\lambda^* > 0$ such that $f(\lambda,t,x) > T$ for $\lambda > \lambda^*, \varepsilon m(a,b) \leq x \leq \frac{1}{\varepsilon}$. Then

$$I(\lambda,x)(t) \geq \int_a^b k(t,s)f(\lambda,s,x)ds \geq T\int_a^b k(t,s)ds$$

(3) Let $f(\lambda,t,x) \geq (\lambda_5 - \varepsilon)x$ for $x < r, \lambda > \lambda^*$. Then for $\lambda > \lambda^*, \|x\| < r$ we have

$$I(\lambda,x)(t) \geq \int_a^b k(t,s)f(\lambda,s,x)ds$$

$$\geq (\lambda_5 - \varepsilon)\int_a^b k(t,s)x(s)ds \geq (\lambda_5 - \varepsilon)m(a,b)\|x\|\int_a^b k(t,s)ds$$

□

Theorem 3.1. *Suppose that (F1)(F2) are satisfied. Then the continuum $\mathcal{C}$ containing $(0,0)$ of the solution set $\sum$ of problem (3.1) has the following properties.*

(i) $\mathcal{C}$ contains a connected closed subset $\mathcal{C}^+ \subset [(0,\infty)\times(P\backslash\{0\})]\bigcup(\{0\}\times P)$.

(ii) $\lambda = 0$ is the bifurcation point of I if $f(\lambda,t,0) \equiv 0$.

(iii) There exists $\lambda_0 > 0$ such that $[\{\lambda\}\times(P\backslash\{0\})]\bigcap\mathcal{C}^+ \neq \emptyset$ for $\lambda \in (0,\lambda_0)$.

Theorem 3.2. *Suppose that (F1)–(F5) are satisfied. Then the continuum $\mathcal{C}$ of $\sum$ containing $(0,0)$ has the following properties.*

(i) $\mathcal{C}$ contains a connected closed subset $\mathcal{C}^+ \subset [(0,\infty)\times(P\backslash\{0\})]\bigcup(\{0\}\times P)$.

(ii) $\lambda = 0$ is the bifurcation point of I if $f(\lambda,t,0) \equiv 0$.

(iii) $\mathcal{C}^+$ meets $\{0\}\times(P\backslash\{0\})$.

(iv) There exists $\lambda_0 > 0$ such that problem (3.1) has at least two nontrivial solutions for $\lambda \in (0,\lambda_0)$.

Corollary 3.1. *Let $f(\lambda,t,x) = \lambda g(t,x) + h(t,x)$ where $g,h : [0,1]\times R^+ \to R^+$ are continuous and $g(t,x) > 0$ for $t \in [0,1], x > 0$. If*

$$\lim_{x\to 0}\frac{h(t,x)}{x} = 0, \lim_{x\to 0}\frac{g(t,x)}{x} = +\infty, \lim_{x\to +\infty}\frac{h(t,x)}{x} = +\infty$$

Uniformly for $t \in [0,1]$, then the conclusions of Theorem 3.1–3.2 hold.

Now we consider a more special type of autonomous problems, namely

$$\begin{cases} -2x''(t) = \lambda g'(x(t)) + h'(x(s)), & t \in (0,1) \\ x(0) = x(1) = 0, & x \in C[0,1] \end{cases} \tag{3.5}$$

where $g, h \in C^1[0,\infty), g(0) = h(0) = 0, g'(x), h'(x) > 0$ for $x > 0$. Let $\lambda \geq 0$ and x be a nontrivial solution to (3.5); i.e.; $x(t) > 0$ for $t \in (0,1)$, and $\|x\| = \max_{t\in[0,1]} |x(t)| = A, x(\omega) = A$. Then $x'(t) \geq 0$ for $t \in (0,\omega)$ and $x'(t) \leq 0$ for $t \in (\omega, 1)$. By integration we have

