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THE PRIMITIVE ELEMENT THEOREM FOR COMMUTATIVE ALGEBRAS

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Abstract. Let $R \subseteq T$ be an extension of commutative rings (with the same 1). We say that $R \subseteq T$ has FIP if the set of R-subalgebras of T is finite. If $R \subseteq T$ has FIP, then T must be algebraic over R; if, in addition, R is a field, then T is a finite-dimensional R-vector space. If $R \subseteq T$ has FIP and T is an integral domain, then either R and T are fields or T is an overring of R. If R is a perfect field, then the main result identifies four exhaustive cases which serve to characterize the condition that $R \subseteq T$ has FIP. Considering extensions $R \subseteq T$ having FIP with T the quotient field of R amounts to studying integral domains R with only finitely many overrings. Such integral domains R are characterized as the semi-quasilocal i-domains of finite Krull dimension having only finitely many integral overrings. This property is interpreted further in case R is either integrally closed or a pseudo-valuation domain. Examples are given to illustrate the sharpness of the results.

1. Introduction

All rings and algebras considered below are commutative with unit. Our starting point is the classical Primitive Element Theorem [2, Theorem 26]: if $K \subseteq L$ is a finite-dimensional field extension, then $L = K[\alpha]$ for some element $\alpha \in L$ if and only if the set of intermediate fields between K and L is finite. To study the underlying properties in the more general context of algebras, we make the following definition. If $R \to T$ is a (unital) ring-homomorphism, we say that $R \to T$ has FIP (for the "finitely many intermediate algebras" property) if the set of R-subalgebras of T is finite. It is evident that a ring-homomorphism $f : R \to T$

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has FIP if and only if the inclusion map $f(R) \hookrightarrow T$ has FIP. For this reason, we henceforth consider FIP only for R-algebras T such that $R \subseteq T$. An exact analogy with the classical result fails, as Proposition 2.1 shows that, even if $R \subseteq T$ is an algebraic extension of rings, the condition that $R \subseteq T$ has FIP is logically independent of the condition that $T = R[\alpha]$ for some $\alpha \in T$. However, some aspects of the classical flavor persist, as it is easy to show that FIP implies algebraicity and, in fact, if the base ring is a field, finite-dimensionality (see Proposition 2.2 (a),(d)). Moreover, if K is a perfect field, our main result, Theorem 3.8, produces four cases whose disjunction characterizes the extensions $K \subseteq T$ which have FIP.

Rather complete information about FIP is available for extensions $R \subseteq T$ where T is a field. In this case, Theorem 2.4 (a) provides the fundamental dichotomy: if $R \subseteq L$ has FIP and L is a field, then either R is a field or L is the quotient field of R. The former option is analyzed completely by Theorem 3.8 (which was mentioned above). The latter option, dealing with integral domains R having only finitely many overrings, is characterized in Theorem 2.6: these R are the semi-quasilocal i-domains (in the sense of [12]) having finite (Krull) dimension and only finitely many integral overrings. This second option is interpreted in Corollary 2.7 when R is a pseudo-valuation domain (in the sense of [9]). When neither R nor T is a field, results on extensions $R \subseteq T$ having FIP are less complete: cf. Corollary 2.3, Theorem 2.4 (b), and Corollary 3.9. These include the fact (Proposition 3.4) that if R is a finite ring, then a ring extension $R \subseteq T$ has FIP if and only if T is finite.

For an integral domain D, Spec(D) denotes the set of prime ideals of D, $\dim(D)$ denotes the Krull dimension of D, $qf(D)$ denotes the quotient field of D, and D' denotes the integral closure of D. Following [10, page 28], we denote the properties of lying-over, going-up, and incomparability by LO, GU, and INC, respectively. Any unexplained material is standard, as in [10].

2. When The Algebra is A Field

We begin by showing that the classical Primitive Element Theorem cannot be naively extended from the context of field extensions to the context of algebras.

Proposition 2.1. *(a) There exists an algebraic ring extension $R \subseteq T$ such that $T = R[\alpha]$ for some $\alpha \in T$ but $R \subseteq T$ does not have FIP.*

(b) There exists an algebraic ring extension $R \subseteq T$ which has FIP but T cannot be generated as an R-algebra by one element.

PROOF. (a) Let R be an infinite-dimensional valuation domain with a height 1 prime ideal P. Pick $\beta \in P \setminus \{0\}$ and put $L = qf(R)$. Evidently, $L = R[\beta^{-1}]$ (cf. [10, Theorem 19]). However, $R \subseteq L$ does not have FIP since $\{R_P : P \in \mathrm{Spec}(R)\}$ is an infinite set of overrings of R.

(b) Let $R = \mathbf{F}_2$ and $T = R \times R \times R$. View $R \subseteq T$ by means of the diagonal embedding $R \to T$. Of course, T is algebraic over R. As T is finite, it is trivial that $R \subseteq T$ has FIP. However, it is straightforward to check that none of the eight elements $\alpha \in T$ satisfies $R[\alpha] = T$. $\square$

Remark. Although the base ring R in Proposition 2.1(b) is a field, such is not the case in Proposition 2.1 (a). Nevertheless, there does exist a ring extension $R \subseteq T$ satisfying the assertion of Proposition 2.1 (a) for which R is a field. A specific example to illustrate this is given by $T = K[X]/(X^4)$, where $R = K$ is an arbitrary infinite field. We defer the verification to Lemma 3.6, where a more general result is established.

The next result collects some useful facts which explain, i.a., the emphasis in Proposition 2.1 (and below) on algebraic algebras.

Proposition 2.2. *(a) Let $R \subseteq T$ have FIP. Then each $\alpha \in T$ is the root of a polynomial in $R[X]$ with a unit coefficient. In particular, T is algebraic over R.*

(b) If $R \subseteq T$ has FIP, then T is a finite-type R-algebra.

(c) If $R \subseteq T$ is an integral extension which has FIP, then T is module-finite over R.

(d) If $K \subseteq T$ has FIP and K is a field, then T is a finite-dimensional K-vector space.

PROOF. (a) Fix α in T. Since $R \subseteq T$ has FIP, there exist positive integers $n < m$ such that $R[\alpha^n] = R[\alpha^m]$. Hence

$$\alpha^n = r_0 + r_1\alpha^m + \cdots + r_d\alpha^{dm}$$

for some $r_0, r_1, \ldots, r_d \in R$, from which the assertions are immediate.

(b) Deny. Pick $t_1 \in T \setminus R$. Inductively find $t_2, t_3, \cdots \in T$ such that $t_{n+1} \in T \setminus R[t_1, \ldots, t_n]$ for each integer $n \geq 1$. It follows that

$$R \subset R[t_1] \subset R[t_1, t_2] \subset \cdots \subset R[t_1, \ldots, t_n] \subset \cdots$$

is a strictly ascending chain of rings, contradicting that $R \subseteq T$ has FIP.

(c) Apply (b).

(d) Apply (c). $\square$

Corollary 2.3. *Let $R \subseteq T$ be an extension of integral domains which has FIP. Let $K = qf(R)$ and $L = qf(T)$, and let A denote the integral closure of R in T. Then:*

(a) For all multiplicatively closed subsets S of R, $R_S \subseteq T_S$ has FIP.
(b) $[L : K] < \infty$, $L = K(\alpha) = K[\alpha]$ for some $\alpha \in L$, and $K \subseteq L$ has FIP.
(c) $L = qf(A)$.
(d) If, in addition, R is integrally closed in T, then T is an overring of R.

