

## ON GENERALIZED RIGIDITY

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ABSTRACT. Concepts of chaotic and of rigid spaces with respect to a given class of mappings are introduced and studied in the paper. A special attention is paid to the classes of open and of monotone mappings. The obtained results are applied to dendrites.

### 1. INTRODUCTION

A nondegenerate topological space  $X$  is said to be:

- (a) *chaotic* if for any two distinct points  $p$  and  $q$  of  $X$  there exists an open neighbourhood  $U$  of  $p$  and an open neighbourhood  $V$  of  $q$  such that no open subset of  $U$  is homeomorphic to any open subset of  $V$ ;
- (b) *strongly chaotic* if for any two distinct points  $p$  and  $q$  of  $X$  there exist open neighborhoods  $U$  of  $p$  and  $V$  of  $q$  respectively such that no open subset of  $U$  is homeomorphic to any subset of  $V$ ;
- (c) *rigid* if it has a trivial autohomeomorphism group, i.e., if the only homeomorphism of  $X$  onto  $X$  is the identity;
- (d) *strongly rigid* if the only homeomorphism of  $X$  into  $X$  is the identity of  $X$  onto itself.

These four concepts were extensively studied in many papers. In [2] a comprehensive list of references is produced and a number of results are presented or recalled, especially those related to curves (i.e., one-dimensional metric continua). Compare also [4] and [5]. In particular, the following proposition is known [5, Proposition 3.13, p. 183].

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**Proposition 1.1.** *For every topological space  $X$  we have the following four implications and none of them can be reversed, even if  $X$  is a dendrite.*

$$\begin{array}{ccc} X \text{ is strongly chaotic} & \Rightarrow & X \text{ is strongly rigid} \\ \Downarrow & & \Downarrow \\ X \text{ is chaotic} & \Rightarrow & X \text{ is rigid.} \end{array}$$

The aim of this paper is to introduce some more general concepts related to the above ones, and to extend earlier results. We also ask some questions related to further research in the area. The leading idea is to replace homeomorphisms in the definitions (a)-(d) above by mappings belonging to a suitable class  $\mathcal{M}$  of mappings between topological spaces.

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## 2. PRELIMINARIES

Given a subset  $A$  of a space  $X$ , we denote by  $\text{cl}(A)$  its closure in  $X$ , and by  $\text{int}(A)$  its interior. A *continuum* means a compact connected metric space, and a *curve* means a one-dimensional continuum. A *dendrite* means a locally connected continuum containing no simple closed curve. Given two points  $p$  and  $q$  of a dendrite  $X$ , we denote by  $pq$  the unique arc from  $p$  to  $q$  in  $X$ .

We shall use the notion of *order of a point* in the sense of Menger-Urysohn (see e.g. [14, §51, I, p. 274]), and we denote by  $\text{ord}(p, X)$  order of the space  $X$  at a point  $p \in X$ . It is well-known (see e.g. [14, §51, p. 274-307]) that the function  $\text{ord}$  takes its values from the set

$$S = \{0, 1, 2, \dots, \omega, \aleph_0, 2^{\aleph_0}\}.$$

Points of order 1 in a space  $X$  are called *end points* of  $X$ ; the set of all end points of  $X$  is denoted by  $E(X)$ . Points of order 2 are called *ordinary points* of  $X$ . It is known that in a dendrite the set of all its ordinary points is a dense subset of the dendrite. And for each  $n \in \{3, 4, \dots, \omega, \aleph_0, 2^{\aleph_0}\}$  points of order  $n$  are called *ramification points* of  $X$ ; the set of all ramification points is denoted by  $R(X)$ . It is known that for each dendrite  $X$  the set  $R(X)$  is at most countable, and that points of order  $\aleph_0$  and  $2^{\aleph_0}$  do not occur in any dendrite.

Given a dendrite  $X$  we decompose it into disjoint subsets of points of a fixed order. Namely for each  $n \in \{1, 2, 3, \dots, \omega\}$  we put

$$R_n(X) = \{p \in X : \text{ord}(p, X) = n\}.$$

We denote by  $\text{comp}(p, X)$  the component of the space  $X$  containing the point  $p$ . By a *free arc*  $A$  in a space  $X$  we mean an arc  $A$  with end points  $x$  and  $y$  such that  $A \setminus \{x, y\}$  is an open subset of  $X$ . In particular, by a *maximal free arc* in a dendrite  $X$  we mean such an arc  $st \subset X$  that  $st \cap (E(X) \cup R(X)) = \{s, t\}$ .

A *mapping* means a continuous transformation. A mapping  $f : X \rightarrow Y$  is said to be:

- a *local homeomorphism* if each point in  $X$  has an open neighborhood  $U$  such that  $f(U)$  is open in  $Y$  and  $f|U : U \rightarrow f(U)$  is a homeomorphism;
- *open* if  $f$  maps each open set in  $X$  onto an open set in  $Y$ ;
- *monotone* if the inverse image of each point of  $Y$  is connected; or, which is equivalent provided  $X$  is a compact Hausdorff space, if the inverse image of each connected subset of  $Y$  is connected;
- *light* if  $f^{-1}(y)$  has one-point components for each  $y \in Y$  (note that if the inverse images of points are compact, this condition is equivalent to the property that they are zero-dimensional).

As usual, the symbol  $\mathbb{N}$  stands for the set of all natural numbers. Let  $\mathcal{M}$  be a class of mappings between topological spaces that contains the class of homeomorphisms. A nondegenerate topological space  $X$  is said to be:

- *semi-chaotic with respect to  $\mathcal{M}$*  (shortly *semi- $\mathcal{M}$ -chaotic*) provided that for any two distinct points  $p$  and  $q$  of  $X$  there are open neighborhoods  $U$  and  $V$  of  $p$  and  $q$  respectively such that for every two open subsets  $U' \subset U$  and  $V' \subset V$  either there is no surjection in  $\mathcal{M}$  from  $U'$  onto  $V'$  or there is no surjection in  $\mathcal{M}$  from  $V'$  onto  $U'$ ;
- *(strongly) chaotic with respect to  $\mathcal{M}$*  (shortly *(strongly)  $\mathcal{M}$ -chaotic*) provided that for any two distinct points  $p$  and  $q$  of  $X$  there exist an open neighbourhood  $U$  of  $p$  and an open neighbourhood  $V$  of  $q$  such that no open subset of  $U$  can be mapped onto any open subset (onto any subset) of  $V$  under a mapping belonging to  $\mathcal{M}$ ;
- *(strongly) rigid with respect to  $\mathcal{M}$*  (shortly *(strongly)  $\mathcal{M}$ -rigid*) provided that the only mapping from  $\mathcal{M}$  of  $X$  onto itself (onto a subspace of  $X$ ) is the identity of  $X$  onto  $X$ .

If we take as  $\mathcal{M}$  the class of homeomorphisms, we get the concepts of chaotic, strongly chaotic, rigid or strongly rigid spaces in the sense of (a), (b), (c) or (d) above, respectively.

Since H. Cook constructed in Section 3 of [7] a continuum  $M_2$  such that the identity is the only mapping of  $M_2$  onto a nondegenerate subcontinuum of  $M_2$

(see Theorem 11 of [7, p. 247]), then it follows that families of continua that are strongly rigid with respect to any class of mappings are nonempty.

Continua which are rigid with respect to the class of all (continuous) mappings, i.e., such that the identity is the only surjection of the continuum onto itself) are called Cook continua ([17, Proposition 29, p. 546]). T. Maćkowiak has shown that there exists a Cook continuum  $X$  which is arc-like, Suslinian and hereditarily divisible by points ([17, Theorem 30, p. 547]). Further, he has constructed a Cook continuum which is arc-like, hereditarily decomposable, and not only strongly rigid but also hereditarily strongly rigid with respect to the class of all mappings, i.e., such that each of its subcontinua is strongly rigid with respect to this class (see [17, Corollary 32, p. 550] and [18, Corollary 6.2, p. 32]).

