Houston Journal of Mathematics

Volume 25, No. 4, 1999

A NOTE ON αCOZERO-COMPLEMENTED SPACES AND αBOREL SETS

ANTHONY W. HAGER
Communicated by Klaus Kaiser

Abstract. α is a regular cardinal number or the symbol ∞, and X is a compact Hausdorff space. It is shown that (Theorem 2.1) X is αcozero-complemented iff each αBorel set differs from an αcozero set by a meagre set. This immediately yields (Corollary 2.2) X is αdisconnected iff each αBorel set differs from a clopen set by a meagre set. (The cases $\alpha = \infty$ in Theorem 2.1, and $\alpha = \omega_1$ in Corollary 2.2 are important theorems in Boolean Algebra). The Boolean algebra of αBorel sets modulo meagre sets is considered as an extension of the clopen algebra on X (when X is Boolean), and compared with the αcut-completion and the αcompletion. (For $\alpha = \infty$, each of the three is the completion.)

1. Preliminaries

Throughout, α denotes a regular cardinal number or the symbol ∞ (the meaning of which will be clear from context), and topological spaces are compact Hausdorff. These assumptions are not always totally needed, but will simplify the discussion. Our main references will be [1, 3, 6].

In a space X, an *αcozero set* is the union of $< \alpha$ cozero sets, and the collection of all these is denoted $\alpha\operatorname{coz} X$. Since X is compact Hausdorff, the cozero sets are the open F_σ's. If $\operatorname{clop} X$ (the collection of all clopen sets) is a basis (i.e., X is a *Boolean* space), each cozero set is the union of $< \omega_1$ clopen sets, and each αcozero set is the union of $< \alpha$ clopen sets. (One may see [1] about cozero sets.) We note that ∞cozero = open, and $\omega_1$cozero = cozero.

1991 *Mathematics Subject Classification.* Primary 54H05, 06E10.

Key words and phrases. αBorel set, αcozero-complemented space, αdisconnected space, αcomplete Boolean Algebra.

X is called *αcozero-complemented*, abbreviated αcc, if for each $U \in \alpha\text{coz}\, X$ there is $V \in \alpha\text{coz}\, X$ with $U \cap V = \emptyset$ and $U \cup V$ dense in X; here we say that "V αcomplements U". Note that any X is ∞cc: if U is ∞cozero, i.e., open, then $V = X - \bar{U}$. The one point compactification of an infinite discrete space is not $\omega_1$cc.

The family of *αBorel sets*, denoted $\alpha\mathcal{B}\, X$, is the σ-field in $\mathcal{P}X$ (the power set of X) generated by $\alpha\text{coz}\, X$. Note that ∞Borel = Borel and ω_1Borel = Baire.

A subset of X which is the union of $< \omega_1$ nowhere dense sets is called *meagre* (or first category), and the collection of all these is denoted MX, or M. This is a σ-ideal in $\alpha\mathcal{B}\, X$. We recall the Baire Category Theorem in the forms: the only meagre open set is $\emptyset$, and the complement of a meagre set is dense (in a compact Hausdorff space).

Let $\mathcal{A}$ be a Boolean algebra with an ideal $\mathcal{I}$, and let $B, G \in \mathcal{A}$. The symmetric difference $B \triangle G$ is $(B - G) \vee (G - B)$, and $B \triangle G \in \mathcal{I}$ means B and G have the same image in the quotient $\mathcal{A}/\mathcal{I}$.

2. αCCSPACES

Theorem 2.1. *X is αcozero-complemented iff for each αBorel set B there is an αcozero set G with $B \triangle G$ meagre.*

Proof. Suppose the condition obtains. Let $U \in \alpha\text{coz}\, X$, and apply the condition to $X - U$: there is $G \in \alpha\text{coz}\, X$ with $(X - U) \triangle G$ meagre. Then $(X - U) - G = X - (U \cup G)$ is meagre, so $U \cup G$ is dense (by the Baire Category Theorem). And $G - (X - U) = G \cap U$ is meagre, and also open thus empty (by the Baire Category Theorem).

Suppose X is αcc. Let $\mathcal{C} = \{S \subseteq X \mid \exists G \in \alpha\text{coz}\, X \text{ with } S \triangle G \text{ meagre}\}$. Clearly, $\mathcal{C} \supseteq \alpha\text{coz}\, X$. We shall show that $\mathcal{C}$ is a σ-field; it follows that $\mathcal{C} \supseteq \alpha\mathcal{B}\, X$, as desired.