PROOF. (a) If E is an R_S-subalgebra of T_S, then $E \cap T$ is an R-subalgebra of T and a straightforward calculation shows that $E = (E \cap T)_S$. Hence, the assignment $E \mapsto E \cap T$ gives an injection from the set of R_S-subalgebras of T_S to the set of R-subalgebras of T. As the latter set is finite, so is the former.

(b), (c): Consider $S = R \setminus \{0\}$. Since L is algebraic over K, the usual clearing-of-denominators trick (as in the proof of [13, Theorem 7, page 264]) shows that L is contained in, and hence, coincides with A_S. Then (c) follows immediately. As for (b), apply (a) to see that $K \subseteq L$ has FIP. By Proposition 2.2 (d), it follows that $[L : K] < \infty$. Therefore, by the Primitive Element Theorem, $L = K(\alpha) = K[\alpha]$ for some $\alpha \in L$.

(d) Since R is integrally closed in T, we have $A = R$. Apply (c). □

Remark. (a) Note that the proof of Corollary 2.3 (a) carries over essentially *verbatim* in case $R \subseteq T$ is an arbitrary ring extension and S is any multiplicatively closed subset of R. One need only interpret $E \cap T$ as $j^{-1}(E)$, where $j : T \to T_S$ is the canonical structure map.

(b) It is interesting to note that there are at least two other proofs of Corollary 2.3 (d). Both depend on more material than the proof given above. These alternate proofs use Corollary 2.3 (a) to reduce to the case of quasilocal R, with $T = R[\alpha]$ for some $\alpha \in T$. The first alternate proof then uses Corollary 2.2 (a) and the proof of the (u, u^{-1}) lemma (cf. [10, Theorem 67]) to obtain the conclusion. The second alternate proof concludes via Zariski's Main Theorem, noting that each prime ideal Q of T is isolated in the fiber above $Q \cap R$, since $R \subseteq T$ satisfies INC (as a consequence of Corollary 2.2 (a) and [5, Theorem, page 38]).

(c) There is another interesting result which illustrates the typically algebraic flavor of extensions having FIP. Let $R \subseteq T$ be rings. Then $R[X] \subseteq T[X]$ has FIP $\Leftrightarrow$ $R[[X]] \subseteq T[[X]]$ has FIP $\Leftrightarrow$ $R = T$. For a proof, observe (in the polynomial case, with power series being treated similarly) that if $R \neq T$, then each positive integer n gives rise to a ring $B_n = R + RX + \cdots + RX^{n-1} + X^n T[X]$ contained between $R[X]$ and $T[X]$, with $B_{n_1} \neq B_{n_2}$ if $n_1 \neq n_2$.

Theorem 2.4 (a) identifies the only two contexts $R \subseteq T$ for which FIP can occur when T is a field.

Theorem 2.4. *(a) If $R \subseteq L$ has FIP and L is a field, then either R is a field or $L = qf(R)$.*

(b) If $R \subseteq T$ has FIP and T is an integral domain, then either R and T are fields or T is an overring of R.

PROOF. (a) Evidently, (a) is a consequence of (b). For the sake of completeness, we include a self-contained proof of (a). Observe that R' is not a field if R is not a field (cf. [10, Theorem 48]). Accordingly, we may assume, without loss of generality, that $R = R'$, that is, that R is integrally closed.

Suppose the assertion fails. Hence, we have proper inclusions $R \subset K = qf(R) \subset L$. Pick $\alpha \in L$. There is no loss of generality in replacing L with $K(\alpha)$. Then, in fact, $L = K[\alpha]$ since Proposition 2.2 (a) yields that α is algebraic over K. By applying clearing-of-denominators to an algebraicity equation of α, we obtain $b \in R \setminus \{0\}$ such that $b\alpha$ is integral over R. Since $K[b\alpha] = K(b\alpha) = K(\alpha) = K[\alpha] = L$, we may suppose that α is integral over R. Then, since R is integrally closed, $n = [L : K]$ is the minimum degree for an integrality equation of α over R (cf. [13, Theorem 4, page 260]).

Choose a nonzero nonunit r of R. For each positive integer k, consider

$$B_k = R + Rr^k\alpha + Rr^k\alpha^2 + \cdots + Rr^k\alpha^{n-1}.$$

It is clear that each B_k is an R-submodule of L which contains R. Moreover, since α satisfies an integrality equation of degree n over R, one easily verifies that each B_k is a ring. Now, as $R \subseteq L$ has FIP, there exist positive integers $i < j$ such that $B_i = B_j$. In particular,

$$r^i\alpha = b_0 + b_1r^j\alpha + b_2r^j\alpha^2 + \cdots + b_{n-1}r^j\alpha^{n-1}$$

for some $b_0, b_1, \dots, b_{n-1} \in R$. Since $\{1, \alpha, \alpha^2, \dots, \alpha^{n-1}\}$ is linearly independent over K, we may equate the coefficients of α and hence conclude that $r^i = b_1r^j$. Because R is an integral domain, it follows that $1 = b_1r^{j-i}$, whence r is a unit of R, the desired contradiction.

(b) Deny. Let $K = qf(R)$ and $L = qf(T)$. Since R is not a field, it follows via LO that the integral closure of R in T is also not a field. Hence, by Corollary 2.3 (d), we may assume that T is integral over R. Writing $T = R[\alpha_1, \dots, \alpha_n]$ by Proposition 2.2 (b) and considering the steps in the tower

$$R \subseteq R[\alpha_1] \subseteq R[\alpha_1, \alpha_2] \subseteq \cdots \subseteq R[\alpha_1, \dots, \alpha_n],$$

we may assume that $n = 1$; that is, $T = R[\alpha]$ for some $\alpha \in T$. If M is a maximal ideal of R, we may replace $R \subseteq T$ by $R_M \subseteq T_{R \setminus M} = R_M[\alpha]$, and so, without loss of generality, (R, M) is quasilocal. Consider a maximal chain of pairwise distinct intermediate rings

$$R = R_0 \subset \cdots \subset R_m = T,$$

with $K_i = qf(R_i)$. For some $1 \leq i \leq m$, $K_{i-1} \neq K_i$ and so, by passing to $R_{i-1} \subset R_i$, we may assume that R and T are adjacent. (In other words, $R \neq T$ and there are no rings properly between R and T. Note that $T = R[\beta]$ for any $\beta \in T \setminus R$.)

An appeal to [11, Corollary 2, page 7] produces the desired contradiction. Alternatively, we may argue as follows. Observe that $MT \neq M$, lest R and T share a common nonzero ideal M (and thus have the same quotient field). By LO, $MT \cap R = M$, and so R is properly contained in $S = R + MT$. By adjacency, $S = T$. However, T is a finitely generated R-module, by Proposition 2.2 (c). Hence, by Nakayama's Lemma, $T = R$, the desired contradiction. □

For the rest of this section, we focus on the second case in the dichotomy described above. To rephrase that case, notice that if R is an integral domain and $L = qf(R)$, then $R \subseteq L$ has FIP if and only if R has only finitely many overrings. Lemma 2.5 prepares the way, while sharpening a point made in the proof of Proposition 2.1 (a).

Lemma 2.5. *(a) Suppose an integral domain R has only finitely many overrings. Then R has only finitely many prime ideals, and so $dim(R) < \infty$ and R is semi-quasilocal.*

(b) Let R be a valuation domain. Then the set of all overrings of R is finite if and only if $dim(R) < \infty$.

