Houston Journal of Mathematics
Volume 26, No. 4, 2000

HEREDITARILY WEAKLY CONFLUENT MAPPINGS ONTO S^1

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Communicated by Andrzej Lelek

Abstract. Results are obtained about the existence and behavior of hereditarily weakly confluent maps of continua onto the unit circle S^1. A simple and useful necessary and sufficient condition is given for a map of a continuum, X, onto S^1 to be hereditarily weakly confluent (HWC). It is shown that when X is arcwise connected, an HWC map of X onto S^1 is monotone with arcwise connected fibers. A number of theorems about HWC irreducible maps of X onto S^1 are proved; for example, such maps are monotone with nowhere dense fibers, and a complete determination of the structure of X is obtained when X admits an HWC irreducible map onto S^1 and X is arcwise connected. Among other results, the arcwise connected semi-locally-connected continua that admit an HWC map onto S^1 are completely determined, and it is shown how the map must be defined.

1. Introduction

A *continuum* is a nonempty compact connected metric space. A *map* is a continuous function. The symbol S^1 denotes the unit circle in the plane; $\mathbb{R}^n$ denotes Euclidean n-space.

Let X and Y be continua. A map $f : X \to Y$ is said to be *weakly confluent* (WC) provided that every subcontinuum of Y is the image under f of a subcontinuum of X. This type of map has been investigated extensively (e.g., see [M2]). Note that WC maps are a fundamental type of map to consider since they are the maps $f : X \to Y$ for which the natural induced map $\hat{f} : C(X) \to C(Y)$ is a surjection (where $C(X)$ and $C(Y)$ are the hyperspaces of X and Y [N1]). Now,

1991 *Mathematics Subject Classification.* Primary 54F15, 54C05, 54E40.
Key words and phrases. compactification, continuum, continuumwise accessible, cyclic element, cyclic element retraction, essential map, hereditarily weakly confluent map, hyperspace, irreducible map, irreducibly essential map, locally connected continuum, monotone map, semi-locally-connected continuum, weakly confluent map.

a map $f : X \to Y$ is *hereditarily weakly confluent* (HWC) provided that for every subcontinuum K of X,

$$f|K : K \to f(K) \text{ is WC.}$$

Note that HWC maps are those maps $f : X \to Y$ for which the induced map $\hat{f} : C(X) \to C(Y)$ is a surjection on $C(K)$ to $C(f(K))$ for all subcontinua K of X. HWC maps were first studied in [Wa] under the name pseudo-monotone (see 1.2 of [M1, p. 124]).

In sections 2 and 3, we obtain general results about HWC maps and irreducible HWC maps of continua onto the unit circle S^1; we use the first theorem in each of these sections throughout most of the rest of the paper. In sections 4 and 5, we present an extensive study of HWC maps of arcwise connected continua onto S^1. In section 6, as an application, we show that there is no HWC map of the cartesian product of any two nondegenerate continua onto S^1. In section 7 we prove a preliminary result about cyclic element retractions for use in section 8. In section 8 we use previous results to investigate HWC maps of arcwise connected semi-locally-connected continua onto S^1; we show that any HWC map of such a continuum onto S^1 has an especially simple form (Theorem 8.4), and we completely determine which such continua can be mapped onto S^1 by an HWC map (Corollary 8.5). In the final section we mention a problem whose solution would be a natural sequel to what we have done.

Several of our theorems limit the types of continua that can be mapped onto S^1 by an HWC map (notably, Theorem 6.2 and Corollary 8.5). In contrast, we point out that *every* nondegenerate continuum can be mapped onto S^1 by a WC map (see 13.68(c) of [N2, p. 309]).

We note the following terminology and notation.

A map $f : X \to S^1$ is *essential* provided that f is not homotopic to a constant map; otherwise, the map f is said to be *inessential.* Note that a map $f : X \to S^1$ is inessential if and only if it has a lift ψ, *i.e.*, ψ is a map of X into the reals $\mathbb{R}^1$ such that

$$f = \exp \circ \psi$$

where exp: $\mathbb{R}^1 \to S^1$ is given by $\exp(t) = (\cos(2\pi t), \sin(2\pi t))$ for all $t \in \mathbb{R}^1$ (Theorem 3 of [K, p. 426]).

We use the notation $f : X \twoheadrightarrow Y$ to mean f maps X onto Y. We use $f|A$ to denote the restriction of a map f to A.

The symbol $\overline{A}$ denotes the closure of A. For $A \subset X$, $Bd(A)$ denotes the boundary of A in X, *i.e.*, $Bd(A) = \overline{A} \cap \overline{(X - A)}$. For $A \subset X$, $\text{int}(A)$ denotes the interior of A in X. For spaces X and Y, $X \times Y$ denotes the cartesian product of X and Y.

Let X and Y be continua. A map $f : X \twoheadrightarrow Y$ is said to be *monotone* provided that $f^{-1}(y)$ is a continuum for each $y \in Y$. We note that if $f : X \twoheadrightarrow Y$ is monotone, then $f^{-1}(C)$ is a continuum whenever C is a subcontinuum of Y (2.2 of [W, p. 138]).

All spaces in this paper are assumed to be metric spaces. A space is said to be *nondegenerate* provided that it consists of more than one point.

Other terminology will be defined at appropriate places or can be found in the references.

2. A Characterization Theorem for HWC Maps onto S^1

We prove Theorem 2.1, which we use often throughout the paper. We include some simple consequences of Theorem 2.1 here.