### 3. OPEN MAPPINGS

Let  $\mathcal{O}$  denote the class of open mappings between topological spaces. We consider as  $\mathcal{M}$  any class of mappings that is contained in the class  $\mathcal{O}$ . For example, the class of homeomorphisms, of local homeomorphisms and of all open mappings are such classes. To make formulations of statements shorter, we shall use the terms *openly* chaotic (rigid, strongly rigid) rather than chaotic (rigid, strongly rigid) with respect to the class  $\mathcal{O}$  of open mappings.

A class  $\mathcal{M}$  of mappings is said to be *hereditary with respect to open subspaces* provided that if a mapping  $f : X \rightarrow Y$  is in  $\mathcal{M}$  and  $X'$  is an open subspace of  $X$ , then the restriction  $f|_{X'} : X' \rightarrow f(X') \subset Y$  also is in  $\mathcal{M}$ . For example, the classes of homeomorphisms, of local homeomorphisms or of open mappings are hereditary with respect to open subspaces, while monotone mappings are not. Note that if  $\mathcal{M} \subset \mathcal{O}$ , then  $\mathcal{M}$  need not be hereditary with respect to open subspaces. In fact, let  $\mathcal{M}$  denote the class of mappings that are monotone and open simultaneously. Further, let  $\mathbb{C}$  and  $\mathbb{R}$  stand for the spaces of complex and of real numbers, respectively, with their natural topologies. If  $f : \mathbb{C} \rightarrow \mathbb{R}$  is defined by  $f(z) = |z|$  for  $z \in \mathbb{C}$ , then  $f \in \mathcal{M}$ . Putting  $X' = \{z \in \mathbb{C} : |\operatorname{Re}(z)| < 1\}$  we see that  $X'$  is an open subspace of  $\mathbb{C}$ , while the restriction  $f|_{X'}$  is open but not monotone, so it is not in the class  $\mathcal{M}$ .

**Theorem 3.1.** *Let a class  $\mathcal{M}$  of mappings be hereditary with respect to open subspaces. If  $\mathcal{M} \subset \mathcal{O}$ , then each  $\mathcal{M}$ -chaotic space is  $\mathcal{M}$ -rigid.*

PROOF. Let a space  $X$  be  $\mathcal{M}$ -chaotic. Then it is chaotic, and thus Hausdorff. Suppose  $X$  is not  $\mathcal{M}$ -rigid. There exists a surjection  $f : X \rightarrow X$  such that  $f \in \mathcal{M}$  and  $f$  is not the identity. Thus there exists a point  $p$  in  $X$  that is distinct from

$f(p) = q$ . Let  $U$  and  $V$  be disjoint open neighborhoods of  $p$  and  $q$  respectively. By continuity of  $f$  at  $p$  there is an open set  $U' \subset U$  such that  $p \in U'$  and  $f(U') \subset V$ . Since  $f$  is open,  $f(U')$  is an open subset of  $V$ , and since  $\mathcal{M}$  is hereditary with respect to open subspaces, the restriction  $f|_{U'} : U' \rightarrow f(U')$  is in  $\mathcal{M}$ . This contradicts the assumption that  $X$  is  $\mathcal{M}$ -chaotic.  $\square$

Note that the classes of homeomorphisms, of local homeomorphisms and of open mappings satisfy the assumptions of Theorem 3.1. Thus we have the following corollary (being a generalization of [2, Proposition 6, p. 221]).

**Corollary 3.2.** *If  $\mathcal{M}$  denotes one of the following classes of mappings: homeomorphisms, local homeomorphisms, or open mappings, then each  $\mathcal{M}$ -chaotic space is  $\mathcal{M}$ -rigid.*

The following observation can easily be verified.

**Observation 3.3.** *If a class  $\mathcal{M}$  of mappings is hereditary with respect to open subspaces, then each strongly  $\mathcal{M}$ -chaotic space is strongly  $\mathcal{M}$ -rigid.*

**Corollary 3.4.** *Let a class  $\mathcal{M}$  of mappings be hereditary with respect to open subspaces. If  $\mathcal{M} \subset \mathcal{O}$  (in particular, if  $\mathcal{M}$  denotes the class of homeomorphisms, of local homeomorphisms, or of all open mappings), then we have the following four implications and none of them can be reversed, even if  $X$  is a dendrite and  $\mathcal{M}$  is the class of homeomorphisms:*

$$(3.4.1) \quad \begin{array}{ccc} X \text{ is strongly } \mathcal{M}\text{-chaotic} & \Rightarrow & X \text{ is strongly } \mathcal{M}\text{-rigid} \\ \downarrow & & \downarrow \\ X \text{ is } \mathcal{M}\text{-chaotic} & \Rightarrow & X \text{ is } \mathcal{M}\text{-rigid.} \end{array}$$

PROOF. The two vertical implications are true by the definitions. The upper horizontal one is just Observation 3.3, and the lower horizontal is Theorem 3.1. It is proved in [5], Proposition 3.13, p. 183] that the considered implications cannot be reversed if  $X$  is a dendrite and  $\mathcal{M}$  is the class of homeomorphisms.  $\square$

**Proposition 3.5.** *Let  $\mathcal{M}$  be a class of mappings between topological spaces. Consider the following four conditions that a topological space  $X$  may satisfy*

- (3.5.1): *for every subset  $U$  of  $X$  and for every mapping  $f : U \rightarrow f(U) \subset X$  with  $f(U)$  being open, if  $f \in \mathcal{M}$ , then  $f$  is the identity on  $U$ ;*
- (3.5.2): *for any two nonempty distinct subsets  $U$  and  $V$  of  $X$  with  $V$  being open there is in  $\mathcal{M}$  no mapping from  $U$  onto  $V$ ;*

**(3.5.3):** for any two nonempty disjoint subsets  $U$  and  $V$  of  $X$  with  $V$  being open there is in  $\mathcal{M}$  no mapping from  $U$  onto  $V$ ;

**(3.5.4):** for any two distinct points  $p$  and  $q$  of  $X$  there exists an open neighbourhood  $U$  of  $p$  and an open neighbourhood  $V$  of  $q$  such that there is in  $\mathcal{M}$  no surjection from a subset of  $U$  onto an open subset of  $V$ .

If the space  $X$  is Hausdorff, then the following three implications are true:

$$(3.5.1) \Rightarrow (3.5.2) \Rightarrow (3.5.3) \Rightarrow (3.5.4).$$

PROOF. All implications are straightforward. Only in the last one do we use the assumption that  $X$  is Hausdorff.  $\square$

**Proposition 3.6.** *Let a class  $\mathcal{M}$  of mappings be hereditary with respect to open subspaces. If  $\mathcal{M} \subset \mathcal{O}$ , then (3.5.4) implies (3.5.1), and therefore conditions (3.5.1) through (3.5.4) of Proposition 3.5 are equivalent.*

PROOF. Assume (3.5.1) does not hold. Then there are a set  $U_0 \subset X$  and a mapping  $f : U_0 \rightarrow f(U_0) \subset X$  with  $f(U_0)$  being an open subset of  $X$  such that  $f \in \mathcal{M}$  and  $f$  is not the identity on  $U_0$ . Thus there is a point  $p_0 \in U_0$  such that  $p_0 \neq f(p_0)$ . Put  $q_0 = f(p_0)$  and note that by (3.5.4) there are open neighborhoods  $U$  and  $V$  of  $p_0$  and  $q_0$  respectively such that

**(3.6.1):** for each subset  $U'$  of  $U$  and for each open subset  $V'$  of  $V$  there is in  $\mathcal{M}$  no surjection  $f'$  from  $U'$  onto  $V'$ .

