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CHERN'S ORTHONORMAL FRAME BUNDLE OF A FINSLER SPACE

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Abstract. The definition of Chern's orthonormal frame bundle $O_F(M)$ of a real strongly convex Finsler space (M, F) and of non-linear connections on $O_F(M)$ is given. It is proved that $O_F(M)$ admits a unique torsion-free non-linear connection and that this connection coincides with the non-linear Finsler connection introduced by S. S. Chern. This fact brings to a new interpretation of Chern's connection and to a simplified proof of the following Chern's theorem: the group of isometries $Iso_F(M)$ of a Finsler space is a Lie group of dimension less or equal to $n + \frac{1}{2}n(n-1)$ $(n = \dim M)$.

1. Introduction

Let M be a smooth manifold of dimension n, endowed with a strongly convex real Finsler metric F on $TM \setminus \{0\}$. The *(local) equivalence problem* for (M, F) consists in characterizing which pairs of points $p, q \in M$ admit a (local) isometry $f : M \to M$ such that $f(p) = q$. The solution to this problem has been given by S. S. Chern in [3] (see also [4], [2]). Let us briefly recall it.

Let $U \subset M$ be a neighborhood which admits a system of coordinates. At any point $p \in U$, Chern uses the Finsler metric F in order to select a special family of coframes, which are mapped into the corresponding family of coframes at the point $f(p)$ by a local isometry $f : U \to U$. The collection $O_F(U)$ of all these special coframes constitutes a trivial bundle over U of dimension $n_o = n + \frac{1}{2}n(n-1)$, it is invariant under the induced actions of local isometries and it admits a set of n_o 1-forms $\mathcal{P} = \{\omega^1, \ldots, \omega^{n_o}\}$, which are linearly independent at all points and which verify the following fundamental property: a local diffeomorphism g of $O_F(U)$ leaves invariant the field of coframes $\mathcal{P} = \{\omega^1, \ldots, \omega^{n_o}\}$ if and only if it

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is the diffeomorphism induced by a local isometry of $U \subset M$. We call it *Chern's absolute parallelism on* $O_F(U)$.

It is well known (see [9]) that if $\mathcal{Q}$ is a given field of coframes on a manifold N, there exists a finite family $\mathcal{F}_{\mathcal{Q}}$ of real valued functions, whose level sets (with the exception of a set of measure zero) coincide almost everywhere with the orbits of the local diffeomorphisms which preserve $\mathcal{Q}$. Moreover, by a celebrated theorem by Kobayashi (see [6]), the group of global diffeomorphisms of N which leave $\mathcal{Q}$ invariant is a Lie group acting freely and effectively on N, it has closed orbits and its dimension is less or equal to $\dim N$.

Since the local isometries of (U, F) are in one to one correspondence with the local diffeomorphisms which leave invariant the field of frames $\mathcal{P}$ on $O_F(M)$, we conclude that the corresponding functions in $\mathcal{F}_{\mathcal{P}}$ plus Kobayashi's theorem can be used to obtain a complete solution to the (local) equivalence problem of (M, F).

Note that the functions in $\mathcal{F}_{\mathcal{P}}$ are components of the curvature tensor of Chern's non-linear connection and of its covariant derivatives.

The main purposes of this paper consist in the following two points:

a) to give the definitions of Chern's orthonormal frames and of Chern's absolute parallelism in a 'coordinate-free' language, based on the terminology used in standard reference texts like e.g. [7] or [9];
b) to characterize Chern's absolute parallelism as the absolute parallelism, which is associated to the unique 'torsion-free' horizontal distribution of the bundle of Chern's orthonormal frames.

From the definitions and from the characterization described in b), it follows immediately that whenever F is the Finsler metric determined by a Riemannian metric g, the bundle $O_F(M)$ coincides with the usual orthonormal frame bundle $O_g(M)$ of (M, g) and Chern's absolute parallelism reduces to the absolute parallelism determined by the Levi-Civita connection on $O_g(M)$. Furthermore, the interpretation of Chern's absolute parallelism $\mathcal{P}$ as absolute parallelism associated to an horizontal distribution on $O_F(M)$ gives a new geometric interpretation of the components of the curvature tensor of Chern's non-linear covariant derivation.

Finally, we believe that the approach to Chern's orthonormal frames and Chern's absolute parallelism we offer here, can be used to give simple explanations for the deep analogies and for the main differences between Finslerian and Riemannian geometries.

We now give a description of the contents of the paper.

In §2 we recall some basic facts of real Finsler metrics. In §3 we introduce the concept of R-sphere bundles, which is a class of objects which slightly generalizes the family of strongly convex Finsler metrics. They are defined as follows.

Let $\{x^1, \ldots, x^n\}$ be a system of coordinates on an open set $U \subset M$ and let $\{x^1, \ldots, x^n, y^1, \ldots, y^n\}$ the corresponding system of coordinates on TM. For any $p \in U$, let $\mathbf{h}$ be the covariant 2-tensor on T_pM given by

$$\mathbf{h}_v = \left.\frac{\partial^2 F}{\partial y^i \partial y^j}\right|_{(p,v)} dx^i dx^j$$

for any $0 \neq v \in T_pM$. A *Chern's orthonormal frame at* p is a frame $\{e_0, e_1, \ldots, e_{n-1}\}$ on T_pM such that:

i) $F(e_0) = 1$;

ii) $\mathbf{h}(e_0, e_i) = 0$ for $i = 1, \ldots, n-1$;

iii) $\mathbf{h}(e_i, e_j) = \delta_{ij}$.

By the properties of a non-degenerate Finsler metric, we may restate ii) and iii) using a 'coordinate-free' language as follows:

ii') the vectors e_i, $i = 1, \ldots n-1$, are tangents at e_0 to the 'Finsler sphere' $S_p = \{v \in T_p \ : F(v) = 1\} \subset T_pM$;

iii') the vectors e_i, $i = 1, \ldots n-1$ are orthonormal w.r.t. to the symmetric bilinear form h_{e_0} on $T_{e_0}S_p$, which is induced by the Hessian $\mathbf{h}$ of F.

At this point it is clear that, in order to define Chern's orthonormal frames, one does not actually need a Finsler metric, but only the bundle $SM \subset TM$, given by the 'Finsler spheres' $S_p = \{v \in T_p \ : F(v) = 1\}$, and the symmetric bilinear forms h_v, $v \in SM$, induced by $\mathbf{h}$ on the tangent spaces of the spheres. Note that the 'Finsler spheres' are diffeomorphic to spheres whenever F is strongly convex. For simplicity, we limit our discussion only to strongly convex Finsler metrics, but the general case can be treated analogously and with very few modifications. Note that if F is strongly convex, the family of bilinear forms h_v determines a Riemannian metric on each Finsler sphere S_p.

