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A STRONG COMPLETENESS CONDITION IN UNIFORM SPACES WITH WELL ORDERED BASES

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Abstract. A natural strengthening of the completeness hypothesis, called $\mathfrak{B}$-completeness, allows us to extend to ω_μ-metric spaces some properties of complete metric spaces which fail to hold if only the completeness condition is assumed. In this way we achieve some results on hyperspace theory, dealing with supercompleteness and with selections of multivalued functions. The notion of $\mathfrak{B}$-completeness is investigated and it is proved that quite important classes of ω_μ-metric spaces have this property.

Introduction

Some properties of complete metric spaces cannot be extended to complete ω_μ-metric spaces. For instance, the classical Michael's theorem on selections [9, Th. 2] does not hold when the range is replaced by a complete ω_μ-metric space (see Ex. 3.1). Moreover, the Hausdorff uniformity on the hyperspace of closed sets of a complete ω_μ-metric space is not necessarily complete [1]. These restraints are due to the fact that, in a complete ω_μ-metric space, a *short* sequence of nested balls may have empty intersection. A strengthening of the completeness condition, called $\mathfrak{B}$-completeness, allows us to obtain results analogous to the metric ones.

In Th. 2.1 we show that the hyperspace of the $\mathfrak{B}$-complete subsets of a complete ω_μ-metric space is complete. In this way we obtain an adjustment of an incorrect statement on supercompleteness [13, Th. 4.4].

In Th. 3.2, we prove the existence of continuous selections for lower semicontinuous multivalued functions with $\mathfrak{B}$-complete values. We use some ideas of

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Michael's theorem; however, transfinite induction works via specific devices which are proper of ω_μ-additive spaces.

A classical example shows that a closed subspace of a $\mathfrak{B}$-complete space is not necessarily $\mathfrak{B}$-complete (Prop. 1.5). On the other hand, Rem. 1 shows that every closed subset is arbitrarily near to a subset which satisfies a condition of $\mathfrak{B}$-completeness.

Important classes of ω_μ-metric spaces, as ω_μ-box product spaces (Ex. 1.4) or function spaces (Section 4), are $\mathfrak{B}$-complete. A previous result shows that $\mathfrak{B}$-complete subspaces are retracts [2]; as a consequence, the role of these subspaces in ω_μ-metric spaces is analogous to that of closed subsets in strongly zero-dimensional metric spaces (Rem. 3).

In Section 4 we prove that any ω_μ-metric space can be isometrically embedded in a $\mathfrak{B}$-complete ω_μ-metric function space. In this way, the completion of a ω_μ-metric space can be obtained without using generalized Baire spaces [10].

1. Telescopic bases

For the sake of simplicity (*e.g.*, to avoid symbols as ${}^{\omega_\mu}2$), the notation ω_μ is replaced by κ. Throughout this paper, we identify every ordinal α with the set of ordinals less than α. Cardinal numbers are identified with initial ordinals; κ always denotes a regular uncountable cardinal.

κ-metric spaces have been introduced in [11] and investigated by several authors. These spaces can be viewed in two ways. The former one uses κ-metrics, taking their values in totally ordered groups of punctual character κ. In the latter one, a κ-metric space is defined as a uniform space which admits a base of uniform coverings $\mathfrak{B} = \{\mathcal{U}_\alpha\}_{\alpha<\kappa}$ which is well ordered (by star-refinement) by κ. These two approaches are uniformly equivalent. We adopt the latter definition, which is more convenient in handling uniform properties.

It is not restrictive to assume that the base $\mathfrak{B}$ consists of clopen partitions [8, p. 133]. Therefore, if $\alpha < \beta$, for every $U \in \mathcal{U}_\alpha$ and $V \in \mathcal{U}_\beta$, then either $U \cap V = \emptyset$ or $V \subseteq U$. A well ordered base of clopen partitions is called a telescopic base.

Obviously, κ-metric spaces are κ-additive uniform spaces, that is, given less than κ-many uniform coverings, they admit a common uniform refinement. As a consequence, κ-metric spaces are topologically κ-additive, that is the intersection of less than κ-many open sets is open. Moreover, they are also ultraparacompact, that is every open covering is refined by a clopen partition [3].