Since  $f$  is continuous at  $p_0$ , there exists an open neighbourhood  $U_1$  of  $p_0$  (in  $X$ ) such that  $f(U_1 \cap U \cap U_0) \subset V$ . Put  $U' = U_1 \cap U \cap U_0$  and  $V' = f(U')$ . Thus  $U'$  is an open subspace of the domain  $U_0$  of  $f$ , and since  $f \in \mathcal{M} \subset \mathcal{O}$ , its image  $V'$  is an open subset of the range  $f(U_0)$  of  $f$ . Since  $f(U_0)$  is open in  $X$ , we see that  $V'$  is open in  $X$ , too. The class  $\mathcal{M}$  being hereditary with respect to open subspaces, the restriction  $f' = f|_{U'} : U' \rightarrow V' = f(U')$  is in  $\mathcal{M}$ . Since  $V'$  is an open subset and  $V' \subset V$ , we have a contradiction to (3.6.1). The proof is complete.  $\square$

Under the assumptions of Proposition 3.6, putting  $U = X$  in condition (3.5.1) we see that if a mapping  $f : X \rightarrow f(X) \subset X$  is in  $\mathcal{M}$ , then  $f$  is the identity (on  $X$ ), and thus  $X$  is strongly  $\mathcal{M}$ -rigid. So, we have proved the following corollary.

**Corollary 3.7.** *Let a class  $\mathcal{M}$  of mappings be hereditary with respect to open subspaces. If  $\mathcal{M} \subset \mathcal{O}$ , then each of the equivalent conditions (3.5.1) through (3.5.4) of Proposition 3.5 implies that a Hausdorff space  $X$  is strongly  $\mathcal{M}$ -rigid.*

The author does not know if the assumption on  $\mathcal{M}$  of being hereditary with respect to open subspaces is indispensable in Theorem 3.1 and Proposition 3.6.

As is known from Corollary 4.9 (of the next chapter), the question of the existence of dendrites that are either (strongly) chaotic or (strongly) rigid with respect to monotone mappings has a negative answer. Concerning a similar problem for the class of open mappings, it is natural to verify first whether or not the known examples of (strongly) chaotic or (strongly) rigid dendrites are either chaotic or rigid with respect to open mappings. To do this recall the following result, which is a part of Theorem 13 of [4].

**Statement 3.8.** *If a dendrite  $X$  satisfies the conditions*

$$(3.8.1) \quad \text{cl}(R(X)) = X$$

and

$$(3.8.2) \quad \text{card } R_n(X) \leq 1 \quad \text{for each } n \in \{3, 4, \dots, \omega\},$$

then  $X$  is chaotic.

The simplest dendrite known to be chaotic (and to which Statement 3.8 can be applied) is one due to Johan J. de Jongh and described by de Groot and Wille in [10, p. 443] (compare also [2, Section 5, p. 227-228]). It satisfies, besides (3.8.1) the following conditions:

$$(3.8.3) \quad \text{card } R_n(X) = 1 \quad \text{for each } n \in \{3, 4, \dots\},$$

$$(3.8.4) \quad R_\omega(X) = \emptyset.$$

However, neither conditions (3.8.1), (3.8.3) and (3.8.4), nor a rough description given in [10], nor one presented in [2], lead to a uniquely determined dendrite, because the constructed dendrite  $X$  depends on a function that assigns the consecutive  $i$ -ods (used in the successive steps of the construction) to midpoints of the maximal free arcs in finite dendrites (i.e., trees) the closure of the union of which is just  $X$  (see [2, p. 228]). Thus we call any of the dendrites obtained in this way to be *of de Groot-Wille type* rather than call it the de Groot-Wille dendrite.

We show that the above mentioned function can be chosen in such a way that the resulting dendrite, being chaotic, is not openly chaotic. To this aim we need another definition of the considered dendrite, namely one in terms of inverse limits. To show the mentioned result we will use the following proposition, an easy proof of which is left to the reader.

**Proposition 3.9.** *For each topological space  $X$  the following conditions are equivalent*

(3.9.1):  $X$  is  $\mathcal{M}$ -chaotic;

(3.9.2):  $X$  is Hausdorff, and for every two nonempty distinct open subsets  $U$  and  $V$  of  $X$  there is in  $\mathcal{M}$  no surjection from  $U$  onto  $V$ ;

(3.9.3):  $X$  is Hausdorff, and for every two nonempty disjoint open subsets  $U$  and  $V$  of  $X$  there is in  $\mathcal{M}$  no surjection from  $U$  onto  $V$ .

**Theorem 3.10.** *There exists a dendrite  $X$  of de Groot-Wille type which is chaotic and not openly chaotic.*

PROOF. The dendrite  $X$  (in the plane) will be defined as

$$(3.10.1) \quad X = \text{cl}\left(\bigcup\{X_n : n \in \mathbb{N}\}\right),$$

where each  $X_n$  is a tree, and  $X_n \subset X_{n+1}$  for each  $n \in \mathbb{N}$ . To show that  $X$  is not openly chaotic we will use condition (3.9.3) of Proposition 3.9, namely we will find an open retraction of  $X$  onto  $Y \subset X$  that maps an open subset of  $X$  to an open subset of  $Y$ . The retraction will be defined as an inverse limit mapping. Therefore it will be convenient to construct three inverse sequences simultaneously: of trees  $X_n$ , of trees  $Y_n \subset X_n$ , and of (open) retractions  $g_n : X_n \rightarrow Y_n$ .

Let us start with the sequence  $\{T_i : i \in \mathbb{N}\}$ , where each  $T_i$  is an  $i$ -od, i.e., the union of  $i$  straight line segments  $A_i^m$  for  $m \in \{1, \dots, i\}$  emanating from one point, called the *origin* of  $T_i$ . We proceed by induction.

Define  $X_1$  as the unit straight line segment with end points  $a$  and  $b$ , and denote by  $c$  its midpoint. We consider  $X_1$  as the union of  $m(1) = 2$  segments:  $ac$  and  $cb$ . For further purposes let  $j_1 = 0$ . Put  $Y_1 = ac \subset X_1$  and define  $g_1 : X_1 \rightarrow Y_1$  by the conditions

$$g_1|Y_1 = \text{id}|Y_1, \quad g_1(b) = a, \quad \text{and} \quad g_1|cb : cb \rightarrow ac \text{ is linear.}$$

Thus  $g_1$  is an open retraction. Let  $x_1$  and  $x_2$  be the midpoints of the segments  $ac$  and  $cb$  contained in  $X_1$ .

Define  $X_2$  as the union of  $X_1$  and of  $j_2 = j_1 + m(1) = 2$  copies of  $T_1$  and  $T_2 = T_{j_2}$  diminished in such a way that the diameter of each copy is less than  $1/2$ , and located in the plane so that, for  $i \in \{1, 2\}$ , the origin of  $T_i$  is identified with  $x_i$  and  $x_i$  is the only common point of  $X_1$  and  $T_i$ . Thus  $X_2$  is a tree which is the union of  $m(2) = 7$  maximal free segments, i.e., segments whose end points, and only end points, belong to the set  $\{c\} \cup E(X_2) \cup R(X_2)$ . Note that  $X_1 \subset X_2$ , and that  $\text{ord}(x_1, X_2) = 3$  and  $\text{ord}(x_2, X_2) = 4$ .