A sphere bundle $SM \subset TM$, endowed with a smooth family of Riemannian metrics on the spheres, is called *R-sphere bundle*. It is clear that the concepts of Chern's orthonormal frames and of Chern's orthonormal frame bundles are easily defined for any R-sphere bundle SM over a manifold M.

In section §3, we also introduce the concept of non-linear connection on a Chern's orthonormal frame bundle $O(SM)$. It is defined as an horizontal distribution which is invariant by the right action of the maximal subgroup of $GL_n(\mathbb{R})$ acting freely on the fibers of $O(SM)$.

We stress that $O(SM)$ is *almost never a principal subbundle*; this is the case only when the sphere bundle SM coincides with the family of unit vectors in TM w.r.t. to a Riemannian metric g and when the Riemannian metrics on the spheres $S_p \in T_pM$ are the Riemannian metrics induced by g.

Nonetheless, the torsion and curvature forms are naturally defined for any non-linear connection on $O(SM)$.

In §4, we consider the R-sphere bundle S^FM associated to a strongly convex Finsler metric F and we consider the associated Chern's orthonormal frame bundle $O_F(M)$. We prove that $O_F(M)$ admits exactly one torsion-free non-linear connection and that such a non-linear connection is invariant under any local isometry of (M, F) (Theorem 4.2). By Kobayashi's theorem (see Proposition 3.3), this implies immediately the following: *the group $Iso(M, F)$ of isometries of a Finsler space (M, F) of dimension n is a Lie group of dimension less or equal to* $n_o = n + \frac{1}{2}n(n-1)$. However this result is not new and it should be considered as first proved by S. S. Chern, being a trivial consequence also of his solution to the local equivalence problem (see [3], p. 102).

Any non-linear connection on $O_F(M)$ determines an absolute parallelism $\{\omega^A\}$ on $O_F(M)$. In §5 we show that the structural equations of the coframes $\{\omega^A\}$ associated to the torsion-free non-linear connection of $O_F(M)$ coincide with those of Chern's absolute parallelism. Since Chern proved the uniqueness of any absolute parallelism which verifies those structural equations, this implies that Chern's and our parallelism coincide.

We conclude observing that if one uses the horizontal spaces of a non-linear connection on $O_F(M)$ to define a parallel transport, a non-linear covariant derivation on M is automatically given (see formula (3.4)). The identity between Chern's and our absolute parallelism on $O_F(M)$ implies that Chern's Finslerian covariant derivation introduced in [2] coincides with the covariant derivation associated to our torsion-free connection $\mathcal{H}$. For this reason, any discussion of the main properties of our non-linear covariant derivation (like metric compatibility, affine relations between tangent spaces given by parallel transport, comparison with other kinds of Finslerian connections, etc.) is automatically reduced to the analysis of the properties of Chern's non-linear covariant derivation. To avoid redundancy, we refer the interested reader to the literature on this topic (in particular to [2]).

Remark. For what concerns a complex Finsler metric F ([1], [5], [8]), one may introduce the concepts of bundle of Finslerian unitary frames and of holomorphic non-linear connection, which are the exact analogue of Chern's orthonormal frames and of Chern's non-linear connection. These two objects reduce to the usual unitary frame bundle and to Chern's connection of an hermitian manifold, in case F is associated to an hermitian metric. We will discuss them in a forthcoming paper.

2. First definitions and preliminaries

Let M be a smooth manifold of dimension n. We denote by $o : M \to TM$ the zero section of TM and we set $\tilde{T}M \overset{\text{def}}{=} TM \setminus o(M)$.

For any $x \in M$ and $v \in T_xM$, the tangent space $T_v(T_xM)$ is naturally identifiable with T_xM itself, being T_xM a vector space. We will use the symbol

$$\imath_v : T_xM \longrightarrow T_v(T_xM)$$

for the identification map. Also, we will use the symbol $\mathcal{D}$ for the so called *dilatation vector field*, i.e. the vector field in $T(T_xM)$ which is equal to

$$\mathcal{D}_v = \imath_v(v) \ . \tag{2.1}$$

Following D. Bao and S. S. Chern, we assume the following definition for a real Finsler metric ([2], p. 135):

Definition 1. A *(real) Finsler metric* on a manifold M is a continuous function

$$F : TM \longrightarrow \mathbb{R}^+$$

satisfying the following properties:

1) F is smooth on $\tilde{T}M$;
2) $F(v) > 0$, for all $v \in \tilde{T}M$;
3) $F(\lambda v) = |\lambda| F(u)$ for any $u \in \tilde{T}M$ and any $\lambda \in \mathbb{R}$.

The *indicatrix at a point* $x \in M$ is the set of vectors $I(x) = \{v \in T_xM \ : \ F(v) < 1\}$. The boundary

$$S_x = \partial I(x) = \{v \in T_xM \ : \ F(v) = 1\}$$

is called *Finsler (pseudo)-sphere at the point* x and it is a smooth hypersurface of T_xM.

A (local) diffeomorphism $f: M \to N$ between two Finsler spaces (M, F) and (N, F') is called *isometry* if $F'(f_* v) = F(v)$, for any $v \in TM$.

We call the function $G = F^2$ the *squared norm of* F ; F is said to be *associated to a Riemannian metric* g if the squared norm of F coincides with the squared norm of g, i.e. for any $v \in TM$

$$G(v) = F^2(v) = g(v, v) \ .$$

It is quite simple to check that property (3) of Definition 1 implies that on any tangent space T_xM

$$\mathcal{D}(G)|_v \equiv 2G_v \ . \tag{2.2}$$

Other useful properties of a Finsler metric can be expressed by considering the following symmetric forms on the tangent spaces TS_x, $x \in M$. Fix a vector $v \in S_x$ and let X_1, X_2, X_3 be three vector fields in $T(T_xM) \simeq T_xM$, which are tangent to S_x in a neighborhood of v. Then let

$$h_v(X_1, X_2) \overset{\text{def}}{=} X_1(X_2(G))|_v \ , \tag{2.3}$$

$$H_v(X_1, X_2, X_3) \overset{\text{def}}{=} X_1(X_2(X_3(G)))|_v \ . \tag{2.4}$$

Lemma 2.1. *The functions* h_v *and* H_v *depend only on the values of the vector fields* X_i *at the point* v *and they are a quadratic and a cubic form on* T_vS_x, *respectively.*

