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FINITISTIC SPACES AND DIMENSION

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Abstract. We shall consider two dimension-like properties on finitistic spaces. We shall prove that there is a universal space for the class of metrizable finitistic spaces of given weight. This answers [10, Question 2] affirmatively. We shall also prove that a Pasynkov's type of factorization theorem for finitistic spaces.

1. Introduction

The concept of finitistic spaces was introduced by Swan [19] for working in fixed point theory and is applied to the theory of transformation groups by using the cohomological structures (cf. [1]).

Let $\mathcal{U}$ be a family of a space X. By the *order* of $\mathcal{U}$ we mean the largest number n such that $\mathcal{U}$ contains n members with non-empty intersection. The order of $\mathcal{U}$ is denoted by $\mathrm{ord}\,\mathcal{U}$. We say a family $\mathcal{U}$ has *finite order* if $\mathrm{ord}\,\mathcal{U} = n$ for some natural number n.

Definition 1. [19] A space X is said to be *finitistic* if every open cover of X has an open refinement with finite order.

By the definition, it is clear that all compact spaces and all finite dimensional (in the sense of the covering dimension dim) paracompact spaces are finitistic spaces. More precisely, we have the following characterization.

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Proposition 1.1. [7], [10] *A paracompact space X is finitistic if and only if there is a compact subspace K of X such that* $\dim F < \infty$ *for every closed subspace F with $F \cap K = \emptyset$.*

In the dimension theoretic points of view, some properties of finitistic spaces are investigated by several authors ([4], [5], [6], [7] and [10]). We shall further dimension theoretic properties of finitistic spaces.

Let $\mathcal{C}$ be a class of topological spaces. We say a topological space X is a *universal space for the class* $\mathcal{C}$ if $X \in \mathcal{C}$ and every $Y \in \mathcal{C}$ is homeomorphic to a subspace of X. In [10], the author asked whether if there is a universal space for the class of metrizable finitistic spaces of given weight. We shall answer the question affirmatively in section 2. The following Pasynkov's factorization theorem is also fundamental in dimension theory: *For every continuous mapping $f : X \to Y$ of a normal space X to a metrizable space Y there is a metrizable space Z and continuous mappings $g : X \to Z$ and $h : Z \to Y$ such that* $\dim Z \leq \dim X$, *$w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$,* where $w(Y)$ and $w(Y)$ denote the weight of Y and Z respectively. In section 3, we shall prove a factorization theorem for finitistic spaces.

For a metric space (X, ρ), a subset A of X and $\varepsilon > 0$ we denote $S_\varepsilon(A) = \{x \in X : \rho(x, A) < \varepsilon\}$. We refer the reader to [9] and [13] for basic results in dimension theory.

2. A universal space theorem

In this section, we shall prove a universal theorem for finitistic spaces.

Theorem 2.1. *Let τ be an infinite cardinal number. Then there is a metric finitistic space $L(\tau)$ of weight τ such that for every metrizable finitistic space X of weight $\leq \tau$ is embedded in $L(\tau)$.*

Proof. It is trivial that the Hilbert cube I^ω is the universal space for the class of separable metrizable finitistic spaces. Hence, we assume that $\tau > \omega$. Let A be a set of the cardinality τ and fix $a^* \in A$. Let $J(\tau) = (I \times A)/\{(0, a) : a \in A\}$ be the hedgehog space of weight τ and $J^*(\tau) = \{(x, a) : x \in I \text{ and } a \in A - \{a^*\}\} \subset J(\tau)$. We denote the point $\{(0, a) : a \in A\}$ of $J(\tau)$ by $\mathbf{0}$. Let $M(\tau) = \{((x_i, a_i))_{i=1}^\infty : (x_1, a_1) \in J^*(\tau)$ and $(x_i, a_i) \in J(\tau)$ for every $i \geq 2$ $\}$ be the subspace of $J(\tau)^\omega$, where $J(\tau)^\omega$ is the countable product of $J(\tau)$. For each n we put

$$M_n(\tau) = \{((x_i, a_i))_{i=1}^\infty \in M(\tau) : \text{ non-zero rational } x_i\text{'s are at most } n\}.$$