Let $\mathcal{U}$, $\mathcal{V}$ be coverings of a set X. We write $\mathcal{U} \prec \mathcal{V}$ to mean that $\mathcal{U}$ refines $\mathcal{V}$. For every $A \subseteq X$, we denote by $\mathcal{U}(A)$ the star of A in $\mathcal{U}$, that is the union of all elements of $\mathcal{U}$ which meet A.

Let $\mathfrak{B}$ be a fixed telescopic base for the κ-metric space X. For every $\mathcal{U}_\alpha \in \mathfrak{B}$ and $x \in X$, the set $\mathcal{U}_\alpha(x)$, which consists of the unique element of $\mathcal{U}_\alpha$ containing x, is called the α-ball about x (hence *balls* are members of some $\mathcal{U}_\alpha$). Observe that, if two balls have a common point, then one of them must be contained in the other one. For a subset A, $\mathcal{U}_\alpha(A)$ is called the α-neighborhood of A.

Example 1.1. A very standard useful κ-metric space can be obtained by equipping $\kappa + 1$ with the following base of uniform coverings:

$$\mathcal{V}_\alpha = \big\{\{0\},\ \{1\},\ \{2\},\ \dots,\ \{\alpha\},\ [\alpha+1,\kappa]\big\}$$

This space, which is denoted by C_κ, is κ-compact, that is every open covering has a subcovering of power less than κ.

Let $\gamma \leq \kappa$ be a limit ordinal. A nest of balls of length γ is a sequence $\{U_{j_\alpha}\}_{\alpha<\gamma}$, where the map $j : \gamma \mapsto \kappa$ is strictly increasing, $U_{j_\alpha} \in \mathcal{U}_{j_\alpha}$ for every $\alpha < \gamma$ and $U_{j_\alpha} \cap U_{j_\beta} \neq \emptyset$ for every α and β. Therefore the family $\{U_{j_\alpha}\}_{\alpha<\gamma}$ is monotone, that is $\alpha < \beta$ implies $U_{j_\alpha} \supseteq U_{j_\beta}$.

Definition 1. Let Y be a subspace of X. A telescopic base $\mathfrak{B}$ of X is said to have the intersection property on Y provided that, if $\{U_{j(\alpha)}\}$ is a nest of balls which meet Y, then the intersection $\bigcap_\alpha U_{j(\alpha)}$ contains a point of Y. We simply say that $\mathfrak{B}$ has the intersection property if it has the intersection property on X.

It is worth noting that uniform completeness in κ-metric spaces is not relevant with regard to nests of balls of length less than κ. Indeed, it is easy to verify that a κ-metric space is complete iff every nest of balls of length κ has non-empty intersection. The following definition is a uniform variant of the notion of d-completeness introduced in [2].

Definition 2. Let $\mathfrak{B}$ be a telescopic base for the κ-metric space X and let Y be a subspace of X. We say that Y is $\mathfrak{B}$-complete provided that $\mathfrak{B}$ has the intersection property on Y.

It is easy to prove that, if X is $\mathfrak{B}$-complete, then every ball is $\mathfrak{B}$-complete. Notice that the space C_κ of Ex. 1.1 is $\mathfrak{B}$-complete.

The proof of the next claim is straightforward.

Proposition 1.2. *A complete κ-metric space is $\mathfrak{B}$-complete if and only if every maximal nest of balls has length κ.*

The next proposition shows that $\mathfrak{B}$-completeness is somewhat independent from the choice of the base $\mathfrak{B}$.

Proposition 1.3. *Assume that the base $\mathfrak{B}$ has the intersection property and let $\mathfrak{D} = \{\mathcal{V}_\alpha\}_{\alpha<\kappa}$ be another telescopic base (which induces the same uniformity). Then there exists a strictly increasing map ι from κ to κ such that the base $\{\mathcal{V}_{\iota(\alpha)}\}_{\alpha<\kappa}$ has the intersection property.*

Proof. By transfinite induction, it is easy to construct increasing maps ι and λ from κ to κ such that, for every $\alpha < \kappa$:

$$\mathcal{V}_{\iota(\alpha+1)} \prec \mathcal{U}_{\lambda(\alpha)} \prec \mathcal{V}_{\iota(\alpha)}$$

Thus, $\alpha < \beta$ implies $\mathcal{V}_{\iota(\beta)} \prec \mathcal{U}_{\lambda(\alpha)}$.