Proof. Since X_1 and X_2 are tangent to the level set $S_x = \{G(v) = 1\}$, we have that

$$h_v(X_1, X_2) = [X_1, X_2](G)|_v + X_2(X_1(G))|_v = h_v(X_2, X_1) \ .$$

Since $h_v(X_1, X_2)$ depends linearly on $X_1|_v$ and it is symmetric, the claim is trivial for h_v. For what concerns H_v, observe that

$$H_v(X_1, X_2, X_3) = X_1 \left[h(X_2, X_3)\right]|_v$$

and hence it is symmetric in the second and third argument. Moreover

$$H_v(X_1, X_2, X_3) = X_3 \left[h(X_1, X_2)\right]|_v - h_v([X_3, X_1], X_2) - h_v(X_1, [X_3, X_2])$$

and therefore it is symmetric w.r.t. to the first and second argument. Being totally symmetric and being linear with respect to $X_1|_v$ the claim is proved also for H_v. □

Note that the quadratic form h_v coincides with the Hessian of the squared norm G; the cubic form H_v coincides with the restriction of the *Minkowsky potential* to the tangent space T_vS_x (for the definition of the Minkowsky potential, see [2] p. 144). Note that H vanishes everywhere if and only if F is associated to a Riemannian metric g: for the proof of this fact, see [3], [2], [1] or just consider the structural equations in §5 of this paper.

F is called *strongly convex* (resp. *non-degenerate*) if for any $x \in M$, h defines a Riemannian (resp. pseudo-Riemannian) metric on $S_x = \partial I(x)$. Note that F is strongly convex if and only if any Finsler (pseudo)-sphere $S_x = \partial I(x)$ is strongly convex and hence diffeomorphic to a sphere.

In all the following, with the terms 'Finsler metric' we will always mean a strongly convex Finsler metric. However, note that with little additional effort it is possible to generalize everything to the case of a non-degenerate Finsler metric.

We conclude this preliminary section recalling some basic facts on linear connections and linear frame bundles (see e. g. [7]). In the following, $V = \mathbb{R}^n$ and $\{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}\}$ is the standard basis of V. $<,>$ is the Euclidean inner product of V. We will also denote by $W = \mathbb{R}^{n-1}$ the subspace $W = span_{\mathbb{R}} < \varepsilon_1, \ldots, \varepsilon_{n-1} >$.

Recall that a *linear frame at a point* $x \in M$ is a linear isomorphism $u : V \to T_xM$ into a tangent space of M. We will always identify a linear frame u at a point $x \in M$ with the corresponding basis $e_i = u(\varepsilon_i) \in T_xM$; furthermore, for any frame u on T_xM, we set $\pi(u) = x$.

The *linear frame bundle* $L(M)$ is the principal $GL_n(\mathbb{R})$-bundle

$$\pi : L(M) \longrightarrow M$$

of all linear frames on M.

A global section $\sigma : M \to L(M)$ is called *absolute parallelism on* M.

The *tautological 1-form* of $L(M)$ is the V-valued 1-form θ defined as

$$\theta_u(X) = u^{-1}(\pi_*(X)) = \sum_{i=0}^{n-1} \theta^i_u(\pi_*(X))\varepsilon_i \ ,$$

where the θ^i_u are the $\mathbb{R}$-valued 1-forms on T_xM which associate to any vector $v \in T_xM$ its components in the frame $u = \{e_0, e_1, \ldots, e_{n-1}\}$. For any subbundle $P \subseteq L(M)$, the restriction of θ on P is called *tautological 1-form of* P and it will be denoted by the same symbol θ.

For any $A \in \mathfrak{gl}_n(\mathbb{R}) = Lie(GL_n(\mathbb{R}))$, the *associated fundamental vector field* is the vector field A^* on $L(M)$, whose flow is

$$\Phi_t^{A^*}(u) = u \circ \exp(tA) \ .$$

A *(linear) connection on a principal subbundle* $P \subseteq L(M)$ is a distribution $\mathcal{C}$, which is complementary to the vertical distribution and which is invariant under the right action of the structural group of P. If G is the structural group of P and $\mathfrak{g} = Lie(G)$, the *connection form* associated to $\mathcal{H}$ is the unique $\mathfrak{g}$-valued 1-form ω which vanishes on the vectors of $\mathcal{C}$ and such that $\omega(A^*) \equiv A$, for any $A \in \mathfrak{g}$.

A connection $\mathcal{C}$ defines a natural absolute parallelism on P as follows. Let us fix a basis $\{E_1, \dots E_N\}$ for the Lie algebra $\mathfrak{g}$. The $\mathfrak{g}$-valued connection form ω is determined by the real valued forms ϖ^a such that

$$\omega_u = \sum_{i=1}^{N} \varpi_u^a E_a \ .$$

From definitions, it follows that the 1-forms $\{\theta^i, \varpi^a\}$ constitute a coframe at any tangent space T_uP, $u \in P$. The frames, which are dual to the coframes $\{\theta^i, \varpi^a\}$, determine a global section $\sigma^{\mathcal{C}} : P \to L(P)$, which is called *absolute parallelism associated to the connection* $\mathcal{C}$.

3. R-sphere bundles and Chern's orthonormal frame bundles

We call *R-sphere bundle over* M a pair (SM, g), formed by a subbundle $SM \subset TM$, with fibers diffeomorphic to the standard sphere S^{n-1}, together with a smooth family of Riemannian metrics g_x, defined on each sphere $S_x = S_xM \subset T_xM$ and depending smoothly on the point $x \in M$.

The main example of an R-sphere bundle is given by the family of Finsler spheres $SM = \{S_x = \partial I_x\}$ associated to a Finsler metric F and endowed with the Riemannian metrics $g_x = h|_{TS_x}$, where h is the Hessian of the squared norm $G = F^2$.

Two R-sphere bundles (SM, g) and (SM', g') over M and M', respectively, are called *isometric* if there exists a diffeomorphism $f : M \to M'$, such that for any $x \in M$ the differential $\hat{f}_x \overset{\text{def}}{=} df_x : T_xM \to T_xM$ induces an isometry $\hat{f}_x : S_xM \to S_{f(x)}M'$.

In the following, any isometry $\psi : S_x \to S_{x'}$ between two spheres of SM will be called *linearly induced* if there exists a linear map $L : T_xM \to T_{x'}M$, so that $L|_{S_x} = \psi$.