$S \in \mathcal{C}$ implies $X - S \in \mathcal{C}$: if $S \in \mathcal{C}$, there is $G \in \alpha\text{coz}\, X$ with $S \triangle G$ meagre. Note that $S \triangle G = (X - S) \triangle (X - G)$. Choose H αcomplementing G; then $H \triangle (X - G)$ is meagre. Since $(X - S) \triangle (X - G)$, $(X - G) \triangle H$ are both meagre, it follows that $(X - S) \triangle H$ is meagre.

$S_1, S_2, \ldots \in \mathcal{C}$ implies $\bigcup S_n \in \mathcal{C}$: if $S_n \in \mathcal{C}$, there is $G_n \in \alpha\text{coz}\, X$ with $S_n \triangle G_n$ meagre. Then, $\bigcup G_n \in \alpha\text{coz}\, X$ since α is regular (or just sequentially regular), and $(\bigcup S_n) \triangle (\bigcup G_n) \subseteq \bigcup (S_n - G_n)$. The latter is meagre, so the former is meagre also. $\square$

X is called α*disconnected* if $\bar{U}$ is open for each $U \in \alpha\mathrm{coz}\,X$. (So, ∞disconnected means extremely disconnected, and $\omega_1$disconnected means basically disconnected.)

Corollary 2.2. *X is αdisconnected iff for each αBorel set B there is a clopen set C with $B \triangle C$ meagre.*

PROOF. Supposes the condition holds. Let $U \in \alpha\mathrm{coz}\,X$, and apply the condition to U: there is clopen C with $U \triangle C$ meagre. Then $U \subseteq C$ (for if not, $U - C$ is open and nonvoid, contradicting the Baire Category Theorem), and $C \subseteq \bar{U}$ (for if not, $C - \bar{U}\ \ldots$). Since C is closed, $C = \bar{U}$; so $\bar{U}$ is open.

Suppose X is αdisconnected. If $U \in \alpha\mathrm{coz}\,X$, then $X - \bar{U}$ αcomplements U, since $\bar{U}$ is open. So X is αcc, and we can use Theorem 2.1. If $B \in \alpha\mathcal{B}\,X$, there is $G \in \alpha\mathrm{coz}\,X$ with $B \triangle G$ meagre; and $\bar{G}$ is clopen. Since $B \triangle G$ and $G \triangle \bar{G}$ are meagre; so is $B \triangle \bar{G}$. $\square$

We quote Halmos [3, p.101] regarding Theorem 2.1 ($\alpha = \infty$) and Corollary 2.2 ($\alpha = \omega_1$). "The (latter) resembles (the former) in many details, in both statement and proof. It is almost certain that the two results are special cases of a common generalization; it is far from certain whether the formulation and proof of such a generalization would yield any new information or save any time."

We may note, that in both Theorem 2.1 and Corollary 2.2 the implications "$\Rightarrow$" do not require assuming that X is compact Hausdorff, and about α, require only *sequential* regularity; and the implications "$\Leftarrow$" do not require assuming anything about α, and about X, require only that the Baire Category Theorem holds. But, further observations below shall need regularity of α, and, that X be Boolean. So, we abandon such commentary.

We consider the meaning of Corollary 2.2 for Boolean algebras. Given the Boolean space X, we have present two Boolean homomorphisms: the injection $\mathrm{clop}\,X \overset{i}{\hookrightarrow} \alpha\mathcal{B}\,X$ and the surjection $\alpha\mathcal{B}\,X \overset{S}{\twoheadrightarrow} \alpha\mathcal{B}\,X/\alpha M$; and the composite si is an injection, by the Baire Category Theorem. By Corollary 2.2 then X is αdisconnected iff $si : \mathrm{clop}\,X \to \alpha\mathcal{B}\,X/\alpha M$ is an isomorphism (onto). Given a Boolean algebra $\mathcal{A}$, we have the Stone representation of $\mathcal{A}$ as $\mathrm{clop}\,S\mathcal{A}$, and $\mathcal{A}$ is αcomplete (meaning, $\bigvee \mathcal{F}$ exists when $|\mathcal{F}| < \alpha$) iff $S\mathcal{A}$ is αdisconnected [6, 22.4]. Thus,