Theorem 2.1. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be a map. Then, f is HWC if and only if for all subcontinua K of X such that $f(K) = S^1$, $f|K : K \twoheadrightarrow S^1$ is essential.*

Proof. First, assume that there is a subcontinuum K of X such that $f(K) = S^1$ and $f|K : K \twoheadrightarrow S^1$ is inessential. Then $f|K$ has a lift ψ, i.e., $f|K = \exp \circ \psi$ (see section 1). Thus, since $f(K) = S^1$, the interval $\psi(K)$ contains a closed subinterval, J, whose length is exactly 2π. Since $\psi : K \twoheadrightarrow \psi(K)$ is WC [R, p. 236], there is a subcontinuum, L, of K such that $\psi(L) = J$. Since the length of J is exactly 2π, clearly

(*) $\exp |J : J \twoheadrightarrow S^1$ is not WC.

Note that since $\psi(L) = J$, $f|L = (\exp|J) \circ (\psi|L)$. Hence, it follows from $(*)$that $f|L : L \twoheadrightarrow S^1$ is not WC. Therefore, f is not HWC. This proves half of Theorem 2.1. The other half follows easily using that essential maps of continua onto S^1 are WC (Lemma 6 of [F, p. 6]) and that all maps of continua onto arcs are WC [R, p. 236]. □

Corollary 2.2. *A map f of a simple closed curve onto S^1 is HWC if and only if f is a homeomorphism.*

PROOF. Obviously, homeomorphisms are HWC. Conversely, it is easy to verify that a map of a simple closed curve onto S^1 that is not one-to- one must map an arc onto S^1; hence, such a map can not be HWC by Theorem 2.1. □

Our next two corollaries involve the following notions: Let X and Y be continua. A map $f : X \twoheadrightarrow Y$ is *irreducible* provided that there is no proper subcontinuum, K of X such that $f(K) = Y$ [W, p. 162]. A map $f : X \to S^1$ is *irreducibly essential* on X provided that f is essential and, for every closed proper subset K of X, $f|K : K \to S^1$ is inessential; equivalently, by 5.51 of [W, p. 223], provided that (#) f is essential and, for every proper subcontinuum K of X, $f|K : K \to S^1$ is inessential.

For the sake of clarity concerning the definitions just given, we make the following observations: A map f of a continuum onto S^1 such that f is both irreducible and essential must be irreducibly essential (use (#) above). However, an irreducibly essential map of a continuum onto S^1 need not be irreducible (*e.g.*, $f : S^1 \twoheadrightarrow S^1$ given by $f(z) = z^2$ for each $z \in S^1$). Nevertheless, we have the following consequence of Theorem 2.1:

Corollary 2.3. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Then, f is irreducible if and only if f is irreducibly essential.*

PROOF. Assume that f is irreducible; then, since f is also essential by Theorem 2.1, f is irreducibly essential by the first observation in the preceding paragraph. The converse follows immediately from Theorem 2.1. □

Corollary 2.4. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be an irreducible map. Then, (1)-(3) are equivalent: (1) f is WC; (2) f is HWC; (3) f is essential.*

PROOF. Since $f : X \twoheadrightarrow S^1$ is irreducible, (2) and (3) are equivalent by Theorem 2.1 and (1) implies (2) using [R, p. 236]. Clearly, (2) implies (1). □

We continue our study of irreducible HWC maps onto S^1 in sections 3 and 5; in relation to Corollary 2.4, see Corollary 3.2. As we will see in sections 4 and 8, our theorems about irreducible HWC maps yield theorems about HWC maps which are not (necessarily) irreducible.

We complete this section with a somewhat general example of HWC maps onto S^1. We use special cases of the example later.

Example 2.5. Let X be a continuum that contains an open subset, U, such that U is homeomorphic to $\mathbb{R}^1$; also, assume that $X - U$ and $\overline{U} - U$ are continua. Let $Y = X/_{(X-U)}$ (the quotient space of X obtained by shrinking $X - U$ to a point, e.g., 3.14 of [N2, p. 41]); let $f : X \twoheadrightarrow Y$ be the quotient map. Since Y is obviously the one-point compactification of $\mathbb{R}^1$, Y is homeomorphic to S^1. We show that

$$f : X \twoheadrightarrow Y \text{ is HWC.}$$

To show this, we use Theorem 2.1, as follows: Let K be a subcontinuum of X such that $f(K) = Y$. Then, $K \supset U$; hence, $K \supset \overline{U}$. Note that any two points of U separate $\overline{U}$ into exactly two components; hence, it is easy to apply 12.40 of [N2, p. 259] to see that $f|\overline{U} : \overline{U} \twoheadrightarrow Y$ is essential. Thus, since $K \supset \overline{U}$, $f|K : K \twoheadrightarrow Y$ is essential. Therefore, by Theorem 2.1, $f : X \twoheadrightarrow Y$ is HWC.

3. A Theorem about Irreducible HWC Maps onto S^1

In the preceding section we proved two results about irreducible HWC maps onto S^1 (Corollary 2.3 and Corollary 2.4). We now prove a theorem which gives much more insight into the behavior of such maps (Theorem 3.1). We will use the theorem in other sections; in addition, the theorem leads to the characterization near the end of this section (Corollary 3.2). With respect to the fact that f is monotone in Theorem 3.1, see Example 3.4.