A frame $u = \{e_0, \ldots, e_{n-1}\} \in L(M)$ at a point $x = \pi(u)$ will be called *Chern's orthonormal frame adapted to* (SM, g) if

$$a)\ \ e_0 \in S_x\ ; \qquad b)\ \ \{e_1, \ldots, e_{n-1}\} \text{ is an orthonormal basis for } T_{e_0}S_x$$

(to be precise, here we identified the vectors e_i, $1 \leq i \leq n-1$, with the corresponding vectors $\imath_{e_0}(e_i) \in T_{e_0}(T_xM)$). The set of Chern's frames is denoted by $O_g(SM)$ and it is called *Chern's orthonormal frame bundle of* (SM, g).

Note that $\pi : O_g(SM) \to M$ is a subbundle of $L(M)$, but in general it is not a principal subbundle, i.e. there is no subgroup of $GL_n(\mathbb{R})$ which acts transitively on the fibers of $O_g(SM)$ over M. Nonetheless, it follows from the definitions that the group $O_{n-1}(\mathbb{R}) = O(W)$ acts freely on each fiber $\mathbb{V}_x = \pi^{-1}(x)$.

The fundamental vector fields A^* on $L(M)$, associated with the elements $A \in \mathfrak{gl}_\mathfrak{n}(\mathbb{R})$, span the subbundle of $TL(M)$ of the vectors which are vertical w.r.t. the fibering $\pi : L(M) \to M$. Hence we may identify any subspace $\mathcal{V}_u \overset{\text{def}}{=} T_u^{\text{vert}} O_g(SM)$ with the subspace

$$\mathfrak{g}_u = \{\ A \in \mathfrak{gl}_\mathfrak{n}(\mathbb{R})\ ,\ A^*_u \in \mathcal{V}_u\}\ . \tag{3.1}$$

For the previous remarks, $\mathfrak{g}_u$ contains $\mathfrak{s}o_{n-1}(\mathbb{R})$ for any frame $u \in O_g(SM)$; moreover, we may claim it is a Lie algebra independent on u if and only if $O_g(SM)$ is a principal subbundle.

Note that any fiber $\mathbb{V}_x$ of the bundle $\pi : O_g(SM) \to M$ coincides with the orthonormal frame bundle $O_{g_x}(S_x)$ of the sphere (S_x, g_x). Hence, for any connection $\mathcal{C}_x$ of $\mathbb{V}_x = O_{g_x}(S_x)$, the associated absolute parallelism $\sigma^{\mathcal{C}}$ determines an isomorphism at any $u \in \mathbb{V}_x$

$$\sigma^{\mathcal{C}}_u : W \oplus \mathfrak{so}_{n-1}(\mathbb{R}) \longrightarrow \mathcal{V}_u \simeq T_u\mathbb{V}_x = T_u(O_{g_x}(S_x))\ .$$

In the following, by $\mathcal{C} = \{\mathcal{C}_x\}$ we mean a family of connections on the fibers $\mathbb{V}_x = O_{g_x}(S_x)$, which is invariant under linearly induced isometries between the spheres. We call such family of connections *linearly invariant*.

An immediate example of linearly invariant family is given by the family given by the Levi-Civita connections of the fibers $\mathbb{V}_x$, since the Levi-Civita connection is invariant under *any* isometry; however other linearly invariant families of connections do exist.

For a fixed linearly invariant family of connections $\mathcal{C}$, any vector $X \in W \oplus \mathfrak{so}_{n-1}(\mathbb{R})$ defines a vertical vector field $\tilde{X}$ on $O_g(SM)$ by

$$\tilde{X}_u = \sigma^{\mathcal{C}}_u(X)\ .$$

The field $\tilde{X}$ is called *generalized fundamental vector field associated with* X *w.r.t.* $\mathcal{C}$.

Note that, independently on the choice of $\mathcal{C}$, the vector fields $\tilde{X}$ associated with some element $X \in \mathfrak{so}_{n-1}(\mathbb{R})$ coincide always with the restriction to $O_g(SM)$ of the fundamental vector field X^* of $L(M)$.

For any $u \in O_g(SM)$, we may combine the isomorphism $\sigma_u^{\mathcal{C}} : W \oplus \mathfrak{so}_{n-1}(\mathbb{R}) \to \mathcal{V}_u$ with the natural identification map

$$\tau_u : \mathfrak{g}_u \subset \mathfrak{gl}_{\mathfrak{n}}(\mathbb{R}) \to \mathcal{V}_u \ , \qquad \tau(A) \stackrel{\text{def}}{=} A_u^* \ ,$$

and we get the isomorphism

$$\varphi_u : W \oplus \mathfrak{so}_{n-1}(\mathbb{R}) \to \mathfrak{g}_u \subset \mathfrak{gl}_{\mathfrak{n}}(\mathbb{R}) \qquad ,$$

$$\varphi_u(A) = \tau_u^{-1}(\tilde{A}_u) \ , \tag{3.2}$$

which clearly coincides with the identity map on $\mathfrak{so}_{n-1}(\mathbb{R})$.

The following Proposition 3.1 can be proved by a simple modification of the arguments used for Prop. VI. 3.1 in [7]. In our statement, for any a (local) diffeomorphism $f : M \to M$, we use the symbol $\hat{f}$ for the associated lifted diffeomorphism of $L(M)$. If furthermore $\hat{f}(O_g(SM)) \subset O_g(SM)$, we use the symbol $\hat{f}'$ for the induced map

$$\hat{f}' : SM = O_g(SM)/O_{n-1}(\mathbb{R}) \longrightarrow SM = O_g(SM)/O_{n-1}(\mathbb{R}) \ .$$

Moreover, we denote by

$$\hat{\pi} : O_g(SM) \to SM = O_g(SM)/O_{n-1}(\mathbb{R}) \ , \qquad \pi_o : SM \to M$$

the natural projection maps.

Proposition 3.1. *Let* $f : M \to M$ *be a (local) isometry for* (SM, g) *and* θ *the restriction to* $O_g(SM)$ *of the tautological form of* $L(M)$*. Furthermore, for any* $X \in W \oplus \mathfrak{so}_{n-1}(\mathbb{R})$ *let* $\tilde{X}$ *be the associated generalized fundamental vector field w.r.t. a linearly invariant family of connections* $\mathcal{C}$ *on the fibers* $\mathbb{V}_x$*. Then*

a) $\hat{f}(O_g(SM)) \subset O_g(SM)$*;*

b) for any generalized fundamental vector field $\tilde{X}$, $\hat{f}_*(\tilde{X}) = \tilde{X}$*;*

c) $\hat{f}^*(\theta) = \theta$.