Corollary 2.3. *The Boolean algebra $\mathcal{A}$ is αcomplete iff and only if $\mathcal{A}$ is isomorphic to $\alpha\mathcal{B}\,S\mathcal{A}/\alpha M$* via *Stone representation and si.*

The case $\alpha = \omega_1$ in Corollary 2.3 implies, or is, the Loomis-Sikorski Theorem, usually stated as: each $\omega_1$complete Boolean algebra is ω_1-representable, i.e., isomorphic to an ω_1-field of sets modulo an ω_1-ideal. The corresponding statement for αcomplete algebras is very false; few α complete algebras are α-representable. See [6, §29]. Whether Corollary 2.3 says anything useful in this regard, I don't know.

We note (Proposition 2.4 below) that αcomplete Boolean algebras *are* α-representable as αFrames: (See [5] for the various definitions, etc.) Given X, $\alpha\mathrm{coz}\, X$ is an αFrame (since α is regular). The codensity congruence on $\alpha\mathrm{coz}\, X$ is: $(U, V) \in \mathit{cdns}$ if $\bar{U} = \bar{V}$. Evidently, X is αdisconnected iff for each αcozero set B there is clopen C with $(B, C) \in \mathit{cdns}$ (cf. Corollary 2.2 above). Since cdns is an αFrame congruence, X/cdns is an αFrame, and the canonical surjection $\alpha\mathrm{coz}\, X \xrightarrow{t} \alpha\mathrm{coz}\, X/\mathit{cdns}$ is an αFrame homomorphism. When X is a Boolean space, we have the lattice injection $\mathrm{clop}\, X \xhookrightarrow{j} \alpha\mathrm{coz}\, X$, the composite tj is also a lattice injection, by the Baire Category Theorem, and tj is onto iff X is αdisconnected (by the remark above). Then, as with Corollary 2.3, we obtain

Proposition 2.4. *The Boolean algebra $\mathcal{A}$ is αcomplete iff $\mathcal{A}$ is lattice-, hence Boolean-, isomorphic to the αFrame $\alpha\mathrm{coz}\, S\mathcal{A}/\mathit{cdns}$* via *Stone representation and tj.*

We note two consequences of αcc, by way of Theorem 2.1, which are exactly similar to the known case of $\alpha = \infty$.

A subalgebra $\mathcal{A}$ of the Boolean algebra $\mathcal{B}$ is α*dense* if, for each $b \in \mathcal{B}$ there is $\mathcal{F} \subseteq \mathcal{A}$ with $|\mathcal{F}| < \alpha$, with $\bigvee \mathcal{F} = b$. ("∞dense" is just called "dense".)

Theorem 2.5.

1. *If X is αcozero-complemented, then $\alpha\mathcal{B}\, X/\alpha M$ is αcomplete.*
2. *For Boolean X, X is αcozero-complemented if and only if* $\mathrm{clop}\, X$ *is αdense in $\alpha\mathcal{B}\, X/\alpha M$ (*via *si).*

Lemma 2.6 (cf. [6, p.75]). *Consider the Boolean surjection $s : \alpha\mathcal{B}\, X \to \alpha\mathcal{B}\, X/\alpha M$. Suppose $\{G_i \mid i \in I\} \subseteq \alpha\mathrm{coz}\, X$, with $|I| < \alpha$. Then, $s(\bigcup_i G_i) = \bigvee_i \{s(G_i)\}$.*

Proof. This uses Banach's Category Theorem [4, pp. 59-85]: In any space Y, if $\{X_i\}_i$ is a family of subsets such that for each i, X_i is open in $\bigcup_i X_i$, and X_i is meagre in Y, then $\bigcup X_i$ is meagre in Y.

To prove the Lemma: First, for each $G_i, G_i - \bigcup_i G_i$ is empty, hence meagre. So $s(\bigcup_i G_i)$ is an upper bound for $\{s(G_i)\}_i$. Second, suppose $s(A)$ is another upper bound: $A \in \alpha\mathcal{B}\,X$, and $G_i - A \in \alpha M$ for each G_i. Since G_i is open, and $G_i - A = (\bigcup_i G_i - A) \cap G_i$, $G_i - A$ is relatively open in $\bigcup_i G_i - A$. Thus, $\bigcup_i G_i - A = \bigcup_i (G_i - A) \in \alpha M$ (by Banach's Theorem). So $s(\bigcup_i G_i) \leq s(A)$. $\square$

Proof of Theorem 2.5.