Define a bonding mapping  $f_1 : X_2 \rightarrow X_1$  by

$$f_1|X_1 = \text{id}|X_1, \quad \text{and} \quad f_1(T_i) = \{x_i\} \text{ for } i \in \{1, 2\},$$



whence it follows that  $f_1$  is a monotone retraction. Define further

$$Y_2 = Y_1 \cup T_1 \subset X_2.$$

Note that  $\text{ord}(c, X_1) = \text{ord}(c, X_2) = 2$ , and thus  $c$  cuts  $X_1$  as well as  $X_2$  into two components. Thus we can write

$$Y_n = \text{cl}(\text{comp}(a, X_n \setminus \{c\})) \quad \text{for } n \in \{1, 2\}.$$

Define  $g_2 : X_2 \rightarrow Y_2$  by the conditions:  $g_2|_{Y_2} = \text{id}|_{Y_2}$ ,  $g_2|_{X_1} = g_1$ , and  $g_2|_{A_2^m} : A_2^m \rightarrow T_1$  for  $m \in \{1, 2\}$  are linear surjections. Thus  $g_2(x_2) = x_1$ , and we see that  $g_2$  is an open retraction. It can be verified that the diagram

$$\begin{array}{ccc} X_1 & \xleftarrow{f_1} & X_2 \\ g_1 \downarrow & & \downarrow g_2 \\ Y_1 & \xleftarrow{f_1|_{Y_2}} & Y_2 \end{array}$$

commutes.

Assume now that a tree  $X_n$  containing  $X_1$  has been defined for some natural number  $n \geq 2$  in such a way that it is the union of finitely many, say  $m(n)$ , straight line segments whose end points, and only end points, are in the set  $\{c\} \cup E(X_n) \cup R(X_n)$ , and that  $X_n$  contains (properly diminished) copies of the first  $j_n$  terms of the sequence  $\{T_i : i \in \mathbb{N}\}$ . We also assume that

**(3.10.2):**  $\text{ord}(c, X_n) = 2$ ,

**(3.10.3):**  $Y_n = \text{cl}(\text{comp}(a, X_n \setminus \{c\}))$ ,

**(3.10.4):**  $g_n : X_n \rightarrow Y_n$  is an open retraction that is linear on each segment contained in  $X_n$  whose end points, and only end points, are in the set  $\{c\} \cup E(X_n) \cup R(X_n)$ .

To define the dendrite  $X_{n+1}$  consider all  $m(n)$  segments mentioned in (3.10.4). We label the midpoints  $x$  of these segments with indices  $i \in \{j_n + 1, \dots, j_{n+1}\}$ , where  $j_{n+1} = j_n + m(n)$ , so that

**(3.10.5):**  $x_{i_1} \in Y_n$  and  $x_{i_2} \in X_n \setminus Y_n$  imply  $i_1 < i_2$  for every  $i_1, i_2 \in \{j_n + 1, \dots, j_{n+1}\}$ .

With each point  $x_i$  so labeled we associate the set  $T_i$  for the indices  $i \in \{j_n + 1, \dots, j_{n+1}\}$ . We take each midpoint  $x_i$  as the origin of a diminished copy  $T_i$  so that the diameter of  $T_i$  is less than  $1/2^{n+1}$  and that  $X_n$  has only the point  $x_i$  in common with the added copy  $T_i$ . All this can clearly be done so carefully that the resulting set  $X_{n+1}$ , when defined as the union of  $X_n$  and of all  $m(n)$  copies  $T_i$  for  $i \in \{j_n + 1, \dots, j_{n+1}\}$ , is a dendrite.

Note that  $\text{ord}(c, X_{n+1}) = 2$ , and define  $Y_{n+1} = \text{cl}(\text{comp}(c, X_{n+1} \setminus \{c\}))$ . Observe that  $\text{ord}(x_i, X_{n+1}) = i + 2$ , whence it follows by (3.10.5) that

**(3.10.6):** if  $x_{i_1} \in Y_n \subset X_{n+1}$  and  $x_{i_2} \in X_n \setminus Y_n \subset X_{n+1}$ , then  $\text{ord}(x_{i_1}, X_{n+1}) < \text{ord}(x_{i_2}, X_{n+1})$ .

It should be stressed here that labeling the midpoints of the considered segments so that condition (3.10.5) (and, consequently, condition (3.10.6)) is satisfied, we have defined the function mentioned before the formulation of Theorem 3.10 that guarantees the existence, for each  $n \in \mathbb{N}$ , of an open mapping from an open subset of  $X_n$  (namely from  $X_n \setminus Y_n$ ) onto  $Y_n \setminus \{c\}$ , whence it will follow that  $X$  is not openly chaotic.

Define  $f_n : X_{n+1} \rightarrow X_n$  by  $f_n|X_n = \text{id}|X_n$  and  $f_n(T_i) = \{x_i\}$  for each  $i \in \{j_n + 1, \dots, j_{n+1}\}$ . Thus  $f_n$  is a monotone retraction. Now define  $g_{n+1} : X_{n+1} \rightarrow Y_{n+1}$  as follows. First, let  $g_{n+1}|X_n = g_n$ . So, for each midpoint  $x_i \in X_n$ , where  $i \in \{j_n + 1, \dots, j_{n+1}\}$ , its image  $g_{n+1}(x_i) = g_n(x_i) \in Y_n \subset Y_{n+1}$  is already determined and, since by (3.10.4) the mapping  $g_n$  is linear on each segment mentioned there,

**(3.10.7):** the point  $g_{n+1}(x_i)$  coincides with  $x_j$  for some  $j \in \{j_n + 1, \dots, j_{n+1}\}$ .

In particular, since  $g_n$  is a retraction by (3.10.4), for each  $i \in \{j_n + 1, \dots, j_{n+1}\}$ , we have  $g_{n+1}(x_i) = x_i$  if  $x_i \in Y_n$ , and, by (3.10.6),

**(3.10.8):**  $\text{ord}(g_{n+1}(x_i), X_{n+1}) < \text{ord}(x_i, X_{n+1})$  if  $x_i \in X_n \setminus Y_n$ .

Second, let  $g_{n+1}|Y_{n+1} = \text{id}|Y_{n+1}$ . It remains to define  $g_{n+1}$  on each copy  $T_i$  in  $X_{n+1}$  whose origin  $x_i$  is in  $X_n \setminus Y_n$ . To this aim let  $x_j = g_{n+1}(x_i) \in Y_n$  according to (3.10.7), and note that  $j < i$  by (3.10.5). Let  $T_j = \bigcup\{A_j^m : m \in \{1, \dots, j\}\}$  and  $T_i = \bigcup\{A_i^m : m \in \{1, \dots, i\}\}$ , where  $A_j^m$  and  $A_i^m$  are arms of  $T_j$  and  $T_i$  respectively. Define  $g_{n+1}|A_i^m : A_i^m \rightarrow A_j^m$  for  $m \in \{1, \dots, j\}$  and  $g_{n+1}|A_i^m : A_i^m \rightarrow A_j^1$  for  $m \in \{j + 1, \dots, i\}$  as linear surjections. Thus the definition of  $g_{n+1} : X_{n+1} \rightarrow Y_{n+1}$  is complete, and it follows that  $g_{n+1}$  is an open retraction. It follows also from the definitions of the mappings that the diagram

$$(3.10.9) \quad \begin{array}{ccc} X_n & \xleftarrow{f_n} & X_{n+1} \\ g_n \downarrow & & \downarrow g_{n+1} \\ Y_n & \xleftarrow{f_n|Y_{n+1}} & Y_{n+1} \end{array}$$

is exact, which means that the diagram commutes and for every  $p_n \in X_n$  and

$q_{n+1} \in Y_{n+1}$  the condition  $g_n(p_n) = (f_n|_{Y_{n+1}})(q_{n+1})$  implies

$$(f_n)^{-1}(p_n) \cap g_{n+1}^{-1}(q_{n+1}) \neq \emptyset$$

(see [13, §3, IV, p. 19]). Finally let

$$(3.10.10) \quad X = \varprojlim \{X_n, f_n\}.$$

Then the continuum  $X$  as defined in (3.10.10) is homeomorphic to the one of (3.10.1) according to [1, Theorem I, p. 348]. Since each  $X_n$  is a dendrite and the bonding mappings are monotone,  $X$  is a dendrite [21, Theorem 4, Part 3, p. 229], and consequently

$$Y = \varprojlim \{Y_n, f_n|_{Y_{n+1}}\}$$

is a subdendrite of  $X$ . Further, it follows that

$$Y = \text{cl} \left( \bigcup \{Y_n : n \in \mathbb{N}\} \right) = \text{cl}(\text{comp}(a, X \setminus \{c\})).$$

Let  $g = \varprojlim g_n$ . Since for each  $n \in \mathbb{N}$  the diagram (3.10.9) is exact, it follows that  $g : X \rightarrow Y$  is continuous [8, Chapter 8, Theorem 3.13, p. 218], surjective (since for each  $n \in \mathbb{N}$  all four mappings in the diagram (3.10.9) are surjective), and open ([9], Theorem 3, p. 58]; see also [22, Theorem 4, p. 61]).