Vice versa, if $\varphi : O_g(SM) \to O_g(SM)$ *is a local diffeomorphism, which verifies b) and c), then there exist two (local) maps* $f : M \to M$ *and* $\tilde{f} : SM \to SM$ *so that:*

d) $f \circ \pi = \pi \circ \varphi$ *and* $\tilde{f} \circ \hat{\pi} = \hat{\pi} \circ \varphi$*;*

e) $\varphi = \hat{f}$*,* f *is a (local) isometry of* (SM, g) *and* $\tilde{f} = \hat{f}'$*.*

We now introduce the concept of non-linear connection.

Definition 2. A *non-linear connection* on $O_g(SM)$ is a distribution $\mathcal{H}$, which verifies the following two properties:

1) $\mathcal{H}$ is complementary to the vertical distribution at all points;
2) for any $h \in O_{n-1}(\mathbb{R})$ and any $u \in O_g(SM)$, $\mathcal{H}_{u\cdot h} = (R_h)_* \mathcal{H}_u$.

The attribute *non-linear* is due to the following fact. If $\mathcal{H}$ is a non-linear connection, for any curve $\gamma : [0,1] \to SM$ and any frame $u \in \hat{\pi}^{-1}(\gamma(0)) \subset O_{g_x}(S_x)$, we may consider the unique lifted curve $\hat{\gamma} : [0,1] \to O_g(SM)$, which projects onto γ, it is always tangent to $\mathcal{H}$ and so that $\hat{\gamma}(0) = u$. Since $\mathcal{H}$ is $O_{n-1}(\mathbb{R})$-invariant, if $u = \{e_0, \ldots, e_{n-1}\} = \hat{\gamma}(0)$ and $u' = \{e'_0, \ldots, e'_{n-1}\} = \hat{\gamma}(1)$, the linear map

$$T_\gamma : T_x M \to T_{x'} M \ , \qquad X = X^i e_i \overset{T_\gamma}{\mapsto} X^i e'_i \tag{3.3}$$

depends only on the curve $\gamma : [0,1] \to SM$ and we call it *parallel transport along* γ.

In case $O_g(SM)$ is a principal subbundle of $L(M)$ and $\mathcal{H}$ is a connection in the usual sense, the parallel transport (3.3) depends only on the projected curve $\gamma_o = \pi_o(\gamma) : [0,1] \to M$. In this case, (3.3) coincides with the parallel transport determined by the (linear) covariant derivation associated to the connection $\mathcal{H}$ (for the definition, see [9]).

In the generic case, (3.3) is the parallel transport associated to the *non-linear covariant derivation* ∇ *of vector fields on* M *along tangent vectors of* SM, which is defined as follows.

Let Y be a local vector field on M defined on a neighborhood of a point x and let $\hat{X} \in T_v SM$, for some $v \in S_x \subset T_x M$. Then we define

$$\nabla_{\hat{X}} Y|_v \overset{\text{def}}{=} u \left[\mathcal{X}(\theta(\mathcal{Y}))|_u - \omega_u(\mathcal{X}) \cdot \theta_u(\mathcal{Y}) \right] \ , \tag{3.4}$$

where u is any Chern's frame belonging to the fiber $\hat{\pi}^{-1}(v)$ over v and $\mathcal{X}$ and $\mathcal{Y}$ are two vector fields on $O_g(SM)$ such that $\hat{\pi}_*(\mathcal{X})_v = \hat{X}$ and $\pi_*(\mathcal{Y}) = Y$. It is simple to check that (3.4) does not depend on the choices of u, $\mathcal{X}$ and $\mathcal{Y}$. Furthermore, for any $Y \in T_x M$ and any curve γ on SM, the parallel transport $T_\gamma(Y)$ defined in (3.3) coincides with the value $Y_{\gamma_o(1)}$ of the unique vector field defined along the curve $\gamma_o = \pi_o(\gamma)$, with vanishing covariant derivative along $X_t = \dot{\gamma}(t)$, for any $t \in [0,1]$.

In analogy with the connections on principal subbundles of $L(M)$, any non-linear connection $\mathcal{H}$ on (SM, g) is defined by a $\mathfrak{g}_u$-valued form ω on $O_g(SM)$, which vanishes on $\mathcal{H}$ and satisfies the condition $\omega(A^*_u) = A$ for any $A \in \mathfrak{g}_u$. Using the identification map (3.2), for any fixed choice of the family of connections $\mathcal{C}$, we may associate to $\mathcal{H}$ the $W \oplus \mathfrak{so}_{n-1}(\mathbb{R})$-valued connection form ϖ

$$\varpi_u = \varphi_u^{-1} \circ \omega_u \ . \tag{3.5}$$

The connection form ϖ is useful to associate $\mathcal{H}$ with an absolute parallelism.

In fact, if $\{E_i\}$ is a basis for $W \oplus \mathfrak{so}_{n-1}(\mathbb{R})$ and if we consider the components ϖ^a, i.e. the real valued forms such that:

$$\varpi = \sum_{a=1}^{N} \varpi^a E_a \ , \quad N = \dim W \oplus \mathfrak{so}_{n-1}(\mathbb{R}) \ ,$$

then the forms $\{\theta^i_u, \varpi^a_u\}$ constitute a coframe at any tangent space $T_u O_g(SM)$. The absolute parallelism given by the frames which are dual to the coframes $\{\theta^i_u, \varpi^a_u\}$ is called *absolute parallelism on $O_g(SM)$ associated with $\mathcal{H}$ and $\mathcal{C}$*. We will denote it by $\sigma^{\mathcal{H},\mathcal{C}}$.

The following Proposition is a direct consequence of Proposition 3.1.

Proposition 3.2. *Let $\mathcal{C}$ be a linearly invariant family of connections on the fibers $\mathbb{V}_x$ of $O_g(SM)$.*

A local diffeomorphism $\varphi : O_g(SM) \to O_g(SM)$ coincides with the lift $\varphi = \hat{f}$ of some isometry f of (SM, g) if and only if it transforms the absolute parallelism $\sigma^{\mathcal{H},\mathcal{C}}$ of some non-linear connection into the absolute parallelism $\sigma^{\mathcal{H}',\mathcal{C}}$ of some other non-linear connection.

Definition 3. A non-linear connection $\mathcal{H}$ on $O_g(SM)$ is called *isometrically invariant* if the absolute parallelism associated with $\mathcal{H}$ and a linearly invariant family of connections $\mathcal{C}$ on the fibers $\mathbb{V}_x$, is invariant under all lifts $\hat{f}$ of the local isometries f of (SM, g).