Point 1. Let $\{B_i \mid i \in I\} \subseteq \alpha\mathcal{B}\,X$, $|I| < \alpha$. For each i, there is $G_i \in \alpha\mathrm{coz}\,X$ with $B_i \triangle G_i \in \alpha M$, by Theorem 2.1. Then, using Lemma 2.6, $s(\bigcup_i G_i) = \bigvee_i s(G_i) = \bigvee_i s(B_i)$, so this last exists.

Point 2. Suppose X is αcc, and $B \in \alpha\mathcal{B}(X)$. Choose $G \in \alpha\mathrm{coz}\,X$ with $B \triangle G \in \alpha M$, by Theorem 2.1. Now $G = \bigcup_i C_i$ for $< \alpha$ clopen sets C_i. We then have, using Lemma 2.6, $s(B) = s(\bigcup_i C_i) = \bigvee_i s(C_i)$ as desired.

Suppose $\mathrm{clop}\,X$ is αdense in $\alpha\mathcal{B}\,X/\alpha M$, and let $U \in \alpha\mathrm{coz}\,X$. By αdensity, $s(X - U) = \bigvee_i s(C_i)$, for some $< \alpha$ clopen sets C_i. Let $V = \bigcup_i C_i$; of course $V \in \alpha\mathrm{coz}\,X$. By Lemma 2.6, $s(V) = \bigvee_i s(C_i)$, so that $s(X - U) = s(V)$, i.e., $(X - U) \triangle V \in \alpha M$. This means V is an α complement for U (using the Baire Category Theorem). $\square$

The converse of Point 1 of Theorem 2.5 fails: An easy counterexample is $X =$ the one-point compactification of a discrete space of cardinal $\geq \alpha$. I don't know what property of X is equivalent to αcompleteness of $\alpha\mathcal{B}\,X/\alpha M$.

3. Comparison with other extensions

$\mathcal{A} \leq \mathcal{B}$ means that $\mathcal{A}$ is a Boolean subalgebra of the Boolean algebra $\mathcal{B}$. The completion of $\mathcal{A}$ is, in its simplest definition, an extension $\mathcal{A} \leq \bar{\mathcal{A}}$, with

1. $\bar{\mathcal{A}}$ complete, and
2. $\mathcal{A}$ dense in $\bar{\mathcal{A}}$. (Such an extension is unique up to isomorphism fixing $\mathcal{A}$.)

We have just seen in Theorem 2.5 that (identifying $\mathcal{A}$ with $\mathrm{clop}\,S\mathcal{A}$, by Stone Representation) $\mathcal{A} \leq \infty\mathcal{B}\,S\mathcal{A}/\infty M$ is a model of $\mathcal{A} \leq \bar{\mathcal{A}}$. (This is standard, and Theorem 2.5 more-or-less replicates the proof in [6, §35]).

The α*completion* of $\mathcal{A}$ (see [6, §35]) is an extension $\mathcal{A} \leq {}^{\alpha}\bar{\mathcal{A}}$, with

1. ${}^{\alpha}\bar{\mathcal{A}}$ αcomplete,
2. $\mathcal{A}$dense in ${}^{\alpha}\bar{\mathcal{A}}$, and
3. the only $\mathcal{C}$ with $\mathcal{A} \leq \mathcal{C} \leq {}^{\alpha}\bar{\mathcal{A}}$ and $\mathcal{C}$ αcomplete is $\mathcal{C} = {}^{\alpha}\bar{\mathcal{A}}$.

Such an extension is unique as before, and it is easily seen that $\bigcap\{\mathcal{C} \mid \mathcal{A} \leq \mathcal{C} \leq \bar{\mathcal{A}}$, $\mathcal{C}$ αcomplete$\}$ is a model ${}^{\infty}\bar{\mathcal{A}}$ of $\bar{\mathcal{A}}$. Of course, ${}^{\infty}\bar{\mathcal{A}} = \bar{\mathcal{A}}$, i.e., for $\alpha = \infty$, Condition 3 can be dropped (and, alternatively, Condition 2 can be dropped [2]).