Let $\left\{V_{\iota(j_\alpha)}\right\}_{\alpha<\gamma}$ be a nest of balls selected in $\{\mathcal{V}_{\iota(\alpha)}\}$. Take the unique element $U_{\lambda(j_\alpha)} \in \mathcal{U}_{\lambda(j_\alpha)}$ such that $V_{\iota(j_{\alpha+1})} \subseteq U_{\lambda(j_\alpha)}$. The sequence $\{U_{\lambda(j_\alpha)}\}$ is a nest of balls of $\mathfrak{B}$. Since the base $\mathfrak{B}$ has the intersection property, the conclusion follows because $U_{\lambda(j_\alpha)} \subseteq V_{\iota(j_\alpha)}$. □

Example 1.4. Let $\{X^\gamma\}_{\gamma<\kappa}$ be a κ-sequence of κ-metric spaces and, for every γ, let $\mathfrak{B}^\gamma = \{\mathcal{U}^\gamma_\alpha : \alpha < \kappa\}$ be a telescopic base of X^γ. The κ-product uniformity on $\prod X^\gamma$ is the coarsest κ-additive uniformity for which all projections are uniformly continuous. This uniformity is κ-metric. A telescopic base $\mathfrak{B}$ consists of the partitions induced by the equivalence relations $\mathcal{R}_\alpha$ defined as follows:

$$x\mathcal{R}_\alpha y \text{ iff } y^\gamma \in \mathcal{U}^\gamma_\alpha(x^\gamma), \quad \forall\gamma < \alpha.$$

It is straightforward to prove that $\prod X^\gamma$ is $\mathfrak{B}$-complete provided that every X^γ is $\mathfrak{B}$-complete.

A space is said to be κ-metrizable if its topology is induced by a κ-metric uniformity. A κ-metrizable space X is said to be b-complete provided that it is $\mathfrak{B}$-complete for some telescopic base $\mathfrak{B}$ which induces the topology of X. Completeness is not equivalent to b-completeness. Indeed, there exists a closed subspace of a $\mathfrak{B}$-complete space which fails to be b-complete. Let ${}^\kappa 2$ be the set of all κ-sequences whose elements are the numbers 0 and 1, equipped with the κ-product uniformity. Let $\mathfrak{B}$ be the base of ${}^\kappa 2$ whose elements are the partitions $\mathcal{U}_\alpha$, where $\mathcal{U}_\alpha$ is the set of equivalence classes of the relation *coinciding on all the coordinates less than* α. As in Ex. 1.4,, ${}^\kappa 2$ is $\mathfrak{B}$-complete. Let F_κ be the closed subspace of ${}^\kappa 2$ consisting of those κ-sequences x for which $\operatorname{supp} x = \{\alpha : x(\alpha) = 1\}$ is finite. The following proposition ensures that F_κ is not b-complete.

Proposition 1.5. *Let $\mathfrak{D} = \{\mathcal{W}_\alpha\}_{\alpha<\kappa}$ be a telescopic family of partitions of clopen subsets of F_κ which induces the topology. Then there exists a countable nest of balls of $\mathfrak{D}$ which has empty intersection.*

Proof. We still denote by $\mathcal{U}_\alpha$ the covering induced on F_κ by the whole covering $\mathcal{U}_\alpha$ of the standard base $\mathfrak{B}$ of ${}^\kappa 2$. We argue by finite induction. Take $\beta_1 < \kappa$, $W_{\beta_1} \in \mathcal{W}_{\beta_1}$ and $x_1 \in W_{\beta_1}$. Choose $\alpha_1 > \max(\operatorname{supp} x_1)$ such that $U_{\alpha_1} = \mathcal{U}_{\alpha_1}(x_1) \subseteq W_{\beta_1}$. Let $x_2 \in U_{\alpha_1}$ such that $x_2(\alpha_1) = 1$. Now, take $\beta_2 > \beta_1$ such that $W_{\beta_2} = \mathcal{W}_{\beta_2}(x_2)$ is contained in U_{α_1}. By proceeding in this way, we get by induction increasing sequences of ordinals β_j, α_j, sequences of *balls* $W_{\beta_j} \in \mathcal{W}_{\beta_j}$, $U_{\alpha_j} \in \mathcal{U}_{\alpha_j}$, and a sequence of points x_j such that:

$$x_j \in U_{\alpha_j}\ , \quad x_{j+1}(\alpha_j) = 1\ , \quad W_{\beta_{j+1}} \subseteq U_{\alpha_j} \subseteq W_{\beta_j}\ .$$