$$x'^2(t) = \lambda g(x) + h(x) - \lambda g(A) - h(A)$$

Hence

$$x'(t) = ö\sqrt{-\lambda g(x) - h(x) + \lambda g(A) + h(A)}$$

where ö$= 1$ for $t \in (0,\omega)$ and ö$= -1$ for $t \in (\omega, 1)$. Write

$$F_{A,\lambda}(x) = \int_0^x \frac{du}{\sqrt{\lambda(g(A) - g(u)) + (h(A) - h(u))}}, x \in (0, A] \tag{3.6}$$

$$x_\lambda(t) = \begin{cases} F^{-1}_{A,\lambda}(t), & t \in (0,\omega) \\ F^{-1}_{A,\lambda}(1-t), & t \in (\omega, 1) \end{cases} \tag{3.7}$$

$$E(\lambda, A) = \int_0^A \frac{du}{\sqrt{\lambda(g(A) - g(u)) + (h(A) - h(u))}}, A > 0 \tag{3.8}$$

If x is a nontrivial solution of (3.5), then by (3.7) we know $\omega = \frac{1}{2}$. Thus we have the following:

Lemma 3.4. *Let $\lambda \geq 0$ and x be a nontrivial solution of (3.5), then $E(\lambda, \|x\|) = \frac{1}{2}$. Conversely, if there exists $\lambda \geq 0, A > 0$ such that $E(\lambda, A) = \frac{1}{2}$, then x_λ is a solution of (3.5), where x_λ is determined by (3.7).*

Lemma 3.5. *$E : R^+ \times (0,\infty) \to (0,\infty)$ is a continuous function. Moreover, E is strictly decreasing with respect to λ.*

Proof. Let $u = At$, then

$$E(\lambda, A) = \int_0^1 \frac{A}{\sqrt{\lambda(g(A) - g(At)) + (h(A) - h(At))}} dt, A > 0 \tag{3.9}$$

Thus for $t \in (\frac{1}{2}, 1)$ by the mean value theorem we have

$$\begin{aligned} & \frac{A}{\sqrt{\lambda(g(A) - g(At)) + (h(A) - h(At))}} \\ = \quad & \frac{A}{\sqrt{\lambda g'(\theta_1 A + (1-\theta_1)At) + h'(\theta_2 A + (1-\theta_2)At)}} \frac{1}{\sqrt{A}\sqrt{1-t}} \\ \leq \quad & C(A)\frac{1}{\sqrt{1-t}} \end{aligned}$$

where $\theta_1, \theta_2 \in [0,1]$ and $C(A)$ is a constant. Hence $E(\lambda, A)$ is continuous. □

Lemma 3.6. *Suppose* $\lim_{x\to 0}\frac{g'(x)}{x}=+\infty$. *Then*

$$\lim_{A\to 0+}\int_0^1 \frac{A}{\sqrt{g(A)-g(At)}}dt=0$$

PROOF. Note that g increases, hence we have $\frac{1}{2}\le\theta_1\le 1$ such that

$$\begin{aligned}\int_0^{\frac{1}{2}} \frac{A}{\sqrt{g(A)-g(At)}}dt &\le \frac{1}{2}\frac{A}{\sqrt{g(A)-g(\frac{A}{2})}}\\ \le\quad & \frac{1}{2}\frac{A}{\sqrt{g'(\theta_1 A)\frac{A}{2}}}\le 2\frac{\sqrt{\theta_1 A}}{\sqrt{g'(\theta_1 A)}}\to 0\end{aligned}$$

Similarly we have $\frac{1}{2}\le t\le\theta_2\le 1$ such that

$$\begin{aligned}\int_{\frac{1}{2}}^1 \frac{A}{\sqrt{g(A)-g(At)}}dt &\le \int_{\frac{1}{2}}^1 \frac{\sqrt{A}}{\sqrt{g'(\theta_2 A)}}\frac{1}{\sqrt{1-t}}dt\\ \le\quad & \sqrt{2}\int_{\frac{1}{2}}^1 \frac{\sqrt{\theta_2 A}}{\sqrt{g'(\theta_2 A)}}\frac{1}{\sqrt{1-t}}dt\end{aligned}$$