Another construction of ${}^{\alpha}\bar{\mathcal{A}}$ is by transfinite iteration of the passage $\mathcal{A} \mapsto {}_{\alpha}\mathcal{A}_1$, where ${}_{\alpha}\mathcal{A}_1$ is the Boolean subalgebra of $\bar{\mathcal{A}}$ generated by the set $\mathcal{S}$ of all $\bigvee \mathcal{F}, \mathcal{F} \subseteq \mathcal{A}$ and $|\mathcal{F}| < \alpha$ ("$\bigvee$" meant in $\bar{\mathcal{A}}$).

Theorem 3.1. *Let $\mathcal{A}$ be a Boolean algebra.*

1. $\alpha\mathcal{B}\, S\mathcal{A}/\alpha M$ *is the* ω_1*completion of* ${}_{\alpha}\mathcal{A}_1$.
2. $\alpha\mathcal{B}\, S\mathcal{A}/\alpha M \leq {}^{\alpha}\bar{\mathcal{A}}$.

Proof. To compress notation, let $X \equiv S\mathcal{A}$, and let $\mathcal{A}(\alpha) \equiv \alpha\mathcal{B}\, X/\alpha M$. With some obvious identifications and suppression of notation, we have

$$\begin{array}{ccccc} \mathcal{A} = \operatorname{clop} X & \leq & \mathcal{A}(\alpha) & \leq \infty\mathcal{B}\, X/\infty M = \bar{\mathcal{A}} \\ \downarrow & & \uparrow & \\ \alpha\operatorname{coz} X & \hookrightarrow & \alpha\mathcal{B} & \end{array}$$

Here, $s(\alpha\operatorname{coz} X)$ is exactly the generating set $\mathcal{S}$ for ${}_{\alpha}\mathcal{A}_1$; this uses Lemma 2.6 and the fact that dense embeddings preserve all joins ([6, 23.1]). Since $\alpha\mathcal{B}\, X$ is $\omega_1$complete and αM is an $\omega_1$ideal (σ-ideal) $\mathcal{A}(\alpha)$ is $\omega_1$complete and the quotient homomorphism s preserves all countable joins ([6, 21.1]).

To prove Point 1, it suffices to show that, if $s(\alpha\operatorname{coz} X) \subseteq \mathcal{C} \leq \mathcal{A}(\alpha)$ and $\mathcal{C}$ is $\omega_1$complete, then $\mathcal{C} = \mathcal{A}(\alpha)$. (Since $\mathcal{A} \leq \mathcal{A}(\alpha) \leq \bar{\mathcal{A}}$, $\mathcal{A}$ is dense in $\mathcal{A}(\alpha)$). Given such $\mathcal{C}$, consider $s^{-1}(\mathcal{C}) \subseteq \alpha\mathcal{B}\, X$. We show that $s^{-1}(\mathcal{C})$ is a σ-field; then $s^{-1}(\mathcal{C}) = \alpha\mathcal{B}\, X$, and $\mathcal{C} = \mathcal{A}(\alpha)$. So let $A \in s^{-1}(\mathcal{C})$. Since s is a homomorphism, $s(X - A) = s(A)' \in \mathcal{C}$ ($(\,)'$ being Boolean complement), so $X - A \in s^{-1}(\mathcal{C})$. Now let $A_1, A_2, \cdots \in s^{-1}(\mathcal{C})$. Since s preserves countable joins, $s(\bigcup_n A_n) = \bigvee_n s(A_n)$ (the latter join in $\mathcal{A}(\alpha)$). Since $\mathcal{C}$ is $\omega_1$complete, $\bigvee_n^{\mathcal{C}} s(A_n)$ exists. The dense embedding $\mathcal{C} \leq \mathcal{A}(\alpha)$ preserves all joins, so $\bigvee_n^{\mathcal{C}} s(A_n) = \bigvee_n s(A_n)$. Thus, $\bigvee_n s(A_n) \in \mathcal{C}$, so $\bigcup_n A_n \in s^{-1}(\mathcal{C})$. So $s^{-1}(\mathcal{C})$ is a σ-field, and Point 1 is shown.