Theorem 3.1. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be HWC and irreducible. Then, f is monotone and each point inverse, $f^{-1}(z)$ for $z \in S^1$, is nowhere dense in X. Furthermore, f is constant on any given nowhere dense subcontinuum of X.*

Proof. We divide the proof of Theorem 3.1 into five steps. Throughout the proof, we assume that $f : X \twoheadrightarrow S^1$ is as in the hypothesis of Theorem 3.1. For Steps 1-3, we assume that $p, q \in S^1$ with $p \neq q$, and that α and β are subarcs of S^1 such that $S^1 = \alpha \cup \beta$ and $\alpha \cap \beta = \{p, q\}$ (as in figure at top of next page).

Step 1. *There is a sequence, $\{E_n\}_{n=1}^{\infty}$, of subcontinua of X satisfying (1)-(4) below:*

(1) *$E_n \subset X - f^{-1}(p)$ for each n;*
(2) *$q \in f(E_n)$ for each n;*
(3) *$lim_{n\to\infty} E_n = X$, hence $\cup_{n=1}^{\infty} E_n$ is dense in X;*
(4) *$f^{-1}(p)$ is nowhere dense in X.*

Proof. Let $\{J_n\}_{n=1}^{\infty}$ be a sequence of arcs in S^1 such that $q \in J_n$ for each $n, p \notin J_n$ for any n, and $\lim_{n\to\infty} J_n = S^1$. Since f is WC, there is, for each n, a subcontinuum E_n of X such that $f(E_n) = J_n$. Since $C(X)$ is compact [N1, p. 7], we may assume (by replacing $\{E_n\}_{n=1}^{\infty}$ with a convergent subsequence if

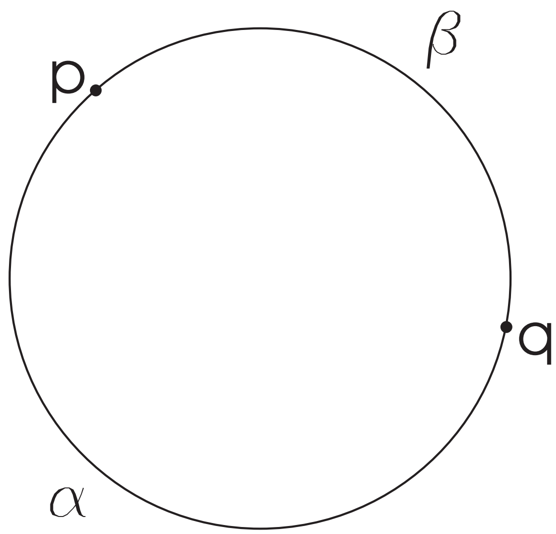

Figure 1. (For Steps 1-3 of Proof of 3.1.)

necessary) that $\{E_n\}_{n=1}^{\infty}$ converges to a subcontinuum L of X. Then, using the uniform continuity of f (compare with 0.49 of [N1, p. 23]),

$$f(L) = f(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} f(E_n) = \lim_{n\to\infty} J_n = S^1.$$

Thus, since $f : X \twoheadrightarrow S^1$ is irreducible and L is a subcontinuum of X, $L = X$. Therefore, since $\lim_{n\to\infty} E_n = L$, $\lim_{n\to\infty} E_n = X$. This proves that the continua E_n satisfy (3) of Step 1. Since $f(E_n) = J_n$ and since $p \notin J_n$ and $q \in J_n$, we see that (1) and (2) hold. Finally, (4) follows from (3) and (1) since $f^{-1}(p)$ is closed in X.

Step 2. *Let* $A = \overline{f^{-1}(\alpha - \{p\})}$ *and* $B = \overline{f^{-1}(\beta - \{p\})}$. *Then,* $A \cap B \cap f^{-1}(p) \neq \varnothing$.

Proof. Fix $s \in \mathbb{R}^1$ such that $\exp(s) = p$. Let t denote the unique point in $[s, s+2\pi]$ such that $\exp(t) = q$. Then, by switching the labeling of α and β if necessary, we may assume that

$$\exp([s,t]) = \alpha \text{ and } \exp([t, s+2\pi]) = \beta.$$

Now, define $\psi_A : A \to [s,t]$ and $\psi_B : B \to [t, s+2\pi]$ as follows:

$$\psi_A = (\exp|[s,t])^{-1} \circ (f|A) \text{ and } \psi_B = (\exp|[t, s+2\pi])^{-1} \circ (f|B).$$

Note the following fact:

(a) ψ_A and ψ_B are lifts of $f|A$ and $f|B$, respectively.

Also, note the facts in (b)-(d) below ((b) follows from (4) of Step 1 since $\alpha \cup \beta = S^1$, (c) follows from the continuity of f using that $\alpha \cap \beta = \{p,q\}$, and (d) follows from the definitions of ψ_A, ψ_B, and t):

(b) $A \cup B = X$;
(c) $A \cap B \subset f^{-1}(p) \cup f^{-1}(q)$;
(d) $\psi_A(x) = t = \psi_B(x)$ for all $x \in f^{-1}(q)$.

Now, suppose that the conclusion of Step 2 is false. Then, by (c), $A \cap B \subset f^{-1}(q)$. Hence, letting

$$\psi(x) = \begin{cases} \psi_A(x), & \text{if } x \in A \\ \psi_B(x), & \text{if } x \in B \end{cases}$$

we see from (d) that ψ is a function. Hence, by (a) and (b), ψ is a lift of f; thus, f is inessential. Therefore, since $f : X \twoheadrightarrow S^1$ is HWC, we have a contradiction to Theorem 2.1.