Then $M_n(\tau)$ is a universal space for the class of n-dimensional metrizable spaces of weight $\leq \tau$. Let

$$I(\tau) = \{((x_i, a_i))_{i=1}^{\infty} \in J(\tau)^{\omega} : a_1 = a^* \text{ and } x_i = 0 \text{ for every } i \geq 2 \ \}.$$

For each k, we shall define a subspace Y_k of $(J(\tau)^{\omega})^{\omega}$. Let $\overline{\mathbf{0}} = (\mathbf{0}, \mathbf{0}, \dots) \in J(\tau)^{\omega}$,

$$Y_1 = M_1(\tau) \times M_2(\tau) \times \{\overline{\mathbf{0}}\} \times \{\overline{\mathbf{0}}\} \times \cdots ,$$

and for each $k \geq 2$,

$$Y_k = \prod_{i=1}^{k-2} I_i(\tau) \times M_{k-1}(\tau) \times M_k(\tau) \times M_{k+1}(\tau) \times \{\overline{\mathbf{0}}\} \times \{\overline{\mathbf{0}}\} \times \cdots ,$$

where $I_i(\tau)$ is a copy of $I(\tau)$ for each $i \leq k-2$. We put

$$L(\tau) = \prod_{i=1}^{\infty} I_i(\tau) \cup \bigcup_{k=1}^{\infty} Y_k,$$

where $I_i(\tau)$ is a copy of $I(\tau)$ for each i.

We shall show that the space $L(\tau)$ is desired.

It is clear that $L(\tau)$ is a metric space of weight τ. We shall show that $L(\tau)$ is finitistic. It should be noticed that $\prod_{i=1}^{\infty} I_i(\tau)$ is homeomorphic to the Hilbert cube. Let U be an open set of $L(\tau)$ which contains $\prod_{i=1}^{\infty} I_i(\tau)$ and U' be the open set of $(J(\tau)^{\omega})^{\omega}$ such that $U' \cap L(\tau) = U$. By the Wallace Theorem ([8, Theorem 3.2.10]), there are a natural number k_0 and a positive number ε such that

$$\prod_{i=1}^{\infty} I_i(\tau) \subset \prod_{i=1}^{k_0} S_{\varepsilon}(I_i(\tau)) \times J(\tau)^{\omega} \times J(\tau)^{\omega} \times \cdots \subset U'.$$

We put $W = \prod_{i=1}^{k_0} S_{\varepsilon}(I_i(\tau)) \times J(\tau)^{\omega} \times J(\tau)^{\omega} \times \cdots$. Then for each $k \geq k_0 + 2$, we have $Y_k \subset W$. Hence we have $Y_k \subset W \cap L(\tau) \subset U' \cap L(\tau) = U$. Therefore, $L(\tau) \setminus U \subset \bigcup_{i=1}^{k_0+1} Y_k$. Since each Y_k is finite dimensional, it follows that $L(\tau) \setminus U$ is finite dimensional. Therefore, $L(\tau)$ is finitistic by Proposition 1.1.

To show the universality of $L(\tau)$, let X be a metrizable finitistic space of weight $\leq \tau$. By Proposition 1.1, there is a compact subspace K of X such that $\dim F < \infty$ for every closed subspace F with $F \cap K = \emptyset$. For each $n \geq 1$, we put

$$U_1 = X \setminus S_{\frac{1}{2}}(K), \text{ and } U_n = S_{\frac{1}{n-1}}(K) \setminus \overline{S_{\frac{1}{n+1}}(K)}.$$

Then we have $X \setminus K = \bigcup_{n=1}^{\infty} U_n$ and $\dim U_n < \infty$ for each n. Without loss of generality, we may assume that $\dim U_n \leq n$ for each n. It is also noticed that

$U_n \cap U_m = \emptyset$ whenever $|n-m| > 1$. Since K is compact, it follows that there is a countable family $\{(F_n, G_n) : n = 1, 2, 3, \dots\}$ of pairs of subsets of X such that

(2.1) F_n is closed in X, G_n is open in X and $F_n \subset G_n$,

(2.2) for each $x \in K$ and a closed set F of X with $x \notin F$ there is n such that $x \in F_n \subset G_n \subset X \setminus F$, and

(2.3) $G_n \subset S_{\frac{1}{n+2}}(K)$ for each n.