Proof. (a) If $P, Q \in \mathrm{Spec}(R)$ are such that $R_P = R_Q$, then $P = PR_P \cap R = QR_Q \cap R = Q$, and so the assignment $P \mapsto R_P$ gives an injection from $\mathrm{Spec}(R)$ to the set of overrings of R. The assertions follow.

(b) Since R is a valuation domain, the assignment $P \mapsto R_P$ gives a one-to-one correspondence between $\mathrm{Spec}(R)$ and the set of overrings of R (cf. (a) and [10, Theorem 65]). Since $\mathrm{Spec}(R)$ is linearly ordered by inclusion, the assertion is immediate. □

Recall from [12] that an integral domain R is called an i-domain if the canonical map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is an injection for each overring T of R; equivalently, if $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an injection and R' is a Prüfer domain; equivalently, if R'_M is a valuation domain for each maximal ideal M of R.

Theorem 2.6. *For an integral domain R, the following conditions are equivalent:*

(1) R has only finitely many overrings;

(2) R is a semi-quasilocal i-domain of finite Krull dimension with only finitely many integral overrings.

Moreover, if the above conditions hold, then R' is a semi-quasilocal Prüfer domain and R' is a finitely generated R-module.

Proof. (1) $\Rightarrow$ (2): Let $L = qf(R)$. Suppose that $R \subset L$ has FIP. Fix a maximal ideal M of R. Since $R_M \subseteq L$ inherits FIP from $R \subseteq L$, it follows from the first assertion in Proposition 2.2 (a) that R'_M is a valuation domain. (See [8, Theorem 5] or [5, Corollary 5].) By the above remarks, R_M is an i-domain. As M is arbitrary, it follows that R is an i-domain, and so R' is a Prüfer domain. Moreover, Lemma 2.5 (a) ensures that $\dim(R)$, $\dim(R') < \infty$ and that R and R' are semi-quasilocal. Also, by applying Proposition 2.2 (c) to the extension $R \subseteq R'$, we see that R' is a finitely generated R-module.

(2) $\Rightarrow$ (1): Assume (2). As $R \subseteq R'$ has FIP, Proposition 2.2 (c) yields that R' is a finitely generated R-module. It follows that if $P \in \mathrm{Spec}(R)$, then there exist only finitely many $Q \in \mathrm{Spec}(R')$ such that $Q \cap R = P$. The hypotheses on R now ensure that R' is a finite-dimensional semi-quasilocal Prüfer, hence Bézout, domain. Therefore, $\mathrm{Spec}(R')$ is a finite set. Since $R \subseteq R'$ satisfies LO (cf. [10, Theorem 44]), $\mathrm{Spec}(R)$ is also finite. Accordingly, by Corollary 2.3 (a), it suffices to prove that if T is an overring of R, then there exists finitely many $P_1, \dots, P_n \in \mathrm{Spec}(R)$ such that $R_S \subseteq T \subseteq R'_S$, where $S = R \setminus (P_1 \cup \cdots \cup P_n)$.

Given T, note that T' is an overring of the Bézout domain R', and so $T' = R'_\Sigma$ for some multiplicatively closed set Σ of R'. As Σ may be assumed saturated without loss of generality, $\Sigma = R' \setminus (Q_1 \cup \cdots \cup Q_n)$ for some finite subset $\{Q_1, \dots, Q_n\}$ of $\mathrm{Spec}(R')$. Put $P_i = Q_i \cap R$ and $S = R \setminus (P_1 \cup \cdots \cup P_n)$.

As elements of T which are units in T' must be units of T (since $T \subseteq T'$ satisfies LO), an easy calculation reveals that $T_S = T$. Hence $R_S \subseteq T_S = T \subseteq T' = R'_\Sigma$. It suffices to show that $R'_\Sigma = R'_S$. Observe that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an injection (since R is an i-domain) and that $R \subseteq R'$ satisfies GU. Thus, by the Prime Avoidance Lemma [10, Theorem 81], the maximal ideals of R'_S are $Q_i R'_S$, for $i = 1, \dots, n$. Therefore, by [7, Corollary 5.2], it follows that $R'_S = R'_\Sigma$. $\square$

We pause to record the facts that an integral domain R satisfying the conditions in Theorem 2.6 need not be quasilocal; and the phrase "with only finitely many integral overrings" cannot be deleted from condition (2) in Theorem 2.6.

Remark. (a) If V and W are distinct one-dimensional valuation domains with the same quotient field K, then $R = V \cap W$ is a Bézout domain with exactly two distinct maximal ideals, and so the only overrings of R are R, V, W, and K (cf. [10, Theorem 107]). More generally, one shows similarly that if $V_1, \dots, V_n$ are finitely many finite-dimensional valuation domains with the same quotient field, then $\cap V_i$ is a Bézout domain with only finitely many overrings.

(b) The phrase "with only finitely many integral overrings" cannot be deleted from condition (2) in Theorem 2.6. In fact, there exists a quasilocal finite-dimensional i-domain R such that R' is a finitely generated R-module although R has infinitely many overrings. For an example, consider $R = k + XF[[X]]$, where $k \subseteq F$ is a finite-dimensional field extension which does not satisfy the conclusion of the Primitive Element Theorem. In view of the lore of the classical $D + M$-construction [7], one needs only to establish the infinitude of the set of overrings. This, in turn, follows by examining the set $\{E_i + XF[[X]]\}$, where $\{E_i\}$ is the (infinite) set of fields contained between k and F. Notice, as might have been predicted from Theorem 2.6, that each of the $E_i + XF[[X]]$ is an integral overring of R. Another way to verify all the assertions concerning R is to invoke Corollary 2.7 below.

The conditions in Theorem 2.6 are interpreted in Corollary 2.7 for a class of quasilocal integral domains. First, we recall some background about pseudo-valuation domains (PVDs). As in [9], an integral domain R is called a PVD if R has a (uniquely determined) valuation overring V such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ as sets. Equivalently, by [1, Proposition 2.6], an integral domain R is a PVD if and only if there is a pullback description $R = V \times_F k$, where (V, M) is a valuation domain, $F = V/M$, and $k = R/M$; moreover, the data in such a description are uniquely determined, with $V = (M : M)$. It is well known that a pseudo-valuation domain R (with the above pullback description) is an i-domain if and only if F is an algebraic field extension of k. It is also evident that a PVD must be quasilocal.

Corollary 2.7. *Let (R, M) be a PVD, with canonical pullback description $R = V \times_F k$. Then the following conditions are equivalent:*

(1) R has only finitely many overrings;

(2) R is an i-domain of finite Krull dimension and $F = k[\alpha]$ for some $\alpha \in F$.

Proof. By Theorem 2.6 and the above remarks, we may assume, without loss of generality, that R is an i-domain of finite Krull dimension and, in particular, that F is an algebraic field extension of k. Since R is a quasilocal i-domain, R' is a valuation domain, and so $R' = V$ [4, Remark 4.8 (a)]. Therefore, by Theorem 2.6, R has only finitely many overrings if and only if $R \subseteq V$ has FIP. (The same conclusion can be reached without appealing to Theorem 2.6. Indeed, as R is a PVD i-domain, it follows from [6, Theorem 1.31] that each overring of R is comparable to $R' = V$ with respect to inclusion; and, since V is a finite-dimensional valuation domain, Lemma 2.5 (b) gives that V has only finitely many overrings.) Now, $R \subseteq V$ has FIP if and only if $R/M = k \subseteq V/M = F$ has FIP because $R = V \times_F k$. It remains only to show that $k \subseteq F$ has FIP if and only if $F = k[\alpha]$ for some $\alpha \in F$. As F is an algebraic field extension of k, this, in turn, follows from Proposition 2.2 (d) and the Primitive Element Theorem. $\square$

Observe that Corollary 2.7 may be viewed as a generalization of Lemma 2.5 (b). Proposition 2.8 (c) provides a generalization of Lemma 2.5 (b) in a different direction. Also, it is interesting to note that in formulating condition (2) in Proposition 2.7, the “missing link” is a field extension $k \subseteq F$ of the kind characterized in the classical Primitive Element Theorem.