Step 3. *There are subcontinua $K(\alpha)$ and $K(\beta)$ of X such that $f(K(\alpha)) = \alpha$, $f(K(\beta)) = \beta$, and $K(\alpha) \cup K(\beta) = X$.*

Proof. Let A and B be as in Step 2. Then, by Step 2, there exists $x_0 \in A \cap B \cap f^{-1}(p)$. For a given integer $i > 0$, let U_i be an open neighborhood in X of x_0 of diameter less than $1/i$. Let

$$G = f^{-1}(\alpha - \{p,q\}), \quad V_i = U_i \cap G, \quad H = f^{-1}(\beta - \{p,q\}), \quad W_i = U_i \cap H.$$

Since $x_0 \in A$ and $U_i - f^{-1}(q)$ is an open neighborhood of x_0, $V_i \neq \varnothing$. Similarly, $W_i \neq \varnothing$. Hence, by (3) of Step 1 (V_i and W_i being open in X), there is a positive integer $n = n(i)$ such that

$$E_n \cap V_i \neq \varnothing \text{ and } E_n \cap W_i \neq \varnothing.$$

Let $a_i \in E_n \cap V_i$ and let $b_i \in E_n \cap W_i$. Note that $a_i \in E_n \cap G$ and $E_n \cap G$ is open in E_n; also, since $b_i \in E_n \cap H$ and $H \cap G = \varnothing$, we know that $E_n \cap G \neq E_n$. Hence, letting A_i denote the component of $\overline{E_n \cap G}$ containing a_i, we have by the Boundary Bumping Theorem in 5.4 of [N2, p. 73] that

$$A_i \cap (E_n - G) \neq \varnothing.$$

Hence, $A_i \cap f^{-1}(\{p, q\}) \neq \varnothing$. Thus, since $A_i \subset E_n$, we see from (1) of Step 1 that $A_i \cap f^{-1}(q) \neq \varnothing$. Therefore, we have produced a subcontinuum A_i of $f^{-1}(\alpha)$ such that $A_i \cap U_i \neq \varnothing$ and $A_i \cap f^{-1}(q) \neq \varnothing$. Similarly, there is a subcontinuum B_i of $f^{-1}(\beta)$ such that $B_i \cap U_i \neq \varnothing$ and $B_i \cap f^{-1}(q) \neq \varnothing$.

Now, having produced A_i and B_i as above for each $i = 1, 2, \ldots$, let $K(\alpha)$, respectively $K(\beta)$, be the limit of a convergent subsequence of $\{A_i\}_{i=1}^{\infty}$, respectively $\{B_i\}_{i=1}^{\infty}$ [N1, p. 7]. Note $(i) - (iv)$ below. Since $K(\alpha)$ and $K(\beta)$ are continua [N1, p. 7], we have that

(i) $K(\alpha)$, and $K(\beta)$ are subcontinua of $f^{-1}(\alpha)$ and $f^{-1}(\beta)$, respectively.

Since $A_i \cap U_i \neq \varnothing$ and $B_i \cap U_i \neq \varnothing$ for each i, $x_0 \in K(\alpha) \cap K(\beta)$. Hence, by (i),

(ii) $K(\alpha) \cup K(\beta)$ is a subcontinuum of X

and, since $f(x_0) = p$,

(iii) $p \in f(K(\alpha))$ and $p \in f(K(\beta))$.

Since $A_i \cap f^{-1}(q) \neq \varnothing$ and $B_i \cap f^{-1}(q) \neq \varnothing$ for each i,

(iv) $q \in f(K(\alpha))$ and $q \in f(K(\beta))$.

By (i), (iii), and (iv), we see that $f(K(\alpha)) = \alpha$ and $f(K(\beta)) = \beta$. Hence,

$$f(K(\alpha) \cup K(\beta)) = \alpha \cup \beta = S^1.$$

Therefore, since f is an irreducible map, we see from (ii) that $K(\alpha) \cup K(\beta) = X$.

Step 4. *The map f is monotone.*

Proof. Let $z \in S^1$. Let $\{\alpha_i\}_{i=1}^{\infty}$ be a sequence of arcs in S^1 satisfying (1)- (3) below:

(1) $z \in \mathrm{int}(\alpha_i)$ for all i;
(2) $\alpha_{i+1} \subset \mathrm{int}(\alpha_i)$ for all i;
(3) $\cap_{i=1}^{\infty} \alpha_i = \{z\}$.

For each i, let p_i and q_i denote the end points of α_i and let β_i be the arc in S^1 such that $\alpha_i \cup \beta_i = S^1$ and $\alpha_i \cap \beta_i = \{p_i, q_i\}$. For each i, let $K(\alpha_i)$ and $K(\beta_i)$ be as guaranteed by Step 3, and let $A_i = K(\alpha_i)$ and $B_i = K(\beta_i)$; therefore,

(4) A_i and B_i are subcontinua of X;
(5) $f(A_i) = \alpha_i$ for all i;
(6) $f(B_i) = \beta_i$ for all i,
(7) $A_i \cup B_i = X$ for all i.