It follows from (2.3) that $\overline{G_n} \cap \overline{U_n} = \emptyset$. For each n there is a continuos mapping $f_n : X \to I(\tau)$ such that $f_n(F_n) \subset \{((1, a^*), \mathbf{0}, \mathbf{0}, \dots)\}$ and $f_n(X \setminus G_n) \subset \{\overline{\mathbf{0}}\}$. Since $M_n(\tau)$ is a universal space for the class of n-dimensional metrizable spaces of weight $\leq \tau$, it follows from [9, Remark 4.2.12] that there is a continuous mapping $g_n : X \to M_n(\tau)$ such that the restriction $g_n|_{U_n}$ is a homeomorphic embedding and $g_n(X \setminus U_n) \subset \{\overline{\mathbf{0}}\}$. Now, we define $h_n : X \to I(\tau) \cup M_n(\tau) (\subset J(\tau)^\omega)$ as follows:

$$h_n(x) = \begin{cases} f_n(x), & \text{if } x \in X \setminus U_n, \\ g_n(x), & \text{if } x \in X \setminus G_n. \end{cases}$$

Then h_n is well-defined and continuous. Let $h : X \to \prod_{n=1}^{\infty} (I_n(\tau) \cup M_n(\tau))$ be the diagonal product of $\{h_n : n = 1, 2, \dots\}$, where $I_n(\tau) = I(\tau)$ for each n. It suffices to show that h is a homeomorphic embedding and $h(X) \subset L(\tau)$. To show that h is an embedding, we shall show that the family $\{h_n : n = 1, 2, \dots\}$ separates points from closed sets (cf. [8, Theorem 2.3.20]). Let $x \in X$ and F be a closed set of X with $x \notin F$. First we suppose that $x \in K$. By (2.2), there is n such that $x \in F_n \subset G_n \subset X \setminus F$. Then $h_n(x) = f_n(x) = ((1, a^*), \mathbf{0}, \mathbf{0}, \dots)$. Since $G_n \cap F = \emptyset$, it follows that $h_n(F) = g_n(F) \subset M_n(\tau)$. It is clear that $((1, a^*), \mathbf{0}, \mathbf{0}, \dots) \notin \overline{M_n(\tau)}$. Hence $h_n(x) \notin \overline{h_n(F)}$. Next, we suppose that $x \notin K$. Let n be a natural number such that $x \in U_n$. By the construction of g_n (cf. [9, Remark 4.2.22]), $g_n(x) \neq \overline{\mathbf{0}}$. Since $g_n|_{U_n}$ is an embedding, $g_n(x) \notin \overline{g_n(F \cap U_n)}$. Therefore, it follows that $g_n(x) \notin \overline{g_n(F)}$, because $\overline{g_n(F)} \subset \overline{g_n(F \cap U_n)} \cup \{\overline{\mathbf{0}}\}$. Since $G_n \cap U_n = \emptyset$, it follows that $x \notin G_n$ and this implies that $h_n(x) = g_n(x)$.

Since $g_n(x) \in M_n(\tau) \setminus \{\overline{\mathbf{0}}\}$, it follows that $g_n(x) \notin I(\tau)$. Hence $h_n(x) = g_n(x) \notin \overline{g_n(F)} \cup I(\tau) \supset \overline{g_n(F)} \cup \overline{f_n(F)} \supset \overline{h_n(F)}$. This implies that h is a homeomorphic embedding.