Take two disjoint open subsets of  $X$ , viz.  $X \setminus Y$  and  $Y \setminus \{c\}$ . Since open mappings are hereditary with respect to open subspaces, we see that the restriction  $g|(X \setminus Y) : X \setminus Y \rightarrow Y \setminus \{c\}$  is open. Then  $X$  is not openly chaotic according to condition (3.9.3) of Proposition 3.9. Finally, it follows from Statement 3.8 that  $X$  is chaotic (for another argument see [2, Section 5, p. 227-228]). The proof is complete.  $\square$

**Question 3.11.** Is the dendrite  $X$  of Theorem 3.10 semi-openly chaotic?

**Question 3.12.** Does there exist a dendrite of de Groot-Wille type which is a) semi-openly chaotic, b) openly chaotic?

**Question 3.13.** Note that each dendrite of de Groot-Wille type satisfies the condition

**(3.13.1):** for each  $k \in \mathbb{N}$  there exists  $m \in \mathbb{N}$  with  $m \geq k$  such that  $R_m(X) \neq \emptyset$ , which is a consequence of (3.8.3). Does there exist an openly chaotic dendrite satisfying (3.13.1)?

The chaotic dendrites considered above contain ramification points of arbitrarily great finite order (compare conditions (3.8.1) and (3.8.2) of Statement 3.8). Another type of chaotic or rigid dendrites are those that have orders of points

bounded. Constructions of such dendrites are known from Theorem 27 and Example 33 of [4], and Theorem 3.7 of [5, p. 185] (a construction of a chaotic dendrite given in [2, Statement 13, p. 231], is a special case of that described in Theorem 27 of [4]). It will be shown below that some of the above quoted results can be generalized from the class of homeomorphisms to that of open mappings. However, to show the generalization, an argument is used that depends on, and is strictly connected with, some structural properties of the dendrites. Thus it is necessary to repeat their construction. It should be stressed, however, that the main idea of this construction is taken from E. W. Miller's paper [20], while some parts of the proof given below imitate the argument presented in the proof of Statement 13 of [2], p. 231-234].

**Theorem 3.14.** *For any two integers  $m$  and  $n$  with  $3 \leq m < n$  there exists a chaotic, strongly rigid and not strongly chaotic dendrite  $X(m, n)$  such that*

- (3.14.1):  $\text{ord}(x, X(m, n)) \in \{1, 2, m, n\}$  for each point  $x \in X(m, n)$ ;
- (3.14.2): if  $\alpha \in \{1, 2, m, n\}$ , then  $\text{cl}(R_\alpha(X(m, n))) = X(m, n)$ ;
- (3.14.3): if  $C$  is a subcontinuum of  $X(m, n)$  such that there is an open mapping  $f : C \rightarrow f(C) \subset X(m, n)$  with  $\text{int}(f(C)) \neq \emptyset$ , then  $C = X(m, n)$  and  $f$  is the identity. Thus  $X(m, n)$  is openly rigid.

PROOF. First we define two auxiliary dendrites  $D_0$  and  $D_1$ . Within a straight line segment  $ab$  ordered from  $a$  to  $b$  by  $<$  we choose points  $a_i$  (where  $i \in \mathbb{N}$ ) so that

$$a_{i+1} < a_i \quad \text{and} \quad \lim_{i \rightarrow \infty} a_i = a.$$

Within each interval  $a_{i+1}a_i$  choose points  $a_{i,j}$  so that

$$a_{i,j} < a_{i,j+1} \quad \text{and} \quad \lim_{j \rightarrow \infty} a_{i,j} = a_i.$$

At each point  $a_i$  and  $a_{i,j}$  erect  $m - 2$  straight line segments mutually disjoint apart from these points and having only these points in common with the segment  $ab$ . Take the segments so that for any positive number  $\varepsilon$  only finitely many of them have length greater than  $\varepsilon$ . The set of points obtained in this way is called  $D_0$ . It is clear that  $D_0$  is a dendrite.

Everything is the same in the definition of  $D_1$  except that the points  $a_{i,j}$  are taken within the intervals  $a_{i+1}a_i$  so that

$$a_{i,j+1} < a_{i,j} \quad \text{and} \quad \lim_{j \rightarrow \infty} a_{i,j} = a_{i+1}.$$

So  $D_1$  is also a dendrite. The point  $a$  is called the *origin* of either  $D_0$  or  $D_1$ , and the straight line segments which we have erected are all referred to as straight line segments of rank 1.

The defined dendrites  $D_0$  and  $D_1$  start an inductive construction of dendrites  $D_{\gamma_1 \dots \gamma_k}$ , where  $k \in \mathbb{N}$  and  $\gamma_1 \dots \gamma_k$  is a zero-one sequence. Assume now that we have defined dendrites  $D_{\gamma_1 \dots \gamma_k}$  for some  $k \in \mathbb{N}$ . Assume furthermore that we have defined the expressions: the origin of  $D_{\gamma_1 \dots \gamma_k}$  and the straight line segments of rank  $k$  of  $D_{\gamma_1 \dots \gamma_k}$ . To define the set  $D_{\gamma_1 \dots \gamma_k 0}$  we proceed as follows. We replace each straight line segment of rank  $k$  of  $D_{\gamma_1 \dots \gamma_k}$  by a copy of  $D_0$  diminished so that the diameter of the copy equals the length of the straight line segment, and located in such a way that the origin of the copy of  $D_0$  is the foot of the erected straight line segment. Furthermore, we do this, as we clearly can, so that the resulting set  $D_{\gamma_1 \dots \gamma_k 0}$  is a dendrite. By the origin of  $D_{\gamma_1 \dots \gamma_k 0}$  we mean the origin of  $D_{\gamma_1 \dots \gamma_k}$ , and by the straight line segments of rank  $k + 1$  of  $D_{\gamma_1 \dots \gamma_k 0}$  we mean the segments of rank 1 of the sets  $D_0$  used in obtaining  $D_{\gamma_1 \dots \gamma_k 0}$  from  $D_{\gamma_1 \dots \gamma_k}$ .

The definition of  $D_{\gamma_1 \dots \gamma_k 1}$  is the same except that in obtaining  $D_{\gamma_1 \dots \gamma_k 1}$  from  $D_{\gamma_1 \dots \gamma_k}$  we use sets  $D_1$  instead of  $D_0$ . The inductive definition of  $D_{\gamma_1 \dots \gamma_k}$  for each  $k \in \mathbb{N}$  is thus finished.

Now define the desired dendrite  $X(m, n)$ . The construction uses the sequence of dendrites

$$D_0, D_{10}, D_{110}, \dots, D_{11\dots 10}, \dots$$

which we re-label in the same order as

$$W_1, W_2, W_3, \dots, W_k, \dots$$

We begin with the dendrite  $W_1$  whose origin is the point  $a$ , and we adjoin to it  $n - 1$  straight line segments  $ab_1, \dots, ab_{n-1}$  so that the only point which any two of the sets  $W_1, ab_1, \dots, ab_{n-1}$  have in common is the point  $a$ . Let us denote the resulting dendrite by  $X_1$ . Observe that just one of the  $n$  distinct arcs  $ab \subset W_1, ab_1, \dots, ab_{n-1}$  contained in  $X_1$  which meet at  $a$  (namely the arc  $ab$ ) has the property that there is a sequence of ramification points on it of Menger-Urysohn order  $m$  which converges to  $a$ .