Let $M>0, A_0>0$ be such that $\frac{g'(A)}{A}>M$ for $0<A<A_0$. Consequently

$$\int_{\frac{1}{2}}^1 \frac{A}{\sqrt{g(A)-g(At)}}dt\le\frac{1}{\sqrt{M}}\int_{\frac{1}{2}}^1\frac{dt}{\sqrt{1-t}}$$

□

Lemma 3.7. *Suppose* $\lim_{x\to +\infty}\frac{h'(x)}{x}=+\infty$. *Then*

$$\lim_{A\to +\infty}\int_0^1 \frac{A}{\sqrt{h(A)-h(At)}}dt=0$$

Proof. Similar to the proof of Lemma 3.6, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{A}{\sqrt{h(A)-h(At)}}dt \\
\leq\ &\int_0^{\frac{1}{2}} \frac{A}{\sqrt{h(A)-h(\frac{A}{2})}}dt = \frac{1}{2}\frac{A}{\sqrt{h'(\theta_1 A)}} \to 0 \\
&\int_{\frac{1}{2}}^1 \frac{A}{\sqrt{h(A)-h(At)}}dt \\
\leq\ &\int_{\frac{1}{2}}^1 \frac{A}{\sqrt{h'(\theta_2 A)}}\frac{1}{\sqrt{A}\sqrt{1-t}} \to 0
\end{aligned}$$

□

Theorem 3.3. *Suppose that the following conditions are satisfied*

$$\lim_{x\to+\infty}\frac{h'(x)}{x} = +\infty, \lim_{x\to 0}\frac{g'(x)}{x} = +\infty \tag{3.10}$$

Then there exists $\lambda^* \in (0,\infty)$ such that problem (3.5) has at least two nontrivial solutions for $0<\lambda<\lambda^*$, and no nontrivial solutions for $\lambda>\lambda^*$.

Proof. By Lemma 3.5–3.7 we know that $E(\lambda, A)$ is continuous with respect to A, $E(\lambda,A) > 0$ for $A>0$, and for fixed $\lambda>0$, $\lim_{A\to 0+} E(\lambda,A) = \lim_{A\to+\infty} E(\lambda,A) = 0$. Let $A_0>0$ be such that for $A>A_0$

$$\int_0^1 \frac{A}{\sqrt{h(A)-h(At)}}dt < \varepsilon$$

Then $E(\lambda,A)<\varepsilon$ for $A>A_0$. Let $C>0$ be such that

$$\int_0^1 \frac{A}{\sqrt{h(A)-h(At)}}dt \leq C, 0<A\leq A_0$$

Then $E(\lambda,A)\leq C\frac{1}{\sqrt{\lambda}}$ for $0<A\leq A_0$. Hence $\lim_{\lambda\to\infty} E(\lambda,A)=0$ uniformly for $A>0$. As a result, equation $E(\lambda,A)=\frac{1}{2}$ has no solutions for λ large enough. □

In order to consider continua of the solution set, we need the following lemma. Let $\sum, \mathcal{C}, \mathcal{C}^+$ be as before, and $\Omega = \{(\lambda,A)\in R^2 : E(\lambda,A)=\frac{1}{2}, \lambda\geq 0, A>0\}$.

Lemma 3.8. *Let S_E be a closed and connected subset of $\sum$. Denote $S_R = \{(\lambda,\|x\|) : (\lambda,x)\in S_E\}$. Then S_E is closed and connected in R^2. Conversely, if $S_R\subset\Omega$ is closed and connected. Let $S_E = \{(\lambda,x_\lambda) : x_\lambda$ is determined by (3.7)$\}$. Then S_R is closed and connected in $R^+\times E$.*

Proof. It suffices to note that the following maps are continuous,

$$R^+ \times E \to R^2 : (\lambda, x) \mapsto (\lambda, \|x\|)$$

$$\Omega \to R^+ \times E : (\lambda, A) \mapsto (\lambda, x_\lambda)$$

where x_λ is determined by (3.7).