Remark. Note that if $\mathcal{C}$ and $\mathcal{C}'$ are two linearly invariant families of connections on the fibers $\mathbb{V}_x$, then the absolute parallelism $\sigma^{\mathcal{H},\mathcal{C}}$ is invariant under isometries of (SM, g) if and only if $\sigma^{\mathcal{H},\mathcal{C}'}$ is.

This implies that the definition of isometrically invariant non-linear connection $\mathcal{H}$ is independent on the choice of $\mathcal{C}$.

It should also to be observed that the choice of $\mathcal{C}$ is necessary only to construct an absolute parallelism on $O_g(SM)$. The definition of the non-linear covariant derivation ∇ associated to a non-linear connection $\mathcal{H}$ is independent on the choice of $\mathcal{C}$.

The importance of the concept of isometrically invariant connection is seen in the so called equivalence problem. In fact, by Proposition 3.2 and Kobayashi's Theorem (Theorem I.3.2 in [6]) we may state the following basic proposition.

Proposition 3.3. *If $O_g(SM)$ admits a isometrically invariant non-linear connection $\mathcal{H}$, then the local isometries of (SM, g) are in one to one correspondence with the local diffeomorphisms of $O_g(SM)$ which preserve the absolute parallelism $\sigma^{\mathcal{H},\mathcal{C}}$.*

In particular, the group $Iso_g(SM)$ of all the isometries of (SM, g) is a Lie group of dimension less or equal to

$$\dim O_g(SM) = \dim V + \dim W + \dim \mathfrak{so}_{n-1}(\mathbb{R}) = n + \frac{1}{2}n(n-1) \ .$$

In the next section, we will show that in case (SM, g) is the R-sphere bundle determined by a Finsler metric on M, then there exists an isometrically invariant connection with vanishing torsion (see the definition below). This will immediately imply the isomorphism between the group of isometries of a Finsler space and the group of automorphisms of an absolute parallelism on Chern's orthonormal frame bundle.

We conclude this section by defining the torsion of a non-linear connection and stating some basic properties. In analogy with the torsion of a (linear) connection (see [9], [6]), we call *torsion at the frame $u \in O_g(SM)$ of the non-linear connection $\mathcal{H}$* the linear map

$$c_u : \Lambda^2 V \longrightarrow V \ ,$$
$$c_u(v_1, v_2) \stackrel{\text{def}}{=} d\theta_u(v_1^H, v_2^H) \ , \tag{3.6}$$

where v_i^H is the unique vector in $\mathcal{H}_u$ which projects onto $u(v_i) \in T_{\pi(u)}M$.

Note that in case $O_g(SM)$ is a principal subbundle of $L(M)$ and $\mathcal{H}$ is invariant under the right action of the full structural group, then c_u depends only on the point $x = \pi(u) \in M$, as it should be. In the generic case, c_u depends on the point $v = \hat{\pi}(u) \in S_x \subset T_xM$.

If ω and ω' are the $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$-valued connection forms of two non-linear connections $\mathcal{H}$ and $\mathcal{H}'$, it is quite simple to check that, for any $u \in O_g(SM)$, there exists a unique linear map $D_u \in \mathrm{Hom}(V, \mathfrak{g}_u)$ such that

$$\omega'_u = \omega_u + D_u \circ \theta_u \ . \tag{3.7}$$

The same arguments used in [9], p. 316 (see also [6], §I.5) bring to the following useful lemma.

Lemma 3.4. *Let $u \in O_g(SM)$ and c_u, c'_u the torsions at u of the non-linear connections $\mathcal{H}$ and $\mathcal{H}'$, respectively. If D_u is the linear map defined in (3.7) then for any $v, w \in V$*

$$c'_u(v,w) - c_u(v,w) = \partial D_u(v,w) \overset{def}{=} D_u(v) \cdot w - D_u(w) \cdot v \ . \tag{3.8}$$

4. Chern's orthonormal frame bundle and the torsion-free connection of a Finsler space

Let (M, F) be a Finsler space and $S_x = S_xM \subset T_xM$ the corresponding Finsler spheres. We will always denote by $S^F M = \bigcup_{x\in M} S_x$ and by h the restriction on the Finsler spheres of the Hessian of the squared norm $G = F^2$ (see (2.3)). By the results in §2, we have:

Proposition 4.1. *If (M, F) is a Finsler space, the pair $(S^F M, h)$ is an R-sphere bundle.*

We call $(S^F M, h)$ the *R-sphere bundle of the Finsler space* (M, F). The corresponding Chern's orthonormal frame bundle $O_F(M) \overset{\text{def}}{=} O_h(S^F M)$ is called *Chern's orthonormal frame bundle of* (M, F) . It is clear that if (M, F) and (N, F') are two Finsler manifolds, a (local) diffeomorphism $f : M \to N$ is an isometry if and only if it is an isometry for the two associated R-sphere bundles. Therefore, by Propositions 3.1 and 3.2, if we can show the existence of an isometrically invariant connection on any Chern's orthonormal frame bundle, the local equivalence problem between M and N is reduced to the equivalence problem between two associated isometrically invariant parallelisms on $O_F(M)$ and $O_{F'}(N)$.

Theorem 4.2. *Any Chern's orthonormal frame bundle $O_F(M)$ admits a unique torsion-free non-linear connection.*

This torsion-free non-linear connection is isometrically invariant.

PROOF. By definition of torsion, a torsion-free non-linear connection is mapped into another torsion-free non-linear connection by the lift of any isometry of (M, F). Therefore if there exists a unique torsion-free non-linear connection, this connection is isometrically invariant.

The proof of existence and uniqueness of a torsion-free non-linear connection is merely a modification of the proof given in [9] for the existence and uniqueness of a torsion-free connection on the orthonormal frame bundle of a Riemannian manifold (Theorem 3.1 in [9]). The main point consists in showing that, for any Chern's frame $u \in O_F(M)$, the operator ∂ defined in (3.8) determines a linear map

$$\partial : \mathrm{Hom}(V, \mathfrak{g}_u) \longrightarrow \mathrm{Hom}(\Lambda^2 V, V) \tag{4.1}$$

which is *injective*. In fact, since $\dim \mathrm{Hom}(V, \mathfrak{g}_u) = \dim \mathrm{Hom}(\Lambda^2 V, V)$, we would then conclude that ∂ induces an isomorphism. By Lemma 3.4, this would imply that for any non-linear connection on $O_F(M)$ with connection form ω, there exists a unique homomorphism D_u at any frame u so that the new connection form $\omega'_u = \omega_u + D_u \circ \theta_u$ corresponds to a torsion-free connection. Since standard arguments guarantee that $O_F(M)$ admits at least one non-linear connection, the previous remarks would give a proof of the existence and uniqueness of the torsion-free connection.