Consider now an arbitrary maximal free arc in  $X_1$ . It is evident from the construction that every such an arc is a straight line segment. Denote the midpoint of this segment by  $x$ . We obtain, of course, a countable set of points  $x$ . With this countable set we associate, in a one-to-one way, the sets  $W_k$  of odd indices  $k$ , i.e.,

$$W_3, W_5, \dots, W_{2r+1}, \dots,$$

and take  $x$  as the origin of the associated set  $W_{2^{r+1}} = W(x)$  in such a way that  $X_1$  and  $W(x)$  have only the point  $x$  in common. Moreover, to the point  $x$  we attach  $n - 3$  straight line segments having  $x$  as one end point and having only  $x$  in common with  $W(x) \cup X_1$ . All this can be clearly done in such a way that the resulting set  $X_2$  is a dendrite. Observe that, for every point  $y$  of  $X_2$  of Menger-Urysohn order  $n$ , just one of the  $n$  essentially distinct arcs of  $X_2$  which meet at  $y$  (namely the arc contained in  $W(x)$  if  $y = x$  or in  $W_1$  if  $y = a$ ) has the property that there is a sequence of ramification points on it of order  $m$  which converges to  $y$ . Now,  $X_3$  is related to  $X_2$  in the same way as  $X_2$  is related to  $X_1$ , except that we make use of sets  $W_{2^{(2k+1)}}$  instead of sets  $W_{2^{k+1}}$ . In general,  $X_{i+1}$  is related to  $X_i$  in the same way as  $X_i$  is related to  $X_{i-1}$  except that we make use of sets  $W_{2^{i-1}(2k+1)}$  instead of sets  $W_{2^{i-2}(2k+1)}$ . It can be observed easily that, for every point  $y$  of  $X_i$  of order  $n$ , just one of the  $n$  essentially disjoint arcs in  $X_i$  which meet at  $y$  has the property mentioned previously. It is well known that such a construction can be carried through so that the closure of the union of the dendrites  $X_i$  successively obtained is itself a dendrite. We may assume then that

$$(3.14.4) \quad X(m, n) = \text{cl} \left( \bigcup \{X_i : i \in \mathbb{N}\} \right)$$

is a dendrite.

Now we intend to prove the needed properties of  $X(m, n)$ . We notice first that any ramification point of  $X(m, n)$  is either of order  $m$  or of order  $n$ . Thus (3.14.1) follows from the construction. The points of order  $n$  are the point  $a$  of  $X_1$  and the points  $x$  which arise at successive stages of the process of construction. We put

$$(3.14.5) \quad K = R_n(X(m, n)).$$

Since for each  $i \in \mathbb{N}$  we take in the construction of  $X_i$  the midpoints  $x$  of all maximal free arcs in  $X_i$ , the set  $K$  is dense in  $X(m, n)$ . Furthermore, notice that the above mentioned property of points of order  $n$  in each  $X_i$  is preserved in  $X$ . Precisely, if  $y$  is in  $K$ , then there is just one of  $n$  arcs in  $X(m, n)$  ending at  $y$  and mutually disjoint out of it, such that it contains a sequence of ramification points of order  $m$  converging to  $y$ . Thus an open neighbourhood about a point  $y \in K$  contains points of order  $m$  in  $X(m, n)$  and, henceforth, the density of the set  $R_m(X(m, n))$  in  $X(m, n)$  follows from the density of the set  $K$ . Consequently, we see that (3.8.1) holds true, which is equivalent, for dendrites, to  $\text{cl}(E(X(m, n))) = X(m, n)$  (see [3, Theorem 2.4, p. 167], cf. [6, Theorem 4.6, p. 10]). The set of points of order 2 is always dense in a dendrite ([19, p. 309]; cf. [14, §51, VI, Theorem 8, p. 302]). Thus (3.14.2) is shown.

It is proved in Theorem 27 of [4] that the dendrite  $X(m, n)$  is chaotic and strongly rigid, and an argument is given in [5, Remark 5.4, p. 185] that it is not strongly chaotic.

Now we prove that (3.14.3) holds. So, let  $C$  be a subcontinuum of  $X(m, n)$  and let  $f : C \rightarrow f(C) \subset X(m, n)$  be an open mapping such that  $\text{int}(f(C)) \neq \emptyset$ . Since the set  $K$  defined by (3.14.5) is dense, we have  $K \cap \text{int}(f(C)) \neq \emptyset$ . Recall that  $f$ , being open, does not increase order of a point of compact spaces ([24, Chapter 8, Corollary 7.31, p. 147]). Since no point of  $X(m, n)$  is of order greater than  $n$  and since  $K$  contains all points of order  $n$  of  $X(m, n)$ , for each point  $v \in K \cap \text{int}(f(C))$  there is a point  $u \in C \cap K$  such that  $f(u) = v$ . Since  $X(m, n)$  is locally connected at  $v$ , we can choose a closed connected neighbourhood  $V$  of  $v$  such that  $V \subset \text{int}(f(C))$ . Denote by  $U$  the component of  $f^{-1}(V)$  to which the point  $u$  belongs. It follows from [24, (7.5) p. 148] that  $f(U) = V$ .

We claim that

(3.14.6)                    the restriction  $f|U : U \rightarrow V$  is open.

Indeed, let  $Z$  be an open subset of  $U$ . Thus there is an open subset  $Z_0$  of  $C$  such that  $Z = Z_0 \cap U$ . Then, since  $f(U) = V$ , we infer that

$$(f|U)(Z) = (f|U)(Z_0 \cap U) = V \cap f(Z_0),$$

and since  $f(Z_0)$  is an open subset of  $f(C)$ , we see that  $(f|U)(Z)$  is an open subset of the range space  $V$ . So, the claim (3.14.6) is proved.

Thus  $f|U : U \rightarrow V$  is an open surjection defined on a dendrite  $U$ . Since every nonconstant open mapping defined on a dendrite is light, (see [6, Corollary 6.15, p. 25]) Theorem 2.4 of [24, p. 188] can be applied which says that if  $f$  is open and light mapping of a compact space, then for each dendrite  $D$  in the range and for each point  $x_0 \in f^{-1}(D)$  there is a dendrite  $E$  in the domain such that  $x_0 \in E$  and  $f|E : E \rightarrow D$  is a homeomorphism (compare also [6, Corollary 6.22, p. 26]). Thereby

**(3.14.7):** there is a dendrite  $E \subset U$  such that  $u \in E$  and that the restriction  $f|E : E \rightarrow V$  is a homeomorphism.