In view of Corollary 2.3 (a), it is natural to ask whether “locally FIP” implies FIP. The next result answers the general question in the negative, while giving a positive answer for $R \subseteq T$ in case R is a semi-quasilocal integrally closed integral domain whose quotient field is contained in T.

Proposition 2.8. *(a) There exists an extension $R \subseteq T$ of integral domains which does not have FIP although $R_P \subseteq T_{R\setminus P}$ has FIP for each $P \in Spec(R)$.*

(b) Let R be an integrally closed integral domain which is properly contained in its quotient field K, and let L be a field extension of K. Then $R \subseteq L$ has FIP if and only if $L = K$ and R is a semi-quasilocal finite-dimensional Prüfer domain.

(c) Let R be an integrally closed semi-quasilocal integral domain, with $K = qf(R)$. Then the following conditions are equivalent:

(1) $R \subseteq K$ has FIP;

(2) R is a finite-dimensional Prüfer domain;

(3) $R_M \subseteq K_{R\setminus M}$ (=K) has FIP for each maximal ideal M of R.

Proof. (a) By Lemma 2.5, it suffices to take R to be any Dedekind domain with infinitely many maximal ideals and $T = qf(R)$.

(b) Combine Theorems 2.4 and 2.6.

(c) Without loss of generality, $R \neq K$. Then (1) $\Leftrightarrow$ (2) by (b), with $L = K$. Moreover, (1) $\Rightarrow$ (3) by Corollary 2.3 (a). Finally, assume (3). For each maximal ideal M of R, Theorem 2.6 shows that R_M is a finite-dimensional valuation domain, and so M has finite height in R. Since R is assumed semi-quasilocal, $\dim(R) < \infty$. Furthermore, since R is locally a valuation domain, R is a Prüfer domain, yielding (2). $\square$

To close the section, we give an example which may be viewed as a companion for the examples in Proposition 2.1. Proposition 2.9 also serves to motivate our main result, Theorem 3.8.

Proposition 2.9. *There exist ring extensions $R \subseteq S$ and $S \subseteq T$ which each have FIP such that $R \subseteq T$ does not have FIP. It can be arranged that R is an integral domain and that T is an overring of R.*

Proof. For any prime number p, let $R = \mathbf{F}_p(X, Y)$, $S = \mathbf{F}_p(X^{1/p}, Y)$, and $T = \mathbf{F}_p(X^{1/p}, Y^{1/p})$. Since $R \subseteq S = R[X^{1/p}]$ and $S \subseteq T = S[Y^{1/p}]$ are finite-dimensional field extensions, it follows from the Primitive Element Theorem that $R \subseteq S$ and $S \subseteq T$ both have FIP. On the other hand, the Primitive Element Theorem ensures that $R \subseteq T$ does not have FIP, since $[R[\alpha] : R] \leq p$ for each $\alpha \in T$ and $[T : R] = p^2$.

Let R, S and T be as above. If $F = T$ denotes the largest of these fields, let $F + M$ be a valuation domain with maximal ideal $M \neq 0$. Put $A = R + M$, $B = S + M$, and $C = T + M$. Note that C is an overring of the integral domain A. Moreover, the extensions $A \subseteq B$ and $B \subseteq C$ inherit FIP from $R \subseteq S$ and $S \subseteq T$, respectively, while $A \subseteq C$ inherits the failure of FIP from $B \subseteq C$. Indeed, it suffices to observe that if $V \subseteq W$ are subrings of F, then [3, Theorem 3.1] gives a one-to-one correspondence between the set of $(V + M)$-subalgebras of $W + M$ and the set of V-subalgebras of W. $\square$

3. When The Base Ring is A Field

In many cases, ring extensions $R \subseteq T$ which have FIP are such that $T = R[\alpha]$ for some $\alpha \in T$. This can be seen by revisiting a standard proof of the classical Primitive Element Theorem. For the sake of completeness, we next record this result.

Proposition 3.1. *Let $R \subseteq T$ have FIP. Suppose that R contains an infinite set S of units of R such that $u - v$ is a unit of R for every pair of distinct elements u, v of S. Then $T = R[\alpha]$ for some $\alpha \in T$.*

Proof. Without loss of generality, $R \neq T$. Reasoning as in the proof of Proposition 2.2 (b), we find $\alpha \in T \setminus R$ such that $R[\alpha]$ is maximal with respect to being a singly-generated R-subalgebra of T. Without loss of generality, we may assume that $\alpha \notin R$. If the assertion fails, choose $\beta \in T \setminus R[\alpha]$. Consider $\mathcal{C} = \{R[\alpha + \gamma\beta] : \gamma \in S\}$. Since $R \subseteq T$ has FIP, $\mathcal{C}$ is finite, and so it follows from the Pigeonhole Principle that there exist distinct $u, v \in S$ such that $R[\alpha + u\beta] = R[\alpha + v\beta]$. Let A denote this ring. As $(u - v)\beta = (\alpha + u\beta) - (\alpha + v\beta) \in A$, we have that $\beta \in A$, since $u - v$ is a unit of A. Then, since $u\beta \in A$, we also have that $\alpha \in A$. It follows from the maximality of $R[\alpha]$ that $A = R[\alpha]$. Therefore, $u\beta = (\alpha + u\beta) - \alpha \in R[\alpha]$. As u is a unit of $R[\alpha]$, we obtain $\beta \in R[\alpha]$, the desired contradiction. □

The next result gives the most important application of Proposition 3.1.

Corollary 3.2. *Let $R \subseteq T$ have FIP. If R contains an infinite field, then $T = R[\alpha]$ for some $\alpha \in T$.*

Next, to prepare for Theorem 3.8, we give several technical results which are of some independent interest.

Proposition 3.3. *(a) If $R \subseteq T$ has FIP and I is an ideal of T, then $R/(I \cap R) \subseteq T/I$ has FIP.*

(b) If $\{T_i\}$ is a family of faithful R-algebras such that $R \subseteq \Pi T_i$ has FIP, then $R \subseteq T_j$ has FIP for each index j.

Proof. (a) As $R/(I \cap R) \cong (R + I)/I$, each ring contained between $R/(I \cap R)$ and T/I takes the form E/I, where E is a uniquely determined ring such that $R + I \subseteq E \subseteq T$. The assertion now follows because $R + I \subseteq T$ inherits FIP from $R \subseteq T$.

(b) Apply (a), with I the kernel of the canonical projection map from ΠT_i onto T_j. □

Part (b) of the next result includes a generalization of Proposition 2.2 (d).

Proposition 3.4. *(a) Let $R \subseteq T = R[\{\alpha_i : i \in I\}]$ be rings such that dim$(R) = 0$. Then T is integral over R if (and only if) each α_i is a root of some polynomial $f_i \in R[X]$ with a unit coefficient.*

(b) If $R \subseteq T$ has FIP and dim$(R) = 0$, then T is integral over R and, in fact, T is a finitely generated R-module.