Finally, we shall show that $h(X) \subset L(\tau)$. Let $x \in K$. Then $x \notin U_n$ for each n. Hence for each n $h_n(x) = f_n(x) \in I(\tau)$. Therefore, we have $h(x) \in$

$\prod_{n=1}^{\infty} I_n(\tau) \subset L(\tau)$. Let $x \in X \setminus K$. There is n_0 such that $x \in U_{n_0}$. If $n \geq n_0+2$, then $x \notin U_n \cup G_n$. Hence $h_n(x) = g_n(x) = \mathbf{0}$. If $n \leq n_0 - 2$, then $x \notin U_n$. Hence $h_n(x) = f_n(x) \in I(\tau)$. If $n_0 - 1 \leq n \leq n_0 + 1$, then $G_n \subset S_{\frac{1}{n+2}}(K) \subset S_{\frac{1}{n_0+1}}(K)$ by (2.3). Furthermore, $U_{n_0} \cap \overline{S_{\frac{1}{n_0+1}}(K)} = \emptyset$. Hence $x \notin G_n$ and hence $h_n(x) = g_n(x) \in M_n(\tau)$. Therefore, $h(x) \in \prod_{n=1}^{n_0-2} I_n(\tau) \times M_{n_0-1}(\tau) \times M_{n_0}(\tau) \times M_{n_0+1}(\tau) \times \{\mathbf{0}\} \times \{\mathbf{0}\} \times \cdots \subset Y_{n_0} \subset L(\tau)$. □

Let $K_n(\tau)$ be the Nagata's universal space for the class of n-dimensional metrizable spaces of weight $\leq \tau$. For a space X and a metric space Y we denote by $C(X,Y)$ the space of all continuous mappings of X to Y endowed with the uniform convergence topology. Pol [17] proved that for an n-dimensional metrizable space X, $\{h \in C(X, J(\tau)^\omega) : h : X \to K_n(\tau) \text{ is an embedding}\}$ is dense in $C(X, J(\tau)^\omega)$. Since $L(\tau)$ is not dense in $(J(\tau)^\omega)^\omega$, it is clear that $\{h \in C(X, L(\tau)) : h \text{ is an embedding}\}$ is not dense in $C(X, (J(\tau)^\omega)^\omega)$. However, we can ask the following.

Question 1. Let X be a metrizable finitistic space of weight $\leq \tau$. Is $\{h \in C(X, L(\tau)) : h \text{ is an embedding}\}$ dense in $C(X, L(\tau))$?

In [16], Pol also proved that if X is a separable metrizable space, then X is finitistic if and only if

$$\{h \in C(X, I^\omega) : \ h \text{ is an embedding and } \overline{h(X)} \setminus h(X) \text{ is countable-dimensional}\}$$

is residual in $C(X, I^\omega)$. By a similar argument to [16, Theorem 4.1] using [18, Proposition 4.3], it is easy to see that for a paracompact finitistic space X $\{f \in C(X, I^\omega) : \overline{f(X)} \setminus f(X) \text{ is countable-dimensional}\}$ is residual in $C(X, I^\omega)$. However, we do not know the converse holds. That is, we can ask the following.

Question 2. Let X be a metrizable (or paracompact) space such that $\{f \in C(X, I^\omega) : \overline{f(X)} \setminus f(X) \text{ is countable-dimensional}\}$ is residual in $C(X, I^\omega)$. Is X finitistic?

3. A factorization theorem

In this section, we shall prove a Pasynkov's type of factorization theorem for finitistic spaces. For a family $\mathcal{U}$ of subsets of a space X and a subset A, we denote $\overline{\mathcal{U}} = \{\overline{U} : U \in \mathcal{U}\}$ and $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$. Further, for coverings $\mathcal{U}$ and $\mathcal{V}$ of a space X let $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

Lemma 3.1. *Let X be a normal finitistic space. Then there is a compact subspace K of X such that $\dim F < \infty$ for every closed set F with $F \cap K = \emptyset$.*

Proof. We notice that the lemma is announced in [10] without the proof. We put

$$P_n = \{x \in X : \text{ there is an open neighborhood } U \text{ of } x \text{ such that } \dim \overline{U} \leq n\}.$$