Let us first compute the subspace $\mathfrak{g}_u$ at a fixed frame u. It amounts to evaluate the elements in $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$, whose associated fundamental vector fields span the vertical tangent space $\mathcal{V}_u \subset T_u O_F(M)$.

Let $\alpha : (-1, 1) \to O_F(M)$ be a curve such that $\alpha_0 = u = \{e_0, \ldots, e_{n-1}\}$ and $\alpha_t = \{e_0(t), \ldots, e_{n-1}(t)\}$ is a frame at $T_x M$ for any t. Let us also use the notation $E_i(t) = \imath_{e_0(t)}(e_i(t)) \in T_{e_0(t)}(T_x M)$. Finally, let us denote by $A^i_j \in \mathfrak{gl}_\mathfrak{n}(\mathbb{R})$ the element such that

$$\dot{\alpha}_0(\varepsilon_j) = u(A^i_j \varepsilon_i) = A^i_j e_i \ .$$

Since $\alpha_t \in O_F(M)$ for all t, the following equations are identically verified ($1 \leq a, b \leq n-1$):

$$G(e_0(t)) \equiv 1 \qquad , \qquad E_a(t)\,(G)\big|_{e_0(t)} \equiv 0 \ , \tag{4.2}$$

$$E_a(t)\,[E_b(t)\,(G)]\big|_{e_0(t)} = h_{e_0(t)}(E_a(t), E_b(t)) \equiv \delta_{ab} \ . \tag{4.3}$$

Now, set $E_i \overset{\mathrm{def}}{=} E_i(0)$. If we differentiate the identities (4.2) and (4.3) at $t = 0$, we easily obtain

$$A^i_0 E_i\,(G)\big|_{e_0} = A^0_0 E_0(G)\big|_{e_0} = A^0_0 \mathcal{D}(G)\big|_{e_0} = 2A^0_0 = 0 \ , \tag{4.2$'$}$$

$$A^i_a E_i\,(G)\big|_{e_0} + A^i_0 E_a\,[E_i\,(G)]\big|_{e_0} =$$

$$= A_a^0 E_0(G)\big|_{e_0} + A_a^b E_b(G)\big|_{e_0} + A_0^0 E_a[E_0(G)]\big|_{e_0} + A_0^b E_a[E_b(G)]\big|_{e_0} =$$

$$= A_a^0 + A_0^a = 0 \ , \qquad (4.2'')$$

$$A_a^i E_i[E_b(G)]\big|_{e_0} + A_b^i E_a[E_i(G)]\big|_{e_0} + A_0^i E_a\{E_b[E_i(G)]\}\big|_{e_0} =$$

$$= A_a^c \delta_{cb} + A_b^c \delta_{ca} + A_0^c H_{abc} + A_0^0 \delta_{ab} =$$

$$= A_a^b + A_b^a + A_0^c H_{abc} \ , \qquad (4.3')$$

where $H_{abc} \stackrel{\text{def}}{=} H_{e_0}(E_a, E_b, E_c) = E_a\{E_b[E_c(G)]\}|_{e_0}$.

From (4.2'), (4.2") and (4.3'), we may characterize $\mathfrak{g}_u$ as follows. For any u let H_u be the trilinear totally symmetric form on V:

$$\hat{H}_u(\varepsilon_i, \varepsilon_j, \varepsilon_k) \stackrel{\text{def}}{=} \begin{cases} H_{e_0}(E_i, E_j, E_k) & i \neq 0, j \neq 0, k \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (4.4)$$

(we recall that $e_0 = u(\varepsilon_0)$ and $E_a = \imath_{e_0}(u(\varepsilon_a))$). Then $\mathfrak{g}_u$ is the subspace of all elements A which verify

$$\boxed{< A \cdot v, w > + < v, A \cdot w > + \hat{H}_u(v, w, A \cdot \varepsilon_0) = 0} \qquad (4.5)$$

Now, in order to conclude, we just have to show that if $D \in \mathrm{Hom}(V, \mathfrak{g}_u)$ and $\partial D = 0$, then $D \equiv 0$.

Using (4.5), by the vanishing of $\hat{H}_u$ whenever one of its arguments equals ε_0, and by the condition $\partial D(v, w) = D(v) \cdot w - D(w) \cdot v = 0$, we have that the following equalities hold:

$$< D(\varepsilon_0) \cdot \varepsilon_0, \varepsilon_j >= - < D(\varepsilon_0) \cdot \varepsilon_j, \varepsilon_0 >= - < D(\varepsilon_j) \cdot \varepsilon_0, \varepsilon_0 >= 0 \ .$$

This implies that $D(\varepsilon_0) \cdot \varepsilon_0 \equiv 0$ and hence

$$< D(\varepsilon_0) \cdot \varepsilon_j, \varepsilon_k >= - < D(\varepsilon_0) \cdot \varepsilon_k, \varepsilon_j >= - < D(\varepsilon_k) \cdot \varepsilon_0, \varepsilon_j >=< D(\varepsilon_k) \cdot \varepsilon_j, \varepsilon_0 >=$$

$$=< D(\varepsilon_j) \cdot \varepsilon_k, \varepsilon_0 >= - < D(\varepsilon_j) \cdot \varepsilon_0, \varepsilon_k >= - < D(\varepsilon_0) \cdot \varepsilon_j, \varepsilon_k >= 0 \ .$$

From this we get that $D(v) \cdot \varepsilon_0 = D(\varepsilon_0) \cdot v = 0$ for any $v \in V$.

Therefore, by (4.5) and the symmetry $D(v) \cdot w = D(w) \cdot v$, we have for any v, w, z, $< D(v) \cdot w, z > + < D(z) \cdot v, w >= 0$. This fact brings to the conclusion by the classical argument of Cartan's lemma:

$$< D(v) \cdot w, z >= - < D(z) \cdot v, w >=< D(w) \cdot z, v >= - < D(v) \cdot w, z >= 0$$

that is $D \equiv 0$. □

5. The structural equations of the torsion-free non-linear connection on $O_F(M)$

In all the following, $\mathcal{H}$ denotes the torsion-free non-linear connection on $O_F(M)$. We take the Levi-Civita connections as linearly invariant family $\mathcal{C}$ of connections on the fibers $\mathbb{V}_x = O_{h_x}(S_x)$. ω and ϖ are the associated $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$-valued and $W \oplus \mathfrak{so}_{n-1}(\mathbb{R})$-valued connection forms, respectively, and $\sigma = \sigma^{\mathcal{H},\mathcal{C}}$ is the absolute parallelism on $O_F(M)$ determined by $\mathcal{H}$ and $\mathcal{C}$.