(c) Let $R \subseteq T$ be rings such that R is finite. Then T is finite if and only if $R \subseteq T$ has FIP.

(d) Let T be a ring. Then T is finite if and only if T has a finite subring R such that $R \subseteq T$ has FIP.

PROOF. (a) For each maximal ideal M of R,

$$R_M \subseteq T_{R \setminus M} = R_M[\{\frac{\alpha_i}{1} : i \in I\}],$$

where $\frac{\alpha_i}{1}$ is a root of the polynomial (with a unit coefficient) in $R_M[X]$ which is induced by f_i. Hence, we may assume without loss of generality that (R, M) is quasilocal, and it suffices to show that for each $i \in I$, $\alpha = \alpha_i$ is integral over R. By hypothesis, α is a root of a polynomial $f = f_i \in R[X]$ having a unit coefficient. In fact, each coefficient of f is either a unit or a nilpotent element, since M is the only prime ideal of R. The constant term of f cannot be the only unit coefficient, lest it be a nilpotent unit. Rewrite the equation $f(\alpha) = 0$ by gathering all the terms with unit coefficients on the left-hand side of the new equation and all the terms with nilpotent coefficients on the right-hand side. (If f has no nilpotent coefficients, then multiplying f by some unit of R produces an integrality polynomial for α, and we are done.) By raising both sides of the new equation to a sufficiently high exponent, we obtain an equation whose right-hand side is 0 and whose left-hand side is therefore a unit multiple of an integrality polynomial for α. This completes the proof.

(b) Combine (a) with Proposition 2.2 (a), (c).

(c) and (d): The "only if" assertions are clear. It suffices to prove the "if" assertion of (c). Since finite rings are zero-dimensional, an application of (b) shows that T is a finitely generated module over the finite ring R, and so T is indeed finite. $\square$

Lemma 3.5. *Let K be an infinite field, and let $K \subseteq T$ be an extension such that T is a reduced ring. Then $K \subseteq T$ has FIP if and only if $T = K[\alpha]$ for some $\alpha \in T$ such that α is algebraic over K.*

PROOF. The "only if" assertion follows by combining Corollary 3.2 and Proposition 2.2 (a). Conversely, suppose that $T = K[\alpha]$, where α is algebraic over K. Since T is a reduced Artinian ring, Wedderburn-Artin Theory (cf. [13, Theorem 3, page 209]) expresses T uniquely as the internal direct product of finitely many fields L_j; that is, $T = L_1 \times \cdots \times L_n$. For each $j = 1, \ldots, n$, the composition $K \hookrightarrow T \to L_j$ allows us to view $K \subseteq L_j$. Now, consider an arbitrary K-subalgebra A of T. Since A is also Artinian, A can be uniquely expressed as the internal direct product of finitely many fields K_i; that is, $A = K_1 \times \cdots \times K_m$. As above, view $K \subseteq K_i$ for each $i = 1, \ldots, m$. Since $T = K[\alpha]$ is generated as a

K-algebra by one element that is integral over K, the same is true of each L_j, as L_j is a homomorphic image of T. In particular, $[L_j : K] < \infty$ and $K \subseteq L_j$ has FIP, by the classical Primitive Element Theorem.

We next claim that for each i, there exists j such that K_i is K-isomorphic to a K-subalgebra of L_j. Suppose that the claim has been established. For each j, let $\{F_{jk}\}$ be the set of K-subalgebras of L_j; this set is finite because $K \subseteq L_j$ has FIP. Note that m is bounded above by $\dim_K(T) < \infty$. Therefore, to prove that $K \subseteq T$ has FIP, the claim reduces our task to showing that for each (j,k), T has only finitely many K-subalgebras isomorphic to the field F_{jk}. Since $[F_{jk} : K] < \infty$, F_{jk} can be generated by finitely many elements as a K-algebra, say $F_{jk} = K[\{a_{jkp}\}]$, where each $a_{jkp} \in L_j$. For each (j,k,p) and each i, the minimum polynomial of a_{jkp} over K has at most finitely many roots in L_i. Hence, for each (j,k), there exist only finitely many K-algebra homomorphisms $F_{jk} \to T$ and, *a fortiori*, only finitely many K-subalgebras of T which are isomorphic to F_{jk}.

To prove the claim, assume without loss of generality that $i = 1$. Consider the inclusion map $\iota : A \hookrightarrow T$. Then $I = \iota(K_1 \times 0 \times \cdots \times 0)$ is nonzero, as ι is an injection. Thus there exists j such that $\pi_j(I)$ is nonzero, where π_j is the canonical projection map $\pi_j : L_1 \times \cdots \times L_n \to L_j$. Hence there is a nonzero map $K_1 \hookrightarrow K_1 \times 0 \times \cdots \times 0 \hookrightarrow T = L_1 \times \cdots \times L_n \to L_j$. Since this map $K_1 \to L_j$ is a nonzero map of fields which preserves multiplication, it sends $1 \in K_1$ to $1 \in L_j$; as it also preserves addition, this map therefore must be an injection. It is easy to check that this map is a K-algebra homomorphism, and so K_1 is indeed isomorphic to a K-subalgebra of L_j. This establishes the claim and completes the proof. $\square$

Lemma 3.6. *Let K be a field and $K \subseteq T$ a ring extension.*

(a) If K is infinite and $K \subseteq T$ has FIP, then T does not contain any nilpotent elements with (nilpotency) index greater than 3.

(b) Suppose that K is infinite and that $T = K[\alpha]$, where α is a nilpotent element of T. Then $K \subseteq T$ has FIP if and only if $\alpha^3 = 0$.

(c) If K is infinite and $K \subseteq T$ has FIP, then T does not contain two nilpotent elements of index 2 which are linearly independent over K.

(d) Suppose that $T = K[\beta] = K[\alpha] \times F$, where β is algebraic over K, $\alpha \in T$ satisfies $\alpha^3 = 0$, and $K \subseteq F$ is a field extension having FIP. Then $K \subseteq T$ has FIP.

(e) Suppose that $T = K[\beta] = K[\alpha] \times K \times \cdots \times K$, where β is algebraic over K, $\alpha \in T$ satisfies $\alpha^3 = 0$, and there are only finitely many factors of K. Then $K \subseteq T$ has FIP.

PROOF. (a) If the assertion fails, T contains a nilpotent element u of index $n \geq 4$. Then $\{1, u, u^2, \ldots, u^{n-1}\}$ is a K-vector space basis of $K[u]$. It is straightforward to check that if $k \in K$, then $B_k = \{a + bu^{n-2} + kbu^{n-1} : a, b \in K\}$ is a K-subalgebra of $K[u]$. Also, $B_{k_1} \neq B_{k_2}$ for $k_1 \neq k_2$, since $u^{n-2} + k_1 u^{n-1} \in B_{k_1} \setminus B_{k_2}$. Since K is infinite, $\{B_k : k \in K\}$ is an infinite collection of intermediate rings between K and T, contradicting that $K \subseteq T$ has FIP.

(b) The "only if" assertion follows from (a). Conversely, suppose that $T = K[\alpha]$ and $\alpha^3 = 0$. If α is nilpotent of index 3, a routine calculation shows that the only ring strictly between K and T is $K[\alpha^2]$. (This conclusion also follows from the fact that the set of such rings is linearly ordered by inclusion [6, Proposition 3.5].) On the other hand, if α has index 2, then there are no intermediate rings strictly between K and T. In either case, it is evident that $K \subseteq T$ has FIP.