Put $K = X \setminus \bigcup_{n=1}^{\infty} P_n$. Suppose that K is not compact. Since K is weakly paracompact, K is not countably compact. Hence, there is a discrete closed subset $\{x_n : n = 1, 2, \dots\}$ of K. Since X is normal, there is a discrete family $\{U_k : k = 1, 2, \dots\}$ of open sets in X such that $x_k \in U_k$. Then $\dim \overline{U_k} = \infty$ for each k. For each k, there is an open family $\mathcal{U}_k$ of X such that $(\cup \mathcal{U}_k) \cap (\cup \mathcal{U}_m) = \emptyset$, $\mathcal{U}_k$ covers $\overline{U_k}$ and there is no open refinement of $\mathcal{U}_k|_{\overline{U_k}}$ of order $\leq k$. It is easy to see that $\mathcal{U} = \bigcup_{k=1}^{\infty} \mathcal{U}_k \cup \{X \setminus \bigcup_{k=1}^{\infty} \overline{U_k}\}$ is an open covering of X which does not have an open refinement of finite order. This is a contradiction.

Next, let F be a closed subset of X which does not meet K. By the point finite sum theorem for dim ([9, Theorem 3.1.13] or [9, Theorem 3.1.14]), it suffices to show that $F \subset P_n$ for some n. Suppose on the contrary. Then we may have a sequence $\{n_k : k = 1, 2, \dots\}$ of natural numbers and a sequence $\{x_k : k = 1, 2, \dots\}$ in F such that $x_{k+1} \in P_{n_{k+1}} \setminus P_{n_k}$. Let x_0 be an accumulation point of $\{x_k : k = 1, 2, \dots\}$. Since F is closed in X, $x_0 \in F$. Hence there is n_0 such that $x_0 \in P_{n_0}$. Let U_0 be an open neighborhood of x_0 such that $\dim \overline{U_0} \leq n_0$. Then there is k such that $n_k > n_0$ and $x_k \in U_0$. By the definition of P_n's, it follows that $x_k \in P_{n_0}$. This contradicts the choice of x_k. Hence the sequence $\{x_k : k = 1, 2, \dots\}$ is discrete in X. As similar to the above argument, we can show that there is an open covering $\mathcal{U}$ of X which has no open refinement of finite order. This is a contradiction. □

If every normal finitistic space is paracompact, then the lemma immediately follows from the proposition in the introduction. Unfortunately, there is a normal finitistic space which is not subparacompact ([3, Example 1.6]). Now, we describe the factorization theorem for finitistic spaces.

Theorem 3.2. *Let X be a normal space, Y a metrizable space and $f : X \to Y$ a continuous mapping. If X is a finitistic space, then there is a metrizable space Z and continuous mappings $g : X \to Z$ and $h : Z \to Y$ such that Z is a finitistic space, $w(Z) \leq w(Y)$, $g(X) = Z$ and $f = hg$*

Proof. By Lemma 3.1, there is a compact subspace K of X such that $\dim F < \infty$ for every closed set F of X with $F \cap K = \emptyset$. Let $\{\mathcal{U}_i\}_{i=0}^{\infty}$ be a sequence of locally finite open coverings of Y such that $\mathcal{U}_0 = \{Y\}$ and for each $i \geq 1$,

$$\text{mesh}\, \mathcal{U}_i < 1/i, \tag{3.1}$$

$$\overline{\mathcal{U}_{i+1}} \text{ is a star refinement of } \mathcal{U}_i. \tag{3.2}$$

Then for each $i \geq 1$ we have

$$\overline{\text{St}(f(K), \mathcal{U}_{i+1})} \subset \text{St}(f(K), \mathcal{U}_i). \tag{3.3}$$

For each k we put $F_k = X \setminus f^{-1}(\text{St}(f(K), \mathcal{U}_k))$. We need the following lemma.