Let us assume for definiteness (by (3.14.7) the argument is similar in the opposite case) that the set  $W_i$  which has the point  $u$  as its origin is of lower index than the set  $W_j$  which has  $v$  as its origin. Consider now an arc  $ub_u \subset U$  which is the only arc of  $n$  arcs ending at  $u$  and pairwise disjoint out of  $u$  that contains a sequence of ramification points of order  $m$  converging to  $u$ . Let  $vb_v \subset V$  have a similar

meaning. Since

$$\text{ord}(u, X(m, n)) = \text{ord}(v, X(m, n)) = n,$$

it is clear by (3.14.7) that there are a subarc  $ub'_u$  of  $ub_u$  and a subarc  $vb'_v$  of  $vb_v$  such that  $f(ub'_u) = vb'_v$ . We can take  $b'_u$  so close to  $u$  and, similarly,  $b'_v$  so close to  $v$  that  $ub'_u$  and  $vb'_v$  are straight line segments. Any ramification point of  $X(m, n)$  on  $ub'_u$  is mapped under  $f$  into a ramification point of  $X(m, n)$  on  $vb'_v$ . If  $W(u) = W_1$ , we see that we have already reached a contradiction, by (3.14.7). For  $W_1 = D_0$  and  $W(v) = W_i = D_{111\dots 10}$ , which means that  $ub'_u$  contains ramification points being limit points of ramification points from the left, while  $vb'_v$  contains no such points. If  $W(u) = W_2$ , we fix our attention upon some one ramification point of  $X(m, n)$  interior to  $ub'_u$ . Let us denote this point by  $s_u$  and put  $s_v = f(s_u) \in vb'_v$ . Consider the straight line segments erected to  $ub'_u$  and  $vb'_v$  at  $s_u$  and  $s_v$  respectively. Denote these straight line segments by  $s_u t_u$  and  $s_v t_v$ . Now, since  $W_2 = D_{10}$ ,  $s_u$  is a limit point along  $s_u t_u$  of ramification points of  $X(m, n)$  which are, in turn, limit points of ramification points of  $X(m, n)$  from below along  $s_u t_u$ , while  $s_v t_v$  contains no such points since  $W(v) = W_i = D_{111\dots 10}$ . It follows from (3.14.7) that the argument exemplified above can be extended to apply to the general case where  $W(u) = W_i$  and  $W(v) = W_j$  for  $i < j$  and  $j < i$ , respectively. Therefore we conclude that  $u = v$  if  $u, v \in K$  and  $f(u) = v$ . Thus  $f$  is the identity on  $K$  which is dense in  $X(m, n)$ , whence it follows that  $f$  is the identity on  $X(m, n)$ .

In particular, if  $C = X(m, n) = f(C)$ , then the condition  $\text{int}(f(C)) \neq \emptyset$  is obviously satisfied, whence it follows that each open autosurjection on  $X(m, n)$  is the identity, and therefore  $X(m, n)$  is openly rigid. The proof is complete.  $\square$

**Remark 3.15.** The assumption  $\text{int}(f(C)) \neq \emptyset$  is essential in (3.14.3) because condition (3.14.2) implies that the set  $R(X(m, n))$  is dense in  $X(m, n)$  and therefore  $X(m, n)$  contains a homeomorphic copy of the standard universal dendrite  $D_3$  of order 3 (see [3, Proposition 3.2, p. 169]), and since by (3.14.1) for each point  $x \in X(m, n)$  the order  $\text{ord}(x, X(m, n))$  is finite, there exists an open mapping  $f : X(m, n) \rightarrow D_3 \subset X(m, n)$ . This shows that the dendrite  $X(m, n)$  being openly rigid is not strongly openly rigid.

**Question 3.16.** Is the dendrite  $X(m, n)$  of Theorem 3.14 openly chaotic?

Our next result generalizes Theorem 3.7 of [5], p. 185]. Namely, the same construction as in that theorem (which is, in fact, a modification of the construction recalled above in the proof of Theorem 3.14, with changing the role of numbers



$m$  and  $n$  in the definition of dendrites  $X(m, n)$  of Theorem 3.14 to get some extra properties needed in a proof) as well as a very similar proof (based on condition (3.14.7)) lead to this more general result. Thus the details of the proof are omitted.

**Theorem 3.17.** *For any two integers  $m$  and  $n$  with  $3 \leq m < n$  there exists a strongly chaotic and openly rigid dendrite  $Y(m, n)$  such that*

- (3.17.1):  $\text{ord}(x, Y(m, n)) \in \{1, 2, m, n\}$  for each point  $x \in Y(m, n)$ ;
- (3.17.2): if  $\alpha \in \{1, 2, m, n\}$ , then  $\text{cl}(R_\alpha(Y(m, n))) = Y(m, n)$ ;
- (3.17.3): for every arc  $A$  in  $Y(m, n)$  we have  $A \cap R_m(Y(m, n)) \neq \emptyset$ .

**Question 3.18.** Is the dendrite  $Y(m, n)$  of Theorem 3.17 openly chaotic?

#### 4. MONOTONE MAPPINGS OF DENDRITES

Since strongly rigid (thus rigid) dendrites as well as strongly chaotic (thus chaotic) ones do exist in profusion (even having various additional properties: see [4, Theorem 27 and Example 33]; [5, Theorem 5.5, p. 185]), it is interesting to know whether or not these existence results can be extended from homeomorphisms to other classes of mappings. Mappings which are relatively close to homeomorphisms are local homeomorphisms. However, each local homeomorphism of a continuum onto a dendrite is a homeomorphism (see [24, Chapter 10, Corollary, p. 199]; for generalizations *a*) with a  $\lambda$ -dendroid as the range space see [15, Corollary 10, p. 858]; *b*) with a tree-like continuum as the domain or the range space see [16, Theorem, p. 64 and Corollary, p. 67]). Thus considering  $\mathcal{M}$  as the class of local homeomorphisms with  $X$  as an arbitrary tree-like continuum is not interesting from our point of view.

Another class of mappings that are generalizations of homeomorphisms is the class of monotone ones. We will show that the mentioned results cannot be extended to this class of mappings: no nondegenerate dendrite is (strongly) chaotic or (strongly) rigid with respect to monotone mappings.

We start our study of monotone mappings of dendrites with a theorem of a more general nature.

**Theorem 4.1.** *Let a compact space  $X$  have the following properties*

- (4.1.1): *the set of all points of  $X$  at which  $X$  is locally connected is dense in  $X$ ;*
- (4.1.2): *for each open connected subset  $C$  of  $X$  and for each monotone mapping  $f: X \rightarrow X$  the restriction  $f|_C: C \rightarrow f(C) \subset X$  is also monotone.*

*Then, if  $X$  is (strongly) chaotic with respect to monotone mappings, then it is also (strongly) rigid with respect to this class of mappings.*

PROOF. Suppose  $X$  is not rigid with respect to monotone mappings. Then there exists a monotone surjection  $f : X \rightarrow X$  distinct from the identity. Thus there is a point  $p \in X$  such that  $p \neq f(p) = q$ . Since  $X$ , being chaotic with respect to monotone mappings, is a Hausdorff space by Proposition 3.9, there are disjoint open neighborhoods  $U$  and  $V$  of points  $p$  and  $q$  respectively. Since  $X$  is locally connected at each point of a dense subset of  $X$ , there is a point  $q' \in V$  at which  $X$  is locally connected. Hence there is an open connected neighbourhood  $V' \subset V$  of  $q'$ . Let  $U' = f^{-1}(V')$ . Then, by continuity of  $f$ , the set  $U'$  is open. Further, since  $X$  is compact and  $f$  is monotone,  $U'$  is connected as the inverse image of a connected set ([24, Chapter 8, (2.2), p. 138]). Hence, by (4.1.2), the restriction  $f|_{U'} : U' \rightarrow V'$  is a monotone surjection. Note that  $U' \neq V'$ , because otherwise we would have  $p \in U' = V' \subset V$  and  $p \in U$ , a contradiction with  $U \cap V = \emptyset$ . Applying the equivalence of conditions (3.9.1) and (3.9.2) of Proposition 3.9 we see that  $X$  is not chaotic with respect to monotone mappings.

Suppose now  $X$  is not strongly rigid with respect to monotone mappings. Then there exists a monotone mapping  $f : X \rightarrow X$  which is not the identity. Thus there is a point  $p \in X$  such that  $p \neq f(p) = q$ . Take two disjoint open neighborhoods  $U$  and  $V$  of points  $p$  and  $q$  respectively. By (4.1.1) and by continuity of  $f$  there is an open connected set  $U' \subset U$  such that  $f(U') \subset V$ . By (4.1.2) the restriction  $f|_{U'} : U' \rightarrow f(U') \subset V$  is also monotone. Thus  $X$  is not strongly chaotic with respect to monotone mappings. The proof is complete.  $\square$

Theorem 4.1 implies the following result.