(c) If not, then T contains nilpotent elements u and v of index 2 which are linearly independent over K. We consider two cases. First, suppose that $uv = 0$. One easily checks that $\{1, u, v\}$ is a K-vector space basis of $K[u, v]$. Then, by an argument similar to that in (a), it follows that, as k runs through K, $C_k = \{a + bu + kbv : a, b \in K\}$ describes an infinite family of rings, contradicting that $K \subseteq T$ has FIP. In the remaining case, $uv \neq 0$. Then $\{1, u, v, uv\}$ is a K-vector space basis of $K[u, v]$. Another routine calculation show that as k runs through K, $D_k = \{a + bu + kbuv : a, b \in K\}$ describes an infinite family of rings, contradicting that $K \subseteq T$ has FIP.

(d) Since β is algebraic over K, $\dim_K(T) < \infty$, and so the assertion is clear if K is finite. Assume henceforth that K is infinite. By Lemma 3.5, we may also suppose that $\alpha \neq 0$. By the proof of (b), the only K-subalgebras of $K[\alpha]$ are $A_1 = K$, $A_2 = K[\alpha^2]$, and $A_3 = K[\alpha]$. (Obviously, if $\alpha^2 = 0$, then this list is redundant.) For each $i = 1, 2, 3$, observe that $B_i = \{(a + b\alpha + c\alpha^2, a) : a, b, c \in K$ and $a + b\alpha + c\alpha^2 \in A_i\}$ is a K-subalgebra of $K[\alpha] \times F$. Note that $B_1 = \triangle = \{(a, a) : a \in K\} \cong K$. Moreover, as E_j runs through the finitely many fields between K and F, $A_i \times E_j$ is also a K-subalgebra of $K[\alpha] \times F$. We claim that the B_i and the $A_i \times E_j$ are the only K-subalgebras of $K[\alpha] \times F$.

To address the claim, suppose that $\triangle \subseteq D \subseteq K[\alpha] \times F$, for some ring D which is not one of the B_i listed above. We first prove that D is not contained in any B_i. Suppose, on the contrary, that D is properly contained in B_3. As projection on the first factor gives a K-algebra isomorphism $B_3 \to A_3$, D is identified with one of the proper K-subalgebras of A_3, namely A_1 or A_2. Therefore, D coincides with either B_1 or B_2, contrary to the choice of D.

The choice of D now ensures that D must contain some element of the form $(a+b\alpha+c\alpha^2, f)$, where $a, b, c \in K$ and $f \neq a$. Then D contains $[(a+b\alpha+c\alpha^2, f)-(a, a)]^3 = (0, (f-a)^3)$. For ease of notation, rename this element as $(0, g)$. Of course, $g \neq 0$. Since F is algebraic over K, we have $g^m + \cdots + a_1 g + a_0 = 0$ for some $a_i \in K$ such that $a_0 \neq 0$. As D contains $(0, g)$, it also contains $(0, g)^m + \cdots + a_1(0, g) = (0, g^m + \cdots + a_1 g) = (0, -a_0)$. However, $-a_0$ is a nonzero element of K, and so it follows that $(0, 1) \in D$. Then $(1, 0) = (1, 1) - (0, 1) \in D$, whence D contains $K \times K$ and $D = D(1, 0) + D(0, 1)$. Notice that first projection $T \to K[\alpha]$ induces a K-algebra isomorphism $D(1, 0) \to D_1 = \{u \in K[\alpha] : (u, v) \in D$ for some $v \in F\}$. Similarly, by second projection, $D(0, 1)$ is isomorphic to the K-subalgebra $D_2 = \{v \in F : (u, v) \in D$ for some $u \in K[\alpha]\}$ of F. Since $D_1 = A_i$ for some i and $D_2 = E_j$ for some j, we conclude $D = D_1 \times D_2 = A_i \times E_j$. This establishes the claim and shows that $K \subseteq K[\alpha] \times F$ has FIP.

(e) As in the proof of (d), we may assume that K is infinite and that $\alpha \neq 0$. By (b), we may assume that at least one factor K appears in the description of T. Induct on the number of such copies of K. The induction basis (i.e., $K \subseteq K[\alpha] \times K$ has FIP) was established in (d).

Let $T = K[\alpha] \times K \times \cdots \times K = K[\alpha] \times K^{n-1}$, where there are $n - 1 \geq 2$ copies of K. Our induction hypothesis is that $K \subseteq S = K[\alpha] \times K^{n-2}$ has FIP. Let $A_1, \dots, A_m$ be the finitely many K-subalgebras of S. For each $i = 1, \dots, m$ and $j = 1, \dots, n-1$, observe that $B_{ij} = \{(a_1 + b\alpha + c\alpha^2, a_2, \dots, a_{n-1}, a_n) : a_1, b, c \in K, (a_1 + b\alpha + c\alpha^2, a_2, \dots, a_{n-1}) \in A_i$, and $a_n = a_j\}$ is a K-subalgebra of T. (Note that the B_{ij} need not be pairwise distinct.) As B_{ij} is isomorphic to a K-subalgebra of A_i and $K \subseteq A_i$ has FIP, each B_{ij} has only finitely many K-subalgebras. In addition, $A_i \times K$ is a K-subalgebra of T for each $i = 1, \dots, m$. It therefore suffices to show that the $A_i \times K$ and the K-subalgebras of the B_{ij} are the only K-subalgebras of T.

To this end, suppose that $\triangle = \{(a, \dots, a) : a \in K\} \subseteq D \subseteq T$ for some ring D which is not contained in any of the B_{ij} listed above. Hence, D must contain some element of the form $(a_1 + b\alpha + c\alpha^2, a_2, \dots, a_n)$, where $a_1, b, c \in K$ and $a_n \neq a_1$. By subtracting $(a_1, \dots, a_1)$ and cubing the difference, we have that D contains an element of the form $(0, b_2, \dots, b_n)$ such that $b_n \neq 0$.

We claim that D contains an element of the form $(0, 0, e_3, \dots, e_n)$ such that $e_n \neq 0$. We show this first in case $b_n = b_2$. Then D contains $u = (0, 1, b'_3, \dots, b'_{n-1}, 1)$. Moreover, since D is not contained in any of the rings B_{ij}, D must contain an element of the form $v = (a'_1 + b'\alpha + c'\alpha^2, a'_2, \dots, a'_n)$, where $a'_1, b', c' \in K$ and $a'_n \neq a'_2$. Since K has at least three elements, we may add a suitable element of $\triangle$ to v if necessary in order to further assume that a_2' and a_n' are each nonzero. By multiplying u and v, we produce an element in D of the form $(0, c_2, \dots, c_n)$ such that $c_2, c_n \neq 0$ and $c_n \neq c_2$. It follows that D contains an element $w = (0, 1, d_3, \dots, d_n)$ where $d_n \neq 0, 1$. We now see that D contains $w^2 - w = (0, 0, e_3, \dots, e_n)$, where $e_n = d_n^2 - d_n \neq 0$ since $d_n \neq 0, 1$.

In the remaining case, $b_n \neq b_2$. If $b_2 = 0$, then we are done, by taking $(0, 0, e_3, \dots, e_n) = (0, b_2, b_3, \dots, b_n)$. Hence, we may assume that $b_2 \neq 0$. Then D contains an element of the form $y = (0, 1, f_3, \dots, f_n)$ such that $f_n \neq 0, 1$. Consequently, D contains $y^2 - y = (0, 0, g_3, \dots, g_n)$ such that $g_n \neq 0$. Thus, the claim has been established for all cases.