Since the elements of $U_{\alpha_{j+1}}$ take value 1 at α_j, the intersection of these sets cannot contain elements with finite support. Therefore, the intersection of the W_{α_j}'s is empty. □

2. Spaces of $\mathfrak{B}$-complete subsets

Let X be a uniform space and let $H(X)$ be the hyperspace of all non-empty closed subsets of X equipped with the Hausdorff uniformity. If X is κ-metric, then $H(X)$ is κ-metric [13]. A net of non-empty closed subsets of X is said to be hyperconvergent if it converges as a net of points in $H(X)$. A filter $\mathcal{F}$ of non-empty closed subsets of X may be regarded as a net in $H(X)$, where the indices are the elements of $\mathcal{F}$ ordered by reverse inclusion. $\mathcal{F}$ is said to be stable if it is a Cauchy net in $H(X)$. X is said to be supercomplete if $H(X)$ is complete. In [8, p. 29] it is proved that X is supercomplete if and only if every stable filter is hyperconvergent.

In contrast with the metric case, a complete κ-metric space is not necessarily supercomplete [1]. However, if we restrict our attention to the subspace $H_{\mathfrak{B}}(X)$, consisting of all non-empty $\mathfrak{B}$-complete subsets of X, we get a result which is analogous to the metric one ($\mathfrak{B} = \{\mathcal{U}_\alpha\}$ is an assigned telescopic base).

Theorem 2.1. *Let X be a complete κ-metric space. Then $H_{\mathfrak{B}}(X)$ is complete.*

Proof. By [8, p. 29], it is enough to prove that every stable filter $\mathcal{F}$ consisting of $\mathfrak{B}$-complete subsets of X converges in the hyperspace $H_{\mathfrak{B}}(X)$. For every $\alpha < \kappa$, there exists $F_\alpha \in \mathcal{F}$ such that $\mathcal{U}_\alpha(F) \supseteq F_\alpha$ for every $F \in \mathcal{F}$ (hence $\mathcal{U}_\alpha(F) \supseteq \mathcal{U}_\alpha(F_\alpha)$). Thus, if $\alpha < \beta$ we have $\mathcal{U}_\alpha(F_\alpha) \supseteq \mathcal{U}_\beta(F_\alpha) \supseteq F_\beta$.

As a consequence, $\alpha < \beta$ implies:

$$\mathcal{U}_\alpha(F_\alpha) \supseteq \mathcal{U}_\beta(F_\beta) \tag{1}$$

$$\mathcal{U}_\alpha(F_\alpha) = \mathcal{U}_\alpha(F_\beta) \tag{2}$$

By (1), the net of neighborhoods $\{\mathcal{U}_\alpha(F_\alpha)\}$ is nested.

We shall prove that $D = \bigcap_{\alpha<\kappa} \mathcal{U}_\alpha(F_\alpha)$ is a non-empty $\mathfrak{B}$-complete subset of X and that the filter $\mathcal{F}$ hyperconverges to D.

Let $\{U_\alpha\}_{\alpha<\gamma}$ be a nest of balls of length less than κ, such that U_α meets F_α, that is $U_\alpha \subseteq \mathcal{U}_\alpha(F_\alpha)$ (along this proof, every nest of balls will be chosen in this way). By (2), the set $\mathcal{U}_\alpha(F_\gamma)$ is equal to $\mathcal{U}_\alpha(F_\alpha)$, so that every U_α meets F_γ. By $\mathfrak{B}$-completeness of F_γ, there exists a point p of F_γ which belongs to every U_α. Therefore the given nest of balls can be extended by adding the ball $\mathcal{U}_\gamma(p)$. Thus, every maximal nest of balls $\{U_\alpha\}$, with $U_\alpha \subseteq \mathcal{U}_\alpha(F_\alpha)$, must have length κ; completeness ensures that its intersection is a point of X. Therefore D is non-empty and it is easy to prove that it consists exactly of the points obtained in this way.

Notice that an α-ball U meets F_α if and only if it meets D, since there exists a maximal nest of balls which contains U as an element. This implies that $\mathcal{U}_\alpha(D)$ coincides with $\mathcal{U}_\alpha(F_\alpha)$ for every α. Therefore, we conclude that the filter $\mathcal{F}$ hyperconverges to D, because for every $F \in \mathcal{F}$, $F \subseteq F_\alpha$, we have $\mathcal{U}_\alpha(D) = \mathcal{U}_\alpha(F_\alpha) \supseteq F$ and moreover $\mathcal{U}_\alpha(F) \supseteq \mathcal{U}_\alpha(F_\alpha) \supseteq D$.