Lemma 3.3. *There are a sequence $\{\mathcal{V}_i\}_{i=0}^{\infty}$ of locally finite cozero set coverings of X and a sequence $\{m_i\}_{i=1}^{\infty}$ of natural numbers such that $\mathcal{V}_0 = \{X\}$, and for each $i \geq 1$,*

(3.4) $m_i = \sum_{j=1}^{i}(\dim F_{j+1} + 1)$,

(3.5) *$\mathcal{V}_i$ is a star refinement of $\mathcal{V}_{i-1} \wedge f^{-1}(\mathcal{U}_{i-1})$,*

(3.6) *$\text{ord}(\mathcal{V}_i|_{F_k}) \leq m_k$ for each $k \leq i$, and*

(3.7) *$\{f^{-1}(U) : U \in \mathcal{U}_i$ and $U \cap f(K) \neq \emptyset\} \subset \mathcal{V}_i$.*

Proof. Since $\dim F_2 < \infty$, there is a locally finite cozero set covering $\mathcal{G}_1$ of F_2 such that $\text{ord}\, \mathcal{G}_1 \leq \dim F_2 + 1$. We put

$$\mathcal{V}_1 = \{G \setminus f^{-1}\left(\overline{\text{St}(f(K), \mathcal{U}_2)}\right) : G \in \mathcal{G}_1\} \cup \{f^{-1}(U) : U \in \mathcal{U}_1 \text{ and } U \cap f(K) \neq \emptyset\}.$$

Then $\mathcal{V}_1$ satisfies the above conditions.

Suppose that $\mathcal{V}_k$ is defined for each $k \leq i$. Let $\mathcal{H}$ be a locally finite cozero set covering of X such that $\mathcal{H}$ is a star refinement of $\mathcal{V}_i \wedge f^{-1}(\mathcal{U}_{i+1})$, where $f^{-1}(\mathcal{U}_{i+1}) = \{f^{-1}(U) : U \in \mathcal{U}_{i+1}\}$. For each $k \leq i+1$ there is a locally finite cozero set covering $\mathcal{G}_k$ of F_{k+1} such that $\mathcal{G}_k$ is a refinement of $\mathcal{H}|_{F_{k+1}}$ and $\text{ord}\, \mathcal{G}_k \leq \dim F_{k+1} + 1$. For each $k \leq i+1$ and each $G \in \mathcal{G}_k$ we put $G^* = G \setminus (F_{k-1} \cup f^{-1}\left(\overline{\text{St}(f(K), \mathcal{U}_{k+1})}\right)$, and

$$\mathcal{V}_{i+1} = \bigcup_{k=1}^{i+1}\{G^* : G \in \mathcal{G}_k\} \cup \{f^{-1}(U) : U \in \mathcal{U}_{i+1} \text{ and } U \cap f(K) \neq \emptyset\}.$$

It is easy to see that $\mathcal{V}_{i+1}$ is a locally finite cozero set covering of X. We shall show that $\mathcal{V}_{i+1}$ satisfies the conditions (3.5) - (3.7). By the construction, it is clear that $\mathcal{V}_{i+1}$ satisfies (3.7). We shall show $\mathcal{V}_{i+1}$ satisfies (3.5). For each $G \in \bigcup_{k=1}^{i+1} \mathcal{G}_k$, there is $V_i \in \mathcal{V}_i$ and $U_G \in \mathcal{U}_{i+1}$ such that $\mathrm{St}(G, \mathcal{H}) \subset V_i \cap f^{-1}(U_G)$.

Case 1. We suppose that $G^* \cap f^{-1}(\mathrm{St}(f(K), \mathcal{U}_{i+1}) = \emptyset$. Then $\mathrm{St}(G^*, \mathcal{V}_{i+1}) \subset \mathrm{St}(G, \mathcal{H}) \subset V_i \cap f^{-1}(U_G) \subset V_i \cap f^{-1}(U_i)$ for some $U_i \in \mathcal{U}_i$.

Case 2. We suppose that $G^* \cap f^{-1}(\mathrm{St}(f(K), \mathcal{U}_{i+1}) \neq \emptyset$. There is $U'_G \in \mathcal{U}_{i+1}$ such that $G^* \cap f^{-1}(U'_G) \neq \emptyset$ and $U'_G \cap f(K) \neq \emptyset$. Since $\mathcal{G}_k$ is a refinement of $\mathcal{H}$ and $\mathcal{H}$ is a star refinement of $f^{-1}(\mathcal{U}_{i+1})$, there is $U_G^{i+1} \in \mathcal{U}_{i+1}$ such that $\mathrm{St}(G^*, \mathcal{H}) \subset f^{-1}(U_G^{i+1})$. By (3.2), there is $U_i \in \mathcal{U}_i$ such that $\mathrm{St}(U_G^{i+1}, \mathcal{U}_{i+1}) \subset U_i$. Then