Repeating the above process, we eventually see that $(0, \dots, 0, 1) \in D$. Hence, D contains $0 \oplus \cdots \oplus 0 \oplus K$. As in the proof of (d), let D_1 and D_2 be the images of D under the projection maps $T \to S$ and $T \to K_n = K$, respectively. Of course, $D \subseteq D_1 \times D_2$; and $D_1 = A_i$ for some i and $D_2 = K$. It suffices to show that $D_1 \times D_2 \subseteq D$. Consider $\delta_1 \in D_1$ and $\delta_2 \in D_2$. Now, $(\delta_1, p) \in D$ for some $p \in K$. Observe that $(0, \delta_2), (0, p) \in 0 \oplus K \subseteq D$. Therefore, D contains $(\delta_1, p) - (0, p) = (\delta_1, 0)$, and so D contains $(\delta_1, 0) + (0, \delta_2) = (\delta_1, \delta_2)$, to complete the proof. □

As usual, it is convenient to let $\overline{L}$ denote the algebraic closure of a field L.

Lemma 3.7. *(a) Let K be a field, $K \subseteq T$ a ring extension, and F a field extension of K. If B_1 and B_2 are K-subalgebras of T which are finite-dimensional K-vector spaces such that $F \otimes_K B_1 = F \otimes_K B_2$ (when viewed canonically inside $F \otimes_K T$), then $B_1 = B_2$. Consequently, if $\overline{K} \subseteq \overline{K} \otimes_K T$ has FIP, then $K \subseteq T$ has FIP.*

(b) If a field extension $K \subseteq L$ has FIP and K is a perfect field, then $\overline{K} \subseteq \overline{K} \otimes_K L$ has FIP.

(c) There exists a field extension $K \subseteq L$ which has FIP although $\overline{K} \subseteq \overline{K} \otimes_K L$ does not have FIP.

Proof. (a) The final assertion follows since the assignment $B \mapsto F \otimes_K B$ would give an injection from the set of K-subalgebras of T to the set of F-subalgebras of $F \otimes_K T$. Now, $F \otimes_K B_1$ and $F \otimes_K B_2$ *can* be canonically viewed as K-subalgebras of $F \otimes_K T$ since F is K-flat. Since $F \otimes_K B_1 = F \otimes_K B_2$, this common algebra also coincides with $F \otimes_K B_1 B_2$, where $B_1 B_2$ denotes the subring of T generated by $B_1 \cup B_2$. It is enough to show that $B_1 = B_1 B_2$ (for then, similarly, we would have that $B_2 = B_1 B_2$, whence $B_1 = B_2$, as asserted). Hence, without loss of generality, we may suppose that $B_1 \subseteq B_2$. By considering the inclusion map $B_1 \hookrightarrow B_2$, we may now conclude that $B_1 = B_2$, since F is a faithfully flat K-module.

(b) Since $K \subseteq L$ has FIP, it follows from Proposition 2.2 (d) that $[L : K] < \infty$. Hence, since K is perfect, L is a separable field extension of K, and so by a standard corollary of the classical Primitive Element Theorem, $L = K(\alpha) = K[\alpha]$ for some $\alpha \in L$. In particular, $L \cong L[X]/(f)$ as K-algebras for some irreducible polynomial $f \in K[X]$. By separability, f factors as $\Pi(X - \beta_i)$ for finitely many pairwise distinct $\beta_i \in \overline{L} = \overline{K}$. Then we have canonical K-algebra isomorphisms

$$\overline{K} \otimes_K L \cong \overline{K} \otimes_K K[X]/(f) \cong \overline{K}[X]/(f) \cong \Pi\overline{K}[X]/(X - \beta_i) \cong \Pi\overline{K}.$$

(The penultimate isomorphism is due to the Chinese Remainder Theorem.) An application of Lemma 3.5 completes the proof.

(c) Consider any prime-power positive integer $q \geq 4$. Put $K = \mathbf{F}_q(X^q)$ and $L = \mathbf{F}_q(X)$. Since $[L : K] < \infty$ and $L = K(X) = K[X]$, it follows from the Primitive Element Theorem that $K \subseteq L$ has FIP. However, as

$$\overline{K} \subseteq \overline{K} \otimes_K L \cong \overline{K} \otimes_K K[Y]/(Y^q - X^q) \cong \overline{K}[Y]/(Y^q - X^q)$$

$$= \overline{K}[Y]/(Y - X)^q = \overline{K}[Y - X]/(Y - X)^q = \overline{K}[Z]/(Z^q),$$

Lemma 3.6 (b) ensures that $\overline{K} \subseteq \overline{K} \otimes_K L$ does not have FIP. □

We now present the titular result.

Theorem 3.8. *Let K be a field. For a ring extension $K \subseteq T$, consider the following four conditions:*

(1) K is finite and T is a finite-dimensional K-vector space;

(2) K is infinite, T is a reduced ring, and $T = K[\alpha]$ for some $\alpha \in T$ which is algebraic over K;

(3) K is infinite and $T = K[\alpha]$ for some $\alpha \in T$ which satisfies $\alpha^3 = 0$;

(4) K is infinite and $T = K[\beta] = K[\alpha] \times K_2 \times \cdots \times K_n$, where β is algebraic over K, $\alpha \in T$ satisfies $\alpha^3 = 0$, and for each $i = 1, \ldots, n$, $K \subseteq K_i$ is a field extension which has FIP.

Then:

(a) If $K \subseteq T$ has FIP, then at least one of conditions (1), (2), (3), (4) holds.

(b) If at least one of conditions (1), (2), or (3) holds, then $K \subseteq T$ has FIP.

(c) If K is a perfect field and condition (4) holds, then $K \subseteq T$ has FIP.

(d) Assume further that K is a perfect field. Then $K \subseteq T$ has FIP if and only if at least one of conditions (1), (2), (3), (4) holds.

PROOF. (a) Assume that $K \subseteq T$ has FIP. If K is finite, then Proposition 2.2 (d) yields condition (1). Thus, we may henceforth assume that K is an infinite field. By Corollary 3.2 and Proposition 2.2 (d), $T = K[\beta]$ for some β which is algebraic over K. If T is reduced, Lemma 3.5 yields condition (2). Thus, we may also assume that T is not reduced. By Wedderburn-Artin Theory, express T uniquely as the internal direct product of finitely many Artinian local rings: $T = A_1 \times \cdots \times A_n$. As in the proof of Lemma 3.5, we may view $K \subseteq A_i$ for each i. Then, by Proposition 3.3 (b), $K \subseteq A_i$ has FIP for each i.

If there exist distinct i, j such that A_i and A_j each fail to be reduced, one obtains a contradiction via Lemma 3.6 (c). Therefore, at least $n-1$ of the A_i are reduced local Artinian K-algebras, hence field extensions of K. As T is not reduced, we may assume that $K[\beta] = T = A_1$ is an Artinian local K-algebra, in which case, it remains only to prove that $T = K[\alpha]$ for some $\alpha \in T$ such that $\alpha^3 = 0$.