Finally, since $\mathcal{U}_\alpha(D) = \mathcal{U}_\alpha(F_\alpha)$, every nest of balls which meet D can be enlarged as above to a maximal nest which converges to a point of D. Thus D is $\mathfrak{B}$-complete. □

Remark 1. Suppose that X is $\mathfrak{B}$-complete with respect to the telescopic base $\mathfrak{B} = \{\mathcal{U}_\alpha\}$. For every α, consider the base $\mathfrak{B}_\alpha = \{\mathcal{U}_\beta : \beta \geq \alpha\}$.
If we define $H_\alpha(X) = H_{\mathfrak{B}_\alpha}(X)$, we obtain an increasing κ-sequence of complete subspaces of $H(X)$.

It is easy to prove that every δ-neighborhood $\mathcal{U}_\delta(Z)$ belongs to $H_\delta(X)$. Since, for every non-empty closed subset Z, the net $\mathcal{U}_\delta(Z)$ hyperconverges to Z, we have that the union of the κ-sequence of complete subspaces $H_\delta(X)$ is dense in $H(X)$. Observe that if $X = C_\kappa$, this union coincides with the whole hyperspace; on the contrary, if $X = {}^\kappa 2$ the closed subset F_κ does not belong to any $H_\delta(X)$.

3. Selections

Let X and Y be topological spaces. A function Φ from X to the subsets of Y is said to be lower semicontinuous if $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is open in X for every open $U \subseteq Y$.

In [9], Michael proved that, if Y is a complete metric space and X is ultraparacompact, then every lower semicontinuous map from X to $H(Y)$ has a continuous selection. One may ask if a similar statement holds when Y is a complete κ-metric space and X is a ultraparacompact κ-additive space. The following example supplies a negative answer.

Example 3.1. Let $X = Y = C_\kappa$. Consider the map $\Phi : C_\kappa \mapsto H(C_\kappa)$ so defined:

$$\Phi(\alpha) = \begin{cases} \{\alpha\} & \text{if } \alpha = 0 \text{ or } \alpha = \kappa, \\ [0, \alpha) & \text{otherwise.} \end{cases}$$

It is easy to prove that the map Φ is lower semicontinuous. Let f be a selection of Φ. Then, for $0 < \alpha < \kappa$, we have $f(\alpha) < \alpha$. By Fodor's lemma [5], there exists an unbounded subset S of κ such that f assumes a constant value $\beta < \kappa$ on S. As a consequence, f cannot be continuous on the point κ.

Notice that every set $\Phi(\alpha)$, defined in Ex. 3.1, is $\mathfrak{B}_\alpha$-complete, where $\mathfrak{B}_\alpha$ denotes the telescopic base consisting of the covering $\mathcal{V}_\beta$ of Ex. 1.1, with $\beta \geq \alpha$. Nevertheless, it does not exist a suitable base $\mathfrak{D}$ such that all sets $\Phi(\alpha)$ are $\mathfrak{D}$-complete. In the following theorem we prove that $\mathfrak{B}$-completeness for an assigned telescopic base $\mathfrak{B} = \{\mathcal{U}_\alpha\}$ of the range Y ensures the existence of a continuous selection.

Theorem 3.2. *Let X be a ultraparacompact κ-additive space and let Y be a complete κ-metric space. Then every lower semicontinuous function Φ from X to $H_{\mathfrak{B}}(Y)$ admits a continuous selection.*

The proof of Th. 3.2 requires two technical lemmas. To prove the former one, argue as in [9, Lemma 2].