To prove Point 2, we want to show that ($\mathcal{A} \leq \mathcal{C} \leq \bar{\mathcal{A}}$ with $\mathcal{C}$ αcomplete implies $\mathcal{A}(\alpha) \leq \mathcal{C}$). Given such $\mathcal{C}$, clearly ${}_{\alpha}\mathcal{A}_1 \subseteq \mathcal{C}$, and the result follows from Point 1. $\square$

Corollary 3.2. *If $\alpha\mathcal{B}\, S\mathcal{A}/\alpha M$ is αcomplete (e.g., if $S\mathcal{A}$ is αcc (Theorem 2.5)) then $\alpha\mathcal{B}\, S\mathcal{A}/\alpha M$ is the αcompletion of $\mathcal{A}$.*

Corollary 3.2 represents an unusual circumstance of ${}^{\alpha}\bar{\mathcal{A}}$ being achieved in a concrete manner (i.e., rather than from without as a large intersection, or from within by transfinite induction, *per* the comments before Theorem 3.1).

On the other hand, there is another "α-generalization" of completeness which is better in this regard. See [2] for the details of what follows.

An αcut in the Boolean algebra $\mathcal{A}$ is a pair $(\mathcal{F}, \mathcal{H})$ of subsets of $\mathcal{A}$ of cardinal $< \alpha$, with $\mathcal{F} \leq \mathcal{H}$ elementwise, and $\bigwedge\{h - f \mid h \in \mathcal{H}, f \in \mathcal{F}\} = 0$. $\mathcal{A}$ is said to be *αcut-complete* if, whenever $(\mathcal{F}, \mathcal{H})$ is an αcut in $\mathcal{A}$, then there is $a \in A$ with $\bigvee \mathcal{F} = a = \bigwedge \mathcal{H}$. Then, for each $\mathcal{A}$, there is $\mathcal{A} \leq \mathcal{A}^\alpha$, with $\mathcal{A}^\alpha$ αcut-complete, and $\mathcal{A}$ αdense in $\mathcal{A}^\alpha$. This $\mathcal{A}^\alpha$ is called the *αcut-completion* of $\mathcal{A}$, is unique up to isomorphism over $\mathcal{A}$, and has this simple description as a subset of the completion $\bar{\mathcal{A}}$:

$\mathcal{A}^\alpha = \{\bigvee \mathcal{F} \mid \text{there is } \mathcal{H} \subseteq \mathcal{A} \text{ for which } (\mathcal{F}, \mathcal{H}) \text{ is an } \alpha\text{cut in } \mathcal{A}\}$. (Clearly, $\mathcal{A}^\infty$ is the completion $\bar{\mathcal{A}}$). Obviously, we have

Proposition 3.3. *For any Boolean algebra $\mathcal{A}$, $\mathcal{A}^\alpha$ is contained in the generating set $\mathcal{S}$ for ${}_\alpha\mathcal{A}_1$, so that $\mathcal{A}^\alpha \leq \alpha\mathcal{B}\, S\mathcal{A}/\alpha M$.*

Observe that $S\mathcal{A}$ is αcc iff each subset of $\mathcal{A}$ of cardinal $< \alpha$ is the left (lower) half of an αcut in $\mathcal{A}$. The following is now obvious or routine.

Proposition 3.4. *For a Boolean algebra $\mathcal{A}$, the following are equivalent.*

1. $\mathcal{A}^\alpha = \mathcal{S}$
2. $\mathcal{A}^\alpha = \alpha\mathcal{B}\, S\mathcal{A}/\alpha M$.
3. *$\mathcal{S}$ is a Boolean algebra (hence an αcomplete algebra, hence $\mathcal{S} = {}^\alpha\bar{\mathcal{A}}$).*
4. *$S\mathcal{A}$ is αcc.*

References

[1] L. Gillman and M. Jerison, *Rings of Continuous Functions* (Van Nostrand, 1960).
[2] A.W. Hager, *α-cut-complete Boolean Algebras*, Algebra Universalis, **39**(1998), 57-70.
[3] P. Halmos, *Lectures on Boolean Algebras* (Van Nostrand, 1963).
[4] K. Kuratowski, *Topology, Vol. I* (Academic Press and Polish Scientific Publishers, 1966).
[5] J.J. Madden, *κ-frames*, J. Pure and Appl. Algebra **70**(1991), 107-127.
[6] R. Sikorski, *Boolean Algebras*, third edition (Springer-Verlag, 1969).

Received February 10, 1998

Mathematics Department, Wesleyan University, Middletown, CT 06457
E-mail address: ahager@wesleyan.edu