Consider now the components θ^i and ω^i_j of θ and ω, i.e. the real valued 1-forms such that

$$\theta = \theta^i \varepsilon_i \ , \qquad \omega = \begin{pmatrix} \omega^0_0 & \omega^1_0 & \dots & \omega^{n-1}_0 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0_{n-1} & \omega^1_{n-1} & \dots & \omega^{n-1}_{n-1} \end{pmatrix} .$$

Clearly, the exterior derivatives $d\theta^i$ and $d\omega^i_j$ can be expressed in terms of the forms θ^i and ω^i_j. In order to find those expressions, we need to find explicitly the generalized fundamental vector fields of $O_F(M)$.

Let us denote by E^i_j the matrix in $\mathfrak{gl}_\mathfrak{n}(\mathbb{R})$ which has a non-trivial entry equal to 1 only in the j-th row and i-th column; let us also assume the convention that indices like a, b, c run just between 1 and $n-1$.

In this way, a basis for $\mathfrak{so}_{n-1}(\mathbb{R})$ is $\{\hat{E}^a_b \overset{\text{def}}{=} E^a_b - E^b_a\}$; the associated generalized fundamental vector fields correspond to the usual $L(M)$ fundamental vector fields $\{(E^a_b - E^b_a)^*\}$ restricted on the subbundle $O_F(M)$.

Using (4.5), we can easily see that the vertical subspace $\mathcal{V}_u$ of a tangent space $T_u O_F(M)$ is spanned by the vectors $(E^a_b - E^b_a)^*_u$ and by

$$\tilde{\varepsilon}_a|_u \overset{\text{def}}{=} (E^a_0 - E^0_a)^*_u - \frac{1}{2} \hat{H}_u(\varepsilon_a, \varepsilon_b, \varepsilon_c)(E^b_c + E^c_b)^*_u \ . \tag{5.1}$$

Note that the distribution $\mathcal{C}_u = span < \tilde{\varepsilon}_a|_u >$ is always complementary in $\mathcal{V}_u$ to the subspace which is vertical w.r.t. $\hat{\pi}$; it is also invariant under the right action of $O_{n-1}(\mathbb{R})$ and we have that

$$\hat{\pi}_*([\tilde{\varepsilon}_a, \tilde{\varepsilon}_b]) \equiv 0 \tag{5.2}$$

for any pair of the vector fields $\tilde{\varepsilon}_a$, $\tilde{\varepsilon}_b$ (note: the checking of (5.2) is quite long, but straightforward). This shows that $\mathcal{C}$ represents the Levi-Civita connection on each fiber $\mathbb{V}_x = \pi^{-1}(x)$ of $O_F(M)$. Furthermore, by checking the action of $O_{n-1}(\mathbb{R})$ on the fields $\tilde{\varepsilon}_a$, it can be realized that at any point u they are mapped by the tautological form of $\mathbb{V}_x$ into the same vectors in W; hence they are the generalized fundamental vector fields associated to a basis $\{\varepsilon_a\}$ for W.

Evaluating θ^i, ω^i_j, $d\theta^i$ and $d\omega^i_j$ on the generalized fundamental vector fields and on the horizontal vector fields $e^H_i \stackrel{\text{def}}{=} \sigma(\varepsilon_i)$, $\varepsilon_i \in V$, the following identities are determined:

$$\omega^i_j + \omega^j_i + \frac{1}{2}\hat{H}_u(\varepsilon_i, \varepsilon_j, \varepsilon_k)(\omega^k_0 - \omega^0_k) = \omega^i_j + \omega^j_i - \hat{H}_u(\varepsilon_i, \varepsilon_j, \varepsilon_k)\omega^0_k = 0 \ , \tag{5.3}$$

$$d\theta^i = -\omega^i_j \wedge \theta^j \ , \tag{5.4}$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \Omega^i_j + \Pi^i_j + \Psi^i_j \ , \tag{5.5}$$

where Ω^i_j, Π^i_j and Ψ^i_j are 2-forms which we may call respectively *horizontal, mixed and vertical parts of the curvature of* $\mathcal{H}$. They are of the form

$$\Omega^i_j = R^i_{jkm}\theta^k \wedge \theta^m \quad , \quad \Pi^i_j = P^i_{jam}\omega^a_0 \wedge \theta^m \quad , \quad \Psi^i_j = V^i_{jab}\omega^a_0 \wedge \omega^b_0$$

for some suitable functions R^i_{jkm}, P^i_{jam} and V^i_{jab}.

Indeed, observe that from $0 = d^2\theta^i = -d\omega^i_j \wedge \theta^j + \omega^i_k \wedge \omega^k_j \wedge \theta^j$ it follows that

$$R^i_{jkm}\theta^k \wedge \theta^m \wedge \theta^j + P^i_{jam}\omega^a_0 \wedge \theta^m \wedge \theta^j + V^i_{jab}\omega^a_0 \wedge \omega^b_0 \wedge \theta^j \equiv 0 \ .$$

This implies some symmetry properties on R^i_{jkm} and P^i_{jam} and moreover that

$$\Psi^i_j = V^i_{jab}\omega^a_0 \wedge \omega^b_0 \equiv 0 \ . \tag{5.6}$$

In case F is associated to a Riemannian metric g, the functions R^i_{jkm} coincide with the components of the classical Riemann curvature tensor in an orthonormal basis. The functions P^i_{jam} vanish identically when the cubic form H vanishes. We call the identities (5.3) - (5.5) the *structural equations* of the torsion-free non-linear connection.

Observe that the Pfaffian system $\{\theta^i, \omega^i_j\}$ on $O_F(M)$ verifies the same structural equations given in [3] and this amounts to state that it *is one of Chern's Pfaffian systems*. In particular, the explicit expression of the functions P^i_{jam} can be obtained by formula (58) in [3].

Furthermore, since different Chern's Pfaffian systems correspond to different definitions of the forms ω^i_j, it is very likely that each of them may be interpreted as a Pfaffian system corresponding to an absolute parallelism, which is still associated to the torsion-free non-linear connection $\mathcal{H}$, but determined via a different choice of the linearly invariant family $\mathcal{C}$ of connections on the fibers $\mathbb{V}_x$.

We conclude noticing that the previous remarks imply also that *the non-linear covariant derivation defined in (3.4) coincides with the non-linear covariant derivation of D. Bao and S. S. Chern given in* [2]. The interested reader is referred to that paper also for an excellent discussion of the concept of parallel transport, the associated non-linear covariant derivation and how this is related to the equations of a geodesic in a Finsler space.

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