Lemma 3.3. *If X is ultraparacompact, Y a topological space, Φ a lower semicontinuous function from X to the non-empty subsets of Y and $\mathcal{U}$ an open covering of Y, then there exists a continuous function f such that $f(x) \in \mathcal{U}\big(\Phi(x)\big)$ for every $x \in X$.*

Lemma 3.4. *Suppose that X and Y satisfy the hypotheses of Th. 3.2. Let $\{f_\alpha\}$ be a γ-sequence of continuous functions from X to Y, with $\gamma < \kappa$, such that the set $\bigcap_{\alpha<\gamma} \mathcal{U}_\alpha\big(f_\alpha(x)\big)$ is non-empty for every $x \in X$. Put $\Psi_\gamma(x) = \bigcap_{\alpha<\gamma} \mathcal{U}_\alpha\big(f_\alpha(x)\big)$ and let V be a ball of Y and x_0 a point of X. Then there exists a neighborhood W of x_0 such that one of the following cases occurs:*

a) $\Psi_\gamma(x) \subseteq V \quad \forall x \in W$*;*
b) $\Psi_\gamma(x) \supseteq V \quad \forall x \in W$*;*

c) $\Psi_\gamma(x) \cap V = \emptyset \quad \forall x \in W$.

PROOF. Observe that $\{\mathcal{U}_\alpha(f_\alpha(x))\}_{\alpha<\gamma}$ is a nest of balls, for every $x \in X$. Let $V \in \mathcal{U}_\delta$ and suppose that $V \cap \mathcal{U}_\alpha(f_\alpha(x_0)) \neq \emptyset$ for every $\alpha < \gamma$.

a) Suppose $\delta < \gamma$. Since $\mathcal{U}_\delta(f_\delta(x_0))$ and V are elements of the same partition meeting each other, they coincide. Let W be a neighborhood of x_0 such that $f_\delta(x) \in V$ for every $x \in W$. Then $\mathcal{U}_\delta(f_\delta(x)) = V$, so that $\Psi_\gamma(x) \subseteq V$ for every $x \in W$.

b) Suppose $\delta \geq \gamma$. Since $\mathcal{U}_\delta$ is a refinement of each $\mathcal{U}_\alpha$, for $\alpha < \gamma$, the set V must be contained in every $\mathcal{U}_\alpha(f_\alpha(x_0))$. Then $f_\alpha(x_0) \in \mathcal{U}_\alpha(V)$. Let W_α be a neighborhood of x_0 such that $f_\alpha(x) \in \mathcal{U}_\alpha(V)$ for every $x \in W_\alpha$. By κ-additivity, the set $W = \cap_{\alpha<\gamma} W_\alpha$ is a neighborhood of x_0 and, for every $x \in W$, we have $f_\alpha(x) \in \mathcal{U}_\alpha(V)$. Consequently, $\mathcal{U}_\alpha(f_\alpha(x)) \supseteq V$ for every $\alpha < \gamma$ and $x \in W$.

The last case occurs when $V \cap \mathcal{U}_\beta(f_\beta(x_0)) = \emptyset$ for some $\beta < \gamma$.

c) There exists a neighborhood W of x_0 such that $f_\beta(x) \in \mathcal{U}_\beta(f_\beta(x_0))$ for every $x \in W$. Then $\mathcal{U}_\beta(f_\beta(x))$ is equal to $\mathcal{U}_\beta(f_\beta(x_0))$, for every $x \in W$, and consequently it is disjoint from V.

□

PROOF OF TH. 3.2. We shall construct a κ-sequence of continuous functions f_α such that, for every $x \in X$:

i) $f_\alpha(x) \in \mathcal{U}_\alpha(\Phi(x))$ for every $\alpha < \kappa$;

ii) $f_\alpha(x) \in \mathcal{U}_\beta(f_\beta(x))$ for every $\beta < \alpha < \kappa$.

From ii), it follows that the κ-sequence $(f_\alpha)_{\alpha<\kappa}$ is uniformly Cauchy. Therefore the κ-sequence f_α converges uniformly to a continuous function f. By taking the limit in ii) as α tends to κ, we have $f(x) \in \mathcal{U}_\beta(f_\beta(x))$ for every $\beta < \kappa$. Furthermore, from i) we have that $\mathcal{U}_\beta(f_\beta(x)) \subseteq \mathcal{U}_\beta(\Phi(x))$. As a consequence, $f(x)$ belongs to $\bigcap_{\beta<\kappa} \mathcal{U}_\beta(\Phi(x)) = \Phi(x)$, for every $x \in X$. In this way we have obtained a continuous selection of Φ.