$$
\begin{aligned}
\mathrm{St}(G^*, \mathcal{V}_{i+1}) &\subset \mathrm{St}(G^*, \mathcal{H}) \cup \mathrm{St}(G^*, f^{-1}(\mathcal{U}_{i+1})) \\
&\subset f^{-1}(U_G^{i+1}) \cup \mathrm{St}(f^{-1}(U_G^{i+1}), f^{-1}(\mathcal{U}_{i+1})) \\
&= \mathrm{St}(f^{-1}(U_G^{i+1}), f^{-1}(\mathcal{U}_{i+1})) \\
&\subset f^{-1}(U_i).
\end{aligned}
$$

On the other hand, $U_i \cap f(K) \supset \mathrm{St}(U_G^{i+1}, \mathcal{U}_{i+1}) \cap f(K) \supset U_G \cap f(K) \neq \emptyset$. Hence $f^{-1}(U_i) \in \mathcal{V}_i \cap f^{-1}(\mathcal{U}_i)$ and hence $f^{-1}(U_i) \in \mathcal{V}_i \wedge f^{-1}(\mathcal{U}_i)$.

On the other hand, by (3.2), it is clear that for each $U \in \mathcal{U}_{i+1}$ with $U \cap f(K) \neq \emptyset$ we have $\mathrm{St}(f^{-1}(U), \mathcal{V}_{i+1}) \subset \mathrm{St}(f^{-1}(U), f^{-1}(\mathcal{U}_{i+1})) \subset f^{-1}(U_i)$ for some $U_i \in \mathcal{U}_i$. Therefore $\mathcal{V}_{i+1}$ is a star refinement of $\mathcal{V}_i \wedge f^{-1}(\mathcal{U}_i)$ and hence the condition (3.5) is satisfied.

We shall show (3.6). Let $k \leq i+1$. For each j with $k+1 \leq j \leq i+1$ and each $G \in \mathcal{G}_j$ we have $G^* \cap F_k = \emptyset$. By the definition of F_{i+1}, it follows that $f^{-1}(\mathrm{St}(f(K), \mathcal{U}_{i+1}) \cap F_k \subset f^{-1}(\mathrm{St}(f(K), \mathcal{U}_{i+1}) \cap F_{i+1} = \emptyset$. Hence,

$$
\mathrm{ord}\, \mathcal{V}_{i+1}|_{F_k} = \sum_{j=1}^{k} \mathrm{ord}\{G^* : G \in \mathcal{G}_j\} \leq \sum_{j=1}^{k} (\dim F_{j+1} + 1) = m_k.
$$

This completes the proof of Lemma 3.3. □

We continue the proof of Theorem 3.2. Let $\{\mathcal{V}_i\}_{i=1}^{\infty}$ be a sequence of locally finite cozero set coverings of X described in Lemma 3.3. Without loss of generality, we may assume that $|\mathcal{V}_i| \leq w(Y)$ for each i. To construct the space Z and continuous mappings g and h, we modify the proof of Pasynkov's factorization theorem ([14], [15]) presented in [9, Theorem 4.2.5] which is obtained by amalgamating the proofs due to [2] and [11]. We outline the construction for the convenience of the reader.

First, we introduce a new topology on X, which is coarser than the original one, by defining an interior operator as follows: For a subset S of X let

$$S^\circ = \{x \in X : \text{there is } i \text{ such that } \mathrm{St}(x, \mathcal{V}_i) \subset S\}.$$