Theorem 3.4. *Suppose (3.10) is satisfied, then there exist $\lambda^*, A^* > 0$ such that $\sum \backslash ((0,\infty) \times \{0\}) \subset [0,\lambda^*] \times B_{R^*}$, and any continuum of $\sum$ will either meet $\{0\} \times P$ twice, or lie in $\{0\} \times P$.*

Proof. By the proof of Theorem 3.3 we know there exists $\lambda^* > 0$ such that $E(\lambda, A) \le \frac{1}{4}$ for $\lambda > \lambda^*, A > 0$. By Lemma 3.7 there exists $A^* > 0$ such that $E(\lambda, A) \le \frac{1}{4}$ for $\lambda \ge 0, A > A^*$. Therefore $\Omega \subset [0,\lambda^*] \times [0, A^*], \sum \backslash ((0,\infty) \times \{0\}) \subset [0,\lambda^*] \times B_{R^*}$. Let $\Omega_0 = \{A > 0 : E(0, A) > \frac{1}{2}\}$. Then Ω_0 is an open set composed of open intervals. Let $J \subset \Omega_0$ be one of its maximal open intervals, then the implicit function theorem implies that there exists a continuous curve $\lambda = \lambda(A) : J \to [0, \lambda^*]$ such that $E(\lambda(A), A) = \frac{1}{2}$. Hence $\{(\lambda(A), A) : A \in J\} \subset \Omega$ is connected. Let $(\lambda, x) \in \sum, \lambda > 0$, then $(0, \|x\|) \in \Omega_0$ since $E(\lambda, A)$ is strictly decreasing with respect to λ. □

Theorem 3.5. *Suppose the following conditions are satisfied*

$$\lim_{x\to 0} \frac{h'(x)}{x} = 0, \lim_{x\to+\infty} \frac{h'(x)}{x} = +\infty, \lim_{x\to 0} \frac{g'(x)}{x} = +\infty \tag{3.11}$$

$$xh'(x) - 2h(x) \text{is strictly increasing for } x > 0 \tag{3.12}$$

Then $\sum = \mathcal{C}$ and $\mathcal{C}^+$ meets $\{0\} \times P$ exactly twice.

Proof. By Theorem 3.2 and Corollary 3.1 we know that $\mathcal{C}^+$ meets $\{0\} \times (P \backslash \{0\})$. Thus by Lemma 3.4, (iii) of Theorem 3.1 and Corollary 3.1 there exists $\lambda_0 > 0$ with $E(\lambda, \|x_\lambda\|) = \frac{1}{2}$, where $(\lambda, x_\lambda) \in \mathcal{C}^+, 0 < \lambda < \lambda_0$. By Lemma 3.5 we know $E(0, \|x_\lambda\|) > \frac{1}{2}$ for $0 < \lambda < \lambda_0$. Hence $(0, \lambda_0) \subset \Omega_0$. Thus by Lemma 3.4 we need only to prove that $E(0, A)$ is strictly decreasing. In fact, let $t \in (0,1)$, $\phi(A) = \frac{h(A)-h(At)}{A^2}$, then by (3.12)

$$A^3\phi'(A) = Ah'(A) - 2h(A) + 2h(At) - Ath'(At) > 0, t \in (0,1)$$

Hence $\phi(A)$ is strictly increasing, and

$$E(0, A) = \int_0^1 \Big[\frac{h(A) - h(At)}{A^2}\Big]^{-\frac{1}{2}} dt$$

is strictly decreasing. Therefore Ω_0 is an open interval. □

Corollary 3.2. *Consider problem (1.1) in the scalar case, i.e.,*

$$\begin{cases} -x'' = \lambda x^q + x^p, & t \in (0,1) \\ x(t) > 0, & t \in (0,1) \\ x(0) = x(1) = 0, \end{cases} \tag{3.13}$$

with $0 < q < 1 < p$. *Then all the conclusions of Theorem 3.1–3.5 hold for (3.13).*

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