Now, fix i. Let $x \in A_{i+1}$. By (5), $f(x) \in \alpha_{i+1}$. Hence, by (2), $f(x) \in \operatorname{int}(\alpha_i)$. Thus, $f(x) \notin \beta_i$. Therefore, by (6) and (7), $x \in A_i$. This proves the following:

(8) $A_{i+1} \subset A_i$ for all i.

Let $L = \cap_{i=1}^{\infty} A_i$. Then, since each A_i is a continuum (by (4)), we have by (8) and 1.8 of [N2, p. 6] that

(9) L is a continuum.

Next, we prove that $L = f^{-1}(z)$. By (1), $z \notin \beta_i$ for any i; hence, by (6) and (7), $f^{-1}(z) \subset A_i$ for all i. Thus, $f^{-1}(z) \subset L$. Also, $L \subset f^{-1}(z)$ since, using (5) and then (3) for the last two equalities below,

$$f(L) = f(\cap_{i=1}^{\infty} A_i) \subset \cap_{i=1}^{\infty} f(A_i) = \cap_{i=1}^{\infty} \alpha_i = \{z\}.$$

Now, having proved that $L = f^{-1}(z)$, we have by (9) that $f^{-1}(z)$ is a continuum. Therefore, we have proved that f is monotone.

Step 5. *The map f is constant on any given nowhere dense subcontinuum of X.*

Proof. Let Y be a proper subcontinuum of X such that f is not constant on Y. We show that Y has nonempty interior in X. Since $f : X \twoheadrightarrow S^1$ is an irreducible map, $f(Y) \neq S^1$. Thus, since f is not constant on Y, $f(Y)$ is an arc in S^1. Let

$$\gamma = \overline{S^1 - f(Y)}.$$

Note that γ is an arc in S^1. By Step 4, $f^{-1}(\gamma)$ is a subcontinuum of X; also, since $\gamma \cap f(Y) \neq \varnothing$, clearly $Y \cap f^{-1}(\gamma) \neq \varnothing$. Therefore, $Y \cup f^{-1}(\gamma)$ is a subcontinuum of X. Furthermore,

$$f(Y \cup f^{-1}(\gamma)) = f(Y) \cup \gamma = S^1.$$

Thus, since f is an irreducible map,

$$Y \cup f^{-1}(\gamma) = X.$$

Hence, $X - f^{-1}(\gamma) \subset Y$. Therefore, since $X - f^{-1}(\gamma)$ is a nonempty open subset of X, Y has nonempty interior in X. This completes the proof of Step 5.

By Steps 4 and 5, and by (4) of Step 1, we have proved Theorem 3.1.

□

The following corollary to Theorem 3.1 supplements Corollary 2.4:

Corollary 3.2. *An irreducible map of a continuum onto S^1 is HWC if and only if it is monotone.*

Proof. The "only if" half is part of Theorem 3.1; the other half follows from Corollary 2.4 and the fact that a monotone map of a continuum onto S^1 is essential (12.66 of [N2, p. 269]). $\square$

Remark 3.3. According to Corollary 2.4, the hypothesis of Theorem 3.1 may be equivalently stated by saying that $f : X \twoheadrightarrow S^1$ is essential and irreducible. As noted above Corollary 2.3, such maps are irreducibly essential, but the converse is false.

There are two conditions each of which is sufficient for an HWC map f of a continuum X onto S^1 to be monotone: f is irreducible (Theorem 3.1), X is arcwise connected (Theorem 4.3). The following example shows that HWC maps of continua onto S^1 are not always monotone:

Example 3.4. Let r denote the radial retraction map of $\mathbb{R}^2 - \{(0,0)\}$ onto S^1 (i.e., $r(z) = z/|z|$ where $|z|$ = distance from z to $(0,0)$). Let H be a half-line in $\mathbb{R}^2 - S^1$ such that $\overline{H} = H \cup S^1$ and $r(H) \neq S^1$ (e.g., Figure 2). Let $f = r|\overline{H}$. We show that f is HWC by using Theorem 2.1: Let K be a subcontinuum of $\overline{H}$ such that $f(K) = S^1$; then, noting that $f(H) \neq S^1$, it follows easily that $K \supset S^1$; thus, since $f|S^1$ is the identity map, $f|K$ is essential; in view of what we have shown, we have by Theorem 2.1 that $f : \overline{H} \twoheadrightarrow S^1$ is HWC. Clearly, f is not monotone.

4. Monotoneity of HWC Maps of Arcwise Connected Continua onto S^1

In Example 3.4 we saw that HWC maps of continua onto S^1 need not be monotone. In this section we prove that any HWC map of an arcwise connected continuum X onto S^1 must be monotone (Theorem 4.3) — moreover, each preimage of a point is arcwise connected even when the map is restricted to an arcwise connected subcontinuum of X (Theorem 4.7).

We use the proposition below to obtain the main results in this section and to prove results in subsequent sections. Since the proposition is not concerned with arcwise connectivity, the proposition is of interest without regard to the emphasis in this section.