The construction of the required κ-sequence of functions is performed by transfinite induction. The Lemma 3.3 provides a function f_0 satisfying i) for $\alpha = 0$. Suppose we have defined the functions f_α satisfying i) and ii), for every $\alpha < \gamma$. Then $\mathcal{U}_\alpha(f_\alpha(x))$ is a nest of balls meeting $\Phi(x)$. By $\mathfrak{B}$-completeness, the closed subset $\Psi_\gamma(x) = \bigcap_{\alpha<\gamma} \mathcal{U}_\alpha(f_\alpha(x))$ meets $\Phi(x)$. Let $\Phi_\gamma(x) = \Phi(x) \cap \Psi_\gamma(x)$.

To prove that Φ_γ is lower semicontinuous at x_0, take a ball V in Y, meeting $\Phi_\gamma(x_0)$. Since V meets $\Psi_\gamma(x_0)$, by Lemma 3.4 there exists a neighborhood W

of x_0 such that either $\Psi_\gamma(x) \subseteq V$ or $\Psi_\gamma(x) \supseteq V$ for every $x \in W$. Let $W_1 \subseteq W$ be a neighborhood of x_0 such that $\Phi(x) \cap V \neq \emptyset$ for every $x \in W_1$. Then, for every $x \in W_1$, we have that either $\Phi_\gamma(x) \subseteq V$ or $\Phi_\gamma(x) \cap V = \Phi(x) \cap (\Psi_\gamma(x) \cap V) = \Phi(x) \cap V$. The lower semicontinuity follows since, in both cases, $\Phi_\gamma(x) \cap V$ is non-empty for every $x \in W_1$.

By using again the Lemma 3.3, we obtain a continuous function f_γ such that $f_\gamma(x) \in \mathcal{U}_\gamma\big(\Phi_\gamma(x)\big)$.
Thus, being $\Phi_\gamma(x) \subseteq \Phi(x)$, we have:

i) $f_\gamma(x) \in \mathcal{U}_\gamma\big(\Phi(x)\big)$.

Since $\Psi_\gamma(x) \subseteq \mathcal{U}_\alpha\big(f_\alpha(x)\big)$, for every $\alpha < \gamma$, we have:

ii) $f_\gamma(x) \in \mathcal{U}_\gamma\big(\mathcal{U}_\alpha\big(f_\alpha(x)\big)\big) = \mathcal{U}_\alpha\big(f_\alpha(x)\big),\ \forall \alpha < \gamma$.

The required construction is so completed. □

Remark 2. The hypotheses of Th. 3.2 can be weakened by requiring that, for every point $p \in X$ there exist a neighborhood U of p and a (uniformly admissible) telescopic base $\mathfrak{B}_U$ such that $\Phi(x)$ is $\mathfrak{B}_U$-complete, for every $x \in U$. One may easily get the conclusion by refining the covering $\{U\}$ with a clopen partition.

4. Embeddings

The completion of a κ-metric space can be obtained by using a sort of Baire κ-metric space (the κ-product of suitable discrete spaces of big power, see *e.g.* [10]). In this section we construct a quite natural isometric embedding from a κ-metric space X to the complete κ-metric space consisting of all continuous functions from X to C_κ. The idea is similar to the one used to embed a metric space X in $C(X, \mathbf{R})$ [7, p. 271].

Let X be a set and Y be a κ-metric space. Given a partition $\mathcal{U}$ of Y, define the partition $\hat{\mathcal{U}}$ of Y^X by the following equivalence relation: $f \sim g$ iff $f(x) \in \mathcal{U}(g(x))$ for every $x \in X$. The *function space* uniformity on the set Y^X [8, p. 49] is generated by all $\hat{\mathcal{U}}$'s, where $\mathcal{U}$ ranges on a base $\mathfrak{B}$ of uniform partitions of Y: we denote by $\hat{\mathfrak{B}}$ this base of Y^X. In this way, Y^X becomes a κ-metric space.

If X is a topological or uniform space, we denote by $C(X,Y)$ or $U(X,Y)$ the closed subspaces of Y^X consisting of all continuous or uniformly continuous functions, respectively. The following proposition ensures that $\mathfrak{B}$-completeness is inherited by these function spaces.