Since T is a local zero-dimensional ring, β must be either a unit or a nilpotent. If β is nilpotent, then it must have index 2 or 3 by Lemma 3.6 (a), and hence condition (3) is satisfied. Thus, we may assume that β is a unit. Since T is not reduced, Lemma 3.6 (a) allows us to choose a nilpotent element u of T having index 2. Of course, $\beta u \neq 0$. Moreover, $\{\beta u, u\}$ must be linearly dependent over K, by Lemma 3.6 (c), and so $\beta u = au$ for some $a \in K$. Then $\gamma = \beta - a$ annihilates u, and so γ is a nonunit of T. Since T is local and zero-dimensional, γ must be a nilpotent element, whence $\gamma^3 = 0$, by Lemma 3.6 (a). Then $T = K[\beta] = K[\beta - a] = K[\gamma]$, yielding condition (3), as desired.

(b) If condition (1) is satisfied, then T is finite and $K \subseteq T$ trivially has FIP. If condition (2) holds, apply Lemma 3.5; and if condition (3) is satisfied, invoke Lemma 3.6 (b).

(c) By Lemma 3.7 (a), it suffices to show that $\overline{K} \subseteq \overline{K} \otimes_K T$ has FIP. Note that $\overline{K} \otimes_K T = \overline{K}[1 \otimes \beta]$ and that $1 \otimes \beta$ is algebraic over $\overline{K}$. Moreover,

$$\overline{K} \otimes_K T \cong (\overline{K} \otimes_K K[\alpha]) \times \prod_{j=2}^{n} (\overline{K} \otimes_K K_j).$$

Now, by the above argument, $\overline{K} \otimes_K K[\alpha] = \overline{K}[1 \otimes \alpha]$, and $(1 \otimes \alpha)^3 = 0$. Also, since K is perfect, we may reason as in the proof of Lemma 3.7 (b) to show that for each $j = 2, \ldots, n$, $\overline{K} \otimes_K K_j$ is $\overline{K}$-algebra isomorphic to the product of $[K_j : K]$ copies of $\overline{K}$. In sum,

$$\overline{K} \otimes_K T = \overline{K}[1 \otimes \beta] \cong \overline{K}[1 \otimes \alpha] \times \overline{K}^m,$$

where $m = \sum_{j=2}^{n}[K_j : K]$. Hence, by Lemma 3.6 (e), $\overline{K} \subseteq \overline{K} \otimes_K T$ has FIP, as desired.

(d) Combine (a), (b) and (c). □

Remark. In condition (4) of Theorem 3.8, it is redundant to suppose that T takes the form $K[\beta]$ for some element β which is algebraic over K. For a proof, note first that $K[\alpha] \cong K[X]/(X^i)$ for some $i = 1, 2, 3$. Next, write $K_j = K[\gamma_j] = K(\gamma_j)$. As K is infinite and nonzero polynomials have only finitely many roots in L, the Pigeonhole Principle provides elements $k_2, \dots, k_n \in K$ such that the minimum polynomials of $\gamma_2 + k_2, \dots, \gamma_n + k_n$ over K are pairwise distinct and unequal to X. Observe that $K_j = K[\gamma_j + k_j]$ for each j. Let g_j denote the minimum polynomial of $\gamma_j + k_j$ over K. Since $\gamma_j + k_j \neq 0$ without loss of generality, $g_j \neq X$. Put $h = X^i g_2 \cdots g_n$. By the Chinese Remainder Theorem, $K[X]/(h) \cong K[X]/(X^i) \times K[X]/(g_2) \times \cdots \times K[X]/(g_n) \cong K[\alpha] \times K[\gamma_2] \times \cdots \times K[\gamma_n] = K[\alpha] \times K_2 \times \cdots \times K_n = T$. Therefore, the canonical image in T of the (h)-coset represented by X is a satisfactory β.

As a complement to Corollary 2.7, we next address FIP in some cases where the base ring is a PVD whose integral closure need not be finitely generated.

Corollary 3.9. *(a) Let $R \subseteq T$ be a ring extension such that $M = (R : T)$ is a maximal ideal of R. Assume either that R/M is finite or T/M is a reduced ring. Assume also that $T = R[\alpha]$ for some $\alpha \in T$ such that α is a root of a polynomial in $R[X]$ with a unit coefficient. Then $R \subseteq T$ has FIP.*

(b) Let $k \subseteq F$ be an algebraic field extension. Let $V = F + M$ be a nontrivial valuation domain with maximal ideal M, and put $R = k + M$. Then $R \subseteq R[\alpha]$ has FIP for each $\alpha \in F$.

Proof. (a) It is evident that $R \subseteq T$ has FIP if and only if $R/M \subseteq T/M$ has FIP. If R/M is finite, the assertion follows from Theorem 3.8 (b), since T/M is a finite-dimensional vector space over R/M. If R/M is infinite and T/M is reduced, the assertion also follows from Theorem 3.8 (b).

(b) Put $T = k[\alpha] + M$. As $k(\alpha) + M = T = R[\alpha]$ and $T/M \cong k(\alpha)$ is reduced, an application of (a) completes the proof. □

Remark. (a) The example in Proposition 2.1 (a) shows that one cannot delete the hypothesis that $M = (R : T)$ is a maximal ideal of R in Corollary 3.9 (a). Indeed, with the notation in the proof of Proposition 2.1 (a), take $T = L$, so that $M = 0$ and $T/M = L$ is evidently reduced. Moreover, $\alpha = \beta^{-1}$ is a root of the polynomial $\beta X - 1$. As noted in Proposition 2.1 (a), $R \subseteq T$ does not have FIP. Of course, M is not a maximal ideal of R since R is not a field.

(b) We next record a result which, while outside the PVD context, retains some of the flavor of Corollary 3.9 (b). Let R be a PID with infinitely many prime ideals. By Theorem 2.6, R has infinitely many overrings; that is, $R \subseteq K$ does not have FIP, where K denotes the quotient field of R. However, $R \subseteq R[\frac{1}{a}]$ does have FIP, for each nonzero nonunit $a \in R$. Indeed, consider rings $R \subseteq T \subseteq R[\frac{1}{a}]$. Since R is a Bézout domain, there exists a saturated multiplicatively closed subset S of R such that $T = R_S$. Also, since R is a UFD, S is generated by a set of prime elements of R. In order to prove that there are only finitely many possible T, it suffices to show that if $p \in S$ is a prime element of R, then p divides a in R. We can write $1/p = r/a^n$ for some $r \in R$, $n \geq 1$. Since p divides a^n and p is a prime element, the assertion follows.

We close with a modest contribution to the study of FIP for algebras over non-perfect fields.

Proposition 3.10. *Let K be a field of characteristic $p = 2, 3$. Let $K \subseteq T$ be a ring extension such that $T = K[\alpha]$ for some element α satisfying $\alpha^p \in K$. Then $K \subseteq T$ has FIP.*

PROOF. Since $\overline{K}$ and T are each K-flat, we can identify each of them with their canonical images in $\overline{K} \otimes_K T$. When this is done, α is identified with $1 \otimes \alpha$, and an easy calculation reveals that $\overline{K} \otimes_K T$ is then identified with $\overline{K}[\alpha]$. Accordingly, by Lemma 3.7 (a), it suffices to prove that $\overline{K} \subseteq \overline{K}[\alpha]$ has FIP. In other words, we may assume that K is algebraically closed. Hence, there exists $b \in K$ such that $b^p = \alpha^p$. As $\mathrm{char}(K) = p$, we have that $(b - \alpha)^p = b^p - \alpha^p = 0$. Thus, $(b - \alpha)^3 = 0$. Since $K[b - \alpha] = K[\alpha] = T$, an application of Lemma 3.6 (b) completes the proof. $\square$

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