Proposition 4.1. *Let M and N be continua such that $M \cap N$ is a continuum that is nowhere dense in M. If $f : M \cup N \twoheadrightarrow S^1$ is HWC and $f|M : M \twoheadrightarrow S^1$ is irreducible, then $f|N$ is a constant map.*

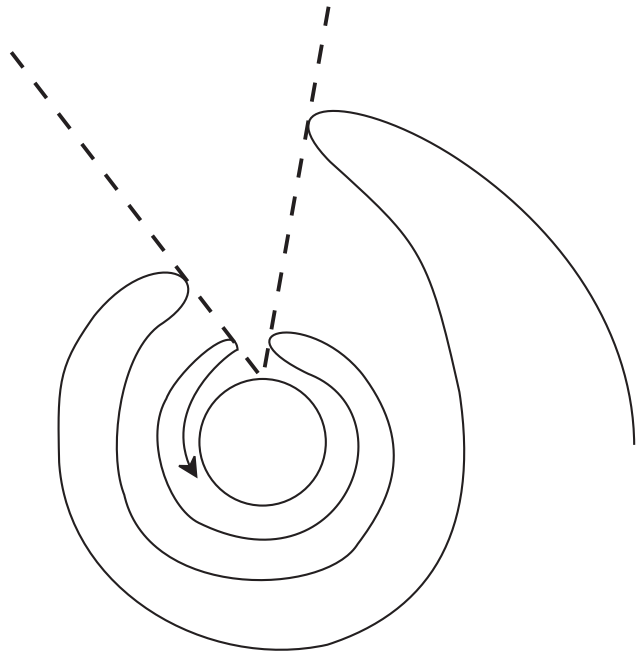

FIGURE 2. (For Example 3.4.)

PROOF. Let $p \in M \cap N$. Note that $f|M : M \twoheadrightarrow S^1$ is HWC as well as irreducible; hence, by the last part of Theorem 3.1,

$$f(M \cap N) = \{f(p)\}.$$

Now, contrary to Proposition 4.1, suppose that $f|N$ is not a constant map. Then, by considering an order arc in the hyperspace $C(N)$ from $M \cap N$ to N (1.8 and 1.11 of [N1]), we see that there is a subcontinuum, Z, of N such that $Z \supset M \cap N$ and $f(Z)$ is an arc in S^1. Since $f(p) \in f(Z)$, there is a subarc, α, of $f(Z)$ such that $f(p)$ is an end point of α. Let $\beta = \overline{S^1 - \alpha}$, and let

$$B = (f|M)^{-1}(\beta).$$

We prove that $B \cap Z$ is a continuum. Since $B \subset M$ and $Z \subset N$, $B \cap Z \subset M \cap N$. The reverse containment also holds: $M \cap N \subset B$ since $f(M \cap N) = \{f(p)\}$ and $f(p) \in \beta$, and $M \cap N \subset Z$ by the way we chose Z. Hence, we have proved that

$$B \cap Z = M \cap N.$$

Therefore, since $M \cap N$ is a continuum (by assumption in Proposition 4.1), we have that

(1) $B \cap Z$ is a continuum.

We prove (2)-(4) below (which lead to a contradiction).

Since β is an arc, clearly $f|B : B \to S^1$ is inessential; also, since Z was chosen so that $f(Z)$ is an arc in S^1, $f|Z : Z \to S^1$ is inessential. Therefore, by (1), we can apply 5.2 of [W, p. 221] to see that

(2) $f|(B \cup Z) : B \cup Z \to S^1$ is inessential.

By the way β was defined, $\beta \cup \alpha = S^1$; since f maps M onto S^1 (assumption in Proposition 4.1), $f(B) = \beta$; and, by the way α was chosen, $f(Z) \supset \alpha$. Therefore,

(3) $f(B \cup Z) = S^1$.

From assumptions in Proposition 4.1, $f|M : M \twoheadrightarrow S^1$ is HWC as well as irreducible. Thus, by Theorem 3.1, $f|M : M \twoheadrightarrow S^1$ is monotone. Hence, by the definition of B, B is a continuum. Thus, since Z was chosen to be a continuum, we see from (1) that

(4) $B \cup Z$ is a continuum.

Finally, since $f : M \cup N \twoheadrightarrow S^1$ is HWC (by assumption), (2)-(4) contradict Theorem 2.1. Therefore, it must be that $f|N$ is a constant map. □

Corollary 4.2. *The result in Proposition 4.1 remains true when the conditions on $M \cap N$ are weakened so as only to require that $M \cap N \neq \varnothing$ and $M \cap N$ is contained in a nowhere dense subcontinuum, L, of M.*

Proof. Apply Proposition 4.1 with N replaced by $N \cup L$. □

We now prove our first main result of this section. (The result will be superseded by Theorem 4.7, but it is included here for use in the proof of Lemma 4.6.)

Theorem 4.3. *Let X be an arcwise connected continuum. If $f : X \twoheadrightarrow S^1$ is HWC, then f is monotone.*

Proof. There is a subcontinuum, M, of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible (4.36(b) of [N2, p. 68]). Let $q \in S^1$. Let $M_q = (f|M)^{-1}(q)$. Since f is HWC, $f|M : M \twoheadrightarrow S^1$ is HWC; hence, by Theorem 3.1, $f|M : M \twoheadrightarrow S^1$ is monotone. Therefore,

(∗) M_q is a continuum.

Now, let $x \in f^{-1}(q) - M_q$. Then, since $x \notin M$ and X is arcwise connected, there is an arc, N_x, in X from x to a point $y \in M$ such that $N_x \cap M = \{y\}$. Hence, by Proposition 4.1, $f|N_x$ is a constant map. Thus, since $x \in N_x$ and $f(x) = q$, $N_x \subset f^{-1}(q)$. Hence, since $y \in N_x \cap M$, $y \in M_q$. Thus, we have proved that any point of $f^{-1}(q) - M_q$ can be joined to a point of M_q by an arc in $f^{-1}(q)$. Therefore, by (∗), $f^{-1}(q)$ is connected. □

The next proposition is concerned with monotone maps that are not necessarily HWC. Like the Proposition 4.1, does not assume arcwise connectivity.