Proposition 4.1. *If Y is $\mathfrak{B}$-complete, then Y^X is $\hat{\mathfrak{B}}$-complete. In this case, the subspaces $C(X,Y)$ and $U(X,Y)$ are $\hat{\mathfrak{B}}$-complete provided that X is a κ-additive topological or uniform space, respectively.*

Proof. We prove that $C(X,Y)$ is $\hat{\mathfrak{B}}$-complete (the proof for Y^X is obtained by assuming X to be discrete). Obviously, every nest of balls of length κ has non-empty intersection, because this space is complete. Let $\{\hat{\mathcal{U}}_\alpha(f_\alpha)\}$ be a nest of balls of length $\gamma < \kappa$. Let $\mathcal{V}_\alpha$ be the preimage of the covering $\mathcal{U}_\alpha$ by the continuous function f_α; the covering $\mathcal{V}_\alpha$ is a clopen partition. By κ-additivity, there exists a clopen partition $\mathcal{V}$ refining all $\mathcal{V}_\alpha$'s. Therefore, for every $V \in \mathcal{V}$ and for every $\alpha < \gamma$, there exists $U_{\alpha,V} \in \mathcal{U}_\alpha$ such that $\mathcal{U}_\alpha(f_\alpha(x)) = U_{\alpha,V}$, for every $x \in V$. For every V, the family $\{U_{\alpha,V}\}_{\alpha<\gamma}$ if a nest of balls on Y. By choosing $y_V \in \bigcap_{\alpha<\gamma}\{U_{\alpha,V}\}$, the function which takes constant value y_V on V is continuous. The conclusion follows in a similar way for the uniform case. □

Let X and Y be κ-metric spaces with telescopic bases $\mathfrak{A} = \{\mathcal{U}_\alpha\}$ and $\mathfrak{B} = \{\mathcal{V}_\alpha\}$ respectively. A function $f: X \mapsto Y$ is said to be an isometry if $f^{\leftarrow}(\mathcal{V}_\alpha) = \mathcal{U}_\alpha$ for every α. Moreover, f is said to be Lypschitz-continuous if $f^{\leftarrow}(\mathcal{V}_\alpha)$ is refined by $\mathcal{U}_\alpha$ for every α. Obviously, a Lypschitz-continuous function is uniformly continuous.

An interesting space is $U(X, C_\kappa)$, where C_κ is defined in Ex. 1.1.

Theorem 4.2. *Every κ-metric space can be isometrically embedded in the function space $U(X, C_\kappa)$.*

Proof. Let $\mathfrak{A} = \{\mathcal{U}_\alpha\}$ be a telescopic base of X and let $\mathfrak{B} = \{\mathcal{V}_\alpha\}$ be the base of C_κ given in Ex. 1.1. For every $x \in X$, let $x^\star$ be the function from X to C_κ defined by:

$$x^\star(z) = \min\{\alpha < \kappa : \ z \notin \mathcal{U}_\alpha(x)\}$$

where $\min \emptyset = \kappa$. Obviously, $x^\star(z) = z^\star(x)$.

Let U be an element of $\mathcal{U}_\alpha$. If $x \notin U$, then $x^\star$ takes a constant value not greater than α on every point of U. If $x \in U$, then $x^\star(y) \in [\alpha+1, \kappa]$ for every $y \in U$. As a consequence, $\mathcal{U}_\alpha \prec x^{\star\leftarrow}(\mathcal{V}_\alpha)$, so that $x^\star$ is Lypschitz-continuous.

Analogously, $y \in \mathcal{U}_\alpha(x)$ iff for every $z \in X$ there exists $V_z \in \mathcal{V}_\alpha$ containing both $x^\star(z)$ and $y^\star(z)$. Hence the map $x \longrightarrow x^\star$ is an isometry. □

Remark 3. We denote by $C(X)$ the vector space of continuous real-valued functions on X. Notice that, if X is κ-additive, it is not restrictive that $\mathbb{R}$ has the discrete topology. If A is a subset of X, a linear extender is a linear map $\Gamma: C(A) \mapsto C(X)$ such that $\Gamma(f)$ extends f for each f in $C(A)$. In [4], Dugundji proved that, for every closed subspace A of a metric space X, there exists a linear extender Γ such that the range of $\Gamma(f)$ is contained in the convex hull of the range of f. Stares and Vaughan proved that the same property does not hold in κ-metric spaces [12]: the counterexample is the space F_κ of Prop. 1.5. However, if the closed subspace A is $\mathfrak{B}$-complete for some telescopic base $\mathfrak{B}$, then the existence of a linear extender follows from the property that A is a retract of X. This result is given in [2] and its proof requires a property weaker than $\mathfrak{B}$-completeness: namely, that every nest of balls in A of length less than κ has non-empty intersection with A. In this way, we have a generalization of the analogous result in strongly zero-dimensional metric spaces [6].

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