Let X' be the topological space with this new topology on the set X. For each i we put $\mathcal{V}_i^\circ = \{V^\circ : V \in \mathcal{V}_i\}$. Then it is easy to see that $\mathcal{V}_i^\circ$ is an open covering of X' and $\mathcal{V}_{i+1}^\circ$ is a star refinement of $\mathcal{V}_i^\circ$. It follows from (3.5) that f is continuous with respect to X'. For x, $y \in X'$ we define an equivalence relation as follows: $x \sim y$ if and only if $y \in \mathrm{St}(x, \mathcal{V}_i)$ for each i. Let $Z = X'/\sim$ and $g : X \to Z$ be the composition of the identity and the natural quotient map $q : X' \to Z$. We define a mapping $h : Z \to Y$ as $h(g(x)) = f(x)$ for each $x \in X$. It follows that h is well-defined and continuous. As shown in the proof of [9, Theorem 4.2.5], Z is a metrizable space of weight $\leq w(Y)$. It is noticed that $\{\mathcal{W}_i\}_{i=1}^\infty$ is a development for Z, where $\mathcal{W}_i = \{g(V^\circ) : V \in \mathcal{V}_i\}$.

Finally, we shall show that Z is a finitistic space. It suffices to see that F is finite dimensional for each closed set F of Z which does not meet $g(K)$. We suppose that $\overline{f(g^{-1}(F))} \cap f(K) \neq \emptyset$. Let $y_0 \in \overline{f(g^{-1}(F))} \cap f(K)$ and $x_0 \in K$ such that $f(x_0) = y_0$. For each k there is $U_{k+1} \in \mathcal{U}_{k+1}$ such that $y_0 \in U_{k+1}$. Then $U_{k+1} \cap f(g^{-1}(F)) \neq \emptyset$. Since $y_0 \in U_{k+1} \cap f(K)$, it follows from (3.7) that $f^{-1}(U_{k+1}) \in \mathcal{V}_{k+1}$. Hence $\mathrm{St}(x_o, \mathcal{V}_{k+1}) \cap g^{-1}(F) \supset f^{-1}(U_{k+1}) \cap g^{-1}(F) \neq \emptyset$. Since $\mathcal{V}_{k+1}$ is a star refinement of $\mathcal{V}_k$, there is $V_k \in \mathcal{V}_k$ such that $\mathrm{St}(f^{-1}(U_{k+1}), \mathcal{V}_{k+1}) \subset V_k$. Then for each $x \in f^{-1}(U_{k+1})$ we have $\mathrm{St}(x, \mathcal{V}_{k+1}) \subset \mathrm{St}(f^{-1}(U_{k+1}), \mathcal{V}_{k+1}) \subset V_k$. Hence $x \in V_k^\circ$, and hence $f^{-1}(U_{k+1}) \subset V_k^\circ$. Therefore $V_k^\circ \cap g^{-1}(F) \supset f^{-1}(U_{k+1}) \cap g^{-1}(F) \neq \emptyset$. It follows that $g(V_k^\circ) \cap F \neq \emptyset$. Thus, for each k we have $\mathrm{St}(g(x_0), \mathcal{W}_k) \cap F \neq \emptyset$. Since $\{\mathcal{W}_i\}_{i=1}^\infty$ is a development for Z, $g(x_0) \in \overline{F} = F$. Hence $g(x_0) \in g(K) \cap F$. This contradicts with $F \cap g(K) = \emptyset$. Therefore, $\overline{f(g^{-1}(F))} \cap f(K) = \emptyset$. Since $f(K)$ is compact and mesh $\mathcal{U}_k < 1/k$ for each k, there is k such that $\overline{f(g^{-1}(F))} \cap \mathrm{St}(f(K), \mathcal{U}_k) = \emptyset$. Then $g^{-1}(F) \subset X \setminus f^{-1}(\mathrm{St}(f(K), \mathcal{U}_k)) = F_k$. For each $x \in F_k$ and each $i \geq k$ it is easy to see that $\mathrm{ord}_{g(x)} \mathcal{W}_i \leq \mathrm{ord}_x \mathcal{V}_i$. It follows from (3.6) that $\mathrm{ord}\, \mathcal{W}_i|_{g(F_k)} \leq \mathrm{ord}\, \mathcal{V}_i|_{F_k} \leq m_k$. Hence $\dim F \leq \dim g(F_k) \leq m_k - 1$ ([12], [9, Theorem 4.2.3] or [13, Corollary to Theorem V.1]) and hence Z is finitistic. $\square$

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