**Theorem 4.2.** *Let a compact space  $X$  satisfy conditions (4.1.1) and (4.1.2), and let  $\mathcal{M}$  denote the class of monotone mappings. Then the four implications (3.4.1) hold.*

Before proving the next result we show an easy lemma.

**Lemma 4.3.** *Let a monotone mapping  $f : X \rightarrow Y$  be defined on a dendrite  $X$ . Then, for each connected subset  $U$  of  $X$ , the restriction  $f|_U : U \rightarrow f(U) \subset Y$  is also monotone.*

PROOF. For each point  $y \in f(U)$  we have  $(f|_U)^{-1}(y) = f^{-1}(y) \cap U$ . Both  $f^{-1}(y)$  and  $U$  are connected subsets of the dendrite  $X$ . Since a continuum is a dendrite if and only if the intersection of two connected sets is connected (see [24, Chapter 5, (1.1) (v), p. 88]), the conclusion follows.  $\square$

Note that each dendrite satisfies condition (4.1.1) and, by Lemma 4.3, also condition (4.1.2). Thus we see by Theorem 4.1 that the implications (3.4.1) of Theorem 4.2 hold true if  $X$  is a dendrite. But it will be seen from further results that for dendrites all of these implications are satisfied vacuously. To prove this recall several definitions and results.

Given a class  $\mathcal{M}$  of mappings, we say that topological spaces  $X_1$  and  $X_2$  are  $\mathcal{M}$ -equivalent provided that there are in  $\mathcal{M}$  surjections from  $X_1$  onto  $X_2$  and from  $X_2$  onto  $X_1$ . If  $\mathcal{M}$  is the class of monotone mappings, then we say that  $X_1$  and  $X_2$  are *monotone equivalent*.

We denote by  $D_3$  the *standard universal dendrite of order 3*, i.e., a dendrite  $X$  characterized by the following two conditions (see [23, Chapter K, p. 137]; [19, Chapter 10, §6, p. 318]; [3, Section 3, p. 167-169])

(4.3.1): each ramification point of  $X$  is of order 3, i.e.,  $R(X) = R_3(X)$ ,

and

(4.3.2): for each arc  $A \subset X$  we have  $\text{cl}(A \cap R(X)) = A$ .

It is known that if a dendrite  $X$  satisfies (4.3.1), then it can be embedded into  $D_3$ .

The following proposition is a particular case of [3, Theorem 6.7, p. 180].

**Proposition 4.4.** *If a dendrite  $X$  satisfies condition (3.8.1), then  $X$  is monotone equivalent to  $D_3$ .*

Let  $\mathcal{M}$  be a class of mappings between topological spaces. A space  $X$  is said to be *homogeneous with respect to  $\mathcal{M}$*  provided that for every two points  $p$  and  $q$  of  $X$  there is a surjective mapping  $f : X \rightarrow X$  such that  $f(p) = q$  and  $f \in \mathcal{M}$ . Kato has proved the following result (see [11, Example 2.4, p. 59] and [12, Proposition 2.4, p. 223]; for a generalization see [3, Theorem 7.1, p. 186]).

**Proposition 4.5** (H. Kato). *The standard universal dendrite  $D_3$  of order 3 is homogeneous with respect to monotone mappings.*

The author is obliged to Professor Alejandro Illanes for the formulation and a fruitful discussion of the following theorem.

**Theorem 4.6** (A. Illanes). *Let a dendrite  $X$  satisfy condition (3.8.1). Then for each point  $p \in X \setminus E(X)$  and for any two components  $C_1$  and  $C_2$  of  $X \setminus \{p\}$  there is a monotone mapping  $f : X \rightarrow X$  such that*

$$(4.6.1) \quad f(C_1) = C_2, \quad f(C_2) = C_1, \quad \text{and} \quad f|(X \setminus (C_1 \cup C_2)) = \text{id}|(X \setminus (C_1 \cup C_2)).$$

PROOF. Since  $p$  is not an end point of  $X$ , it is a cut point [24, Chapter 5, (1.1), (ii), p. 88], and thus there are at least two components of  $X \setminus \{p\}$ . Take two of them,  $C_1$  and  $C_2$ , and note that

$$\text{cl}(C_i) = C_i \cup \{p\} \quad \text{for } i \in \{1, 2\},$$

and that, by (3.8.1),

$$(4.6.2) \quad \text{cl}(R(\text{cl}(C_i))) = \text{cl}(C_i) \quad \text{for } i \in \{1, 2\}.$$

By (4.6.2) and Proposition 4.4 each of the dendrites  $\text{cl}(C_1)$  and  $\text{cl}(C_2)$  is monotone equivalent to  $D_3$ . In particular there are monotone surjective mappings  $g_1 : \text{cl}(C_1) \rightarrow D_3$  and  $g_2 : D_3 \rightarrow \text{cl}(C_2)$ . Let  $q \in g_2^{-1}(p)$ . By Proposition 4.5 there is a monotone mapping  $g : D_3 \rightarrow D_3$  such that  $g(g_1(p)) = q$ . Defining  $f_1 = g_2 \circ g \circ g_1$  we see that  $f_1 : \text{cl}(C_1) \rightarrow \text{cl}(C_2)$  is a monotone surjection with  $f_1(p) = p$ . Similarly we can find a monotone surjection  $f_2 : \text{cl}(C_2) \rightarrow \text{cl}(C_1)$  such that  $f_2(p) = p$ . Define  $f : X \rightarrow X$  putting

$$f|_{C_i} = f_i \quad \text{for } i \in \{1, 2\} \quad \text{and} \quad f|_{X \setminus (C_1 \cup C_2)} = \text{id}|_{(X \setminus (C_1 \cup C_2))}.$$

It is obvious that  $f$  is a continuous monotone mapping satisfying condition (4.6.1). The proof is finished.  $\square$

It is observed in Proposition 25 of [4] that each rigid dendrite has a dense set of ramification points. In fact, if (3.8.1) does not hold for a dendrite  $X$ , the closure of each component of  $X \setminus \text{cl}(R(X))$  is a free arc  $A$ , and therefore the needed homeomorphism  $h : X \rightarrow X$  can be defined so that  $h|_{X \setminus A}$  is the identity and  $h|_A$  is not. Thus we have an observation.

**Observation 4.7.** *If a dendrite  $X$  does not satisfy condition (3.8.1), then there exists a homeomorphism of  $X$  onto itself which is not the identity.*

**Corollary 4.8.** *For each dendrite  $X$  there exists a monotone surjection  $f : X \rightarrow X$  which is not the identity.*

PROOF. Consider two cases. If (3.8.1) holds, the conclusion follows by Theorem 4.6. If (3.8.1) is not satisfied, then there is a free arc  $A$  in  $X \setminus \text{cl}(R(X))$  and therefore the needed monotone surjection can be defined as a homeomorphism  $h : X \rightarrow X$  such that  $h|_{\text{cl}(R(X))}$  is the identity, and  $h|_A : A \rightarrow A$  is not.  $\square$

**Corollary 4.9.** *Any dendrite is not rigid (and thus by (3.4.1) neither strongly rigid, nor chaotic nor strongly chaotic) with respect to monotone mappings.*

Investigating rigidity phenomena for monotone mappings of acyclic continua, in particular for acyclic curves, it is natural (in the light of Corollary 4.9) to extend the area of interest to curves which are not necessarily locally connected. A class of such curves, that includes dendrites, is the one of *dendroids*, i.e., arcwise connected and hereditarily unicoherent continua. So, the following question seems to be both interesting and natural.

**Question 4.10.** Do there exist dendroids which are (strongly) chaotic or (strongly) rigid with respect to monotone mappings?

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