Proposition 4.4. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be monotone. Let K and L be disjoint subcontinua of X such that $f(K) = f(L)$ is nondegenerate. Then there is a subcontinuum, Y, of X such that $f|Y : Y \twoheadrightarrow S^1$ is inessential.*

Proof. First, we show that there are subcontinua, K' and L', of K and L (respectively) such that $f(K') = f(L')$ is an arc, α, in S^1. To prove this, assume that $f(K) = f(L) = S^1$. Let $a \in K$ and $b \in L$ such that $f(a) = f(b)$. Consider order arcs in $C(K)$ and $C(L)$ from $\{a\}$ and $\{b\}$ to K and L, respectively (1.8 and 1.11 of [N1]); then we see that there are subcontinua, G and H, of K and L, respectively, such that $f(G)$ and $f(H)$ are arcs and $f(G) \cap f(H)$ contains an arc, α. Hence, by [R, p. 236], there are subcontinua, K' and L', of G and H, respectively, such that $f(K') = \alpha$ and $f(L') = \alpha$. Therefore, we have produced K', L', and α as required.

Next, let p and q denote the end points of α, let $r \in \mathrm{int}(\alpha)$, and let α_p and α_q denote the subarcs of α from p to r and from q to r, respectively. Then, since $f(K') = \alpha$ and $f(L') = \alpha$, we have by [R, p. 236] that there are subcontinua, K'_p and L'_q, of K' and L', respectively, such that $f(K'_p) = \alpha_p$ and $f(L'_q) = \alpha_q$. Let

$$\beta = (S^1 - \alpha) \cup \{p, q\}, \quad B = f^{-1}(\beta), \text{ and } Y = B \cup K'_p \cup L'_q.$$

We show that Y satisfies the conclusion of the proposition.

Since f is monotone, B is a continuum; thus, since $B \cap K'_p \neq \varnothing$ and $B \cap L'_q \neq \varnothing$, Y is a continuum. Since $f(B) = \beta$ and $f(K'_p \cup L'_q) = \alpha$, clearly $f(Y) = S^1$.

Finally, $f|Y : Y \twoheadrightarrow S^1$ is inessential, which we prove as follows: Note that $f(B \cup K'_p) = \beta \cup \alpha_p$, $f(B \cup L'_q) = \beta \cup \alpha_q$, and that $\beta \cup \alpha_p$ and $\beta \cup \alpha_q$ are arcs; thus, $f|(B \cup K'_p)$ and $f|(B \cup L'_q)$ are inessential as maps into S^1. Also, since $K \cap L = \varnothing$, clearly $K'_p \cap L'_q = \varnothing$ and, hence, $(B \cup K'_p) \cap (B \cup L'_q) = B$. Therefore, since B is a continuum, $f|Y : Y \twoheadrightarrow S^1$ is inessential by 5.2 of [W, p. 221]. □

The corollary below shows what Proposition 4.4 says about HWC maps on arcwise connected continua. We use the corollary in the proof of Lemma 4.6 and in later sections.

Corollary 4.5. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let K and L be subcontinua of X such that $f(K) = f(L)$ is nondegenerate. Then, $K \cap L \neq \varnothing$.*

Proof. By Theorem 4.3, f is monotone. Thus, if $K \cap L = \varnothing$, we can then apply Proposition 4.4 to contradict Theorem 2.1. Therefore, $K \cap L \neq \varnothing$. □

For the most general version of Proposition 4.4 for HWC maps, see Remark 4.9.

Lemma 4.6. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Then, f is monotone when restricted to any arc in X.*

Proof. Suppose that there is an arc, A, in X such that $f|A$ is not monotone. Then, obviously, $f|A$ is not a constant map and, by Theorem 2.1, $f(A) \neq S^1$. Hence, $f(A)$ is an arc, α, in S^1. Now, since A and α are arcs and $f|A : A \twoheadrightarrow \alpha$ is not monotone, it is evident that there are disjoint subarcs, K and L, of A such that $f(K) = f(L)$ is a subarc of α. This contradicts Corollary 4.5. □

Theorem 4.7. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. If Y is an arcwise connected subcontinuum of X, then $f|Y$ is monotone — moreover, $(f|Y)^{-1}(p)$ is arcwise connected for each $p \in f(Y)$.*

Proof. We only need to prove the second part of the conclusion (since it implies the first part). Let $p \in f(Y)$, and let y, $z \in (f|Y)^{-1}(p)$ such that $y \neq z$. Then, since Y is arcwise connected, there is an arc, A, in Y from y to z. By Lemma 4.6, $f|A$ is monotone; thus, since y and z are the end points of A and $f(y) = f(z) = p$, it follows easily that $f(A) = \{p\}$. Hence, since $A \subset Y$, $A \subset (f|Y)^{-1}(p)$. Therefore, we have proved that $(f|Y)^{-1}(p)$ is arcwise connected for each $p \in f(Y)$. □

We note two definitions for the corollary that follows. A continuum is *hereditarily arcwise connected* provided that each of its subcontinua is arcwise connected. A map $f : X \to Y$, where X and Y are continua, is *hereditarily monotone* provided that for every subcontinuum K of X, $f|K : K \to f(K)$ is monotone (see [M1] or [M2, p. 16]).

Corollary 4.8. *Let X be a hereditarily arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be a map. Then, f is HWC if and only if f is hereditarily monotone.*

PROOF. If f is HWC, then f is hereditarily monotone by Theorem 4.7. The converse is evident for maps between continua in general. □

Remark 4.9. The proof of Corollary 4.5 shows that the following general version of Proposition 4.4 for HWC maps is true: *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be a monotone HWC map. Let K and L be subcontinua of X such that $f(K) = f(L)$ is nondegenerate. Then, $K \cap L \neq \varnothing$.*

5. Subcontinua On Which HWC Maps of Arcwise Connected Continua onto S^1 Are Irreducible

Any map $f : X \twoheadrightarrow Y$, where X and Y are continua, is irreducible on some subcontinuum, M, of X (4.36(b) of [N2, p. 68]). The results in this section focus on M when the map $f : X \twoheadrightarrow S^1$ is HWC and X is arcwise connected: Theorem 5.2 and Corollary 5.4 show how f behaves on M, Theorem 5.3 exhibits the structure of M, Theorem 5.6 characterizes M when $M = X$, and Theorem 5.7 shows that M is unique.

Lemma 5.1. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible. Then, for any points p_1, $p_2 \in S^1$ with $p_1 \neq p_2$, there is an arc in M from a point of $(f|M)^{-1}(p_1)$ to a point of $(f|M)^{-1}(p_2)$.*

PROOF. Since X is arcwise connected, there is an arc in X from a point of $(f|M)^{-1}(p_1)$ to a point of $(f|M)^{-1}(p_2)$. Hence, there is an arc, A, in X with end points e_1 and e_2 such that

(1) $A \cap (f|M)^{-1}(p_i) = \{e_i\}$ for each i.

We show that $A \subset M$ (which proves the lemma).

Let $<$ denote the simple ordering for A such that $e_1 < e_2$.

Now, suppose to the contrary that $A \not\subset M$. Then there is a subarc, B, of A with end points b_1, $b_2 \in M$ such that $B \cap M = \{b_1, b_2\}$ and $b_1 < b_2$. Let $b \in B - \{b_1, b_2\}$, and let B_i denote the subarc of B from b_i to b for each i. Then, since $f|M : M \twoheadrightarrow S^1$ is irreducible, we see from Proposition 4.1 that $f|B_i$ is a constant map for each i. Therefore, since $B = B_1 \cup B_2$ is connected, we see that

(2) $f|B$ is a constant map.

It follows easily from (1) that $f(A)$ is an arc, α, in S^1 from p_1 to p_2. Let $\gamma = \overline{S^1 - \alpha}$, and let $M_\gamma = (f|M)^{-1}(\gamma)$. Since $f|M : M \twoheadrightarrow S^1$ is HWC and irreducible, $f|M$ is monotone by Theorem 3.1; hence,

(3) M_γ is a continuum.

Next, we define a continuum K that we will use to obtain a contradiction.

For each i, let A_i be the subarc of A from e_i to b_i (note: $e_i \neq b_i$ for each i since, e.g., if $e_1 = b_1$ then, by (2), $f(e_1) = f(b_2)$ which, since $e_1 < b_2$, contradicts (1)). By (1) and the fact that $\alpha \cap \gamma = \{p_1, p_2\}$, we see that

(4) $A_i \cap M_\gamma = \{e_i\}$ for each i.

Now, let $K = A_1 \cup A_2 \cup M_\gamma$. By (3) and (4), K is a continuum.

We prove that $f(K) = S^1$. Clearly, $K \cup B = A \cup M_\gamma$. Hence,

$$f(K) \cup f(B) = f(A) \cup f(M_\gamma) = \alpha \cup \gamma = S^1;$$

also, since $b_i \in K$, we see from (2) that $f(B) \subset f(K)$. Therefore, $f(K) = S^1$.

We have proved that K is a subcontinuum of X and that $f(K) = S^1$. Therefore, by Theorem 2.1, $f|K : K \twoheadrightarrow S^1$ is essential. However, $f|K : K \twoheadrightarrow S^1$ is inessential, which we prove as follows: $f(A_i) \subset \alpha$ for each i and $f(M_\gamma) = \gamma$; hence, $f|A_i : A_i \to S^1$ (each i) and $f|M_\gamma : M_\gamma \to S^1$ are inessential; thus, by (4) and 5.2 of [W, p. 221],

$$f|(A_i \cup M_\gamma) : A_i \cup M_\gamma \to S^1 \text{ is inessential for each } i;$$

therefore, since $(A_1 \cup M_\gamma) \cap (A_2 \cup M_\gamma) = M_\gamma$, we see from (3) and 5.2 of [W, p. 221] that $f|K : K \twoheadrightarrow S^1$ is inessential.

Therefore, having arrived at the self-contradictory statements in the preceding paragraph, we can now conclude that $A \subset M$. □

Theorem 5.2. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible. Then there is at most one point $p \in S^1$ such that $(f|M)^{-1}(p)$ is nondegenerate.*

Proof. For any three points a, b, c of S^1, we write abc to mean the arc in S^1 from a to c that contains b (in its interior). We let $g = f|M$.

Now, let $p, q \in S^1$ such that $p \neq q$. We show that $g^{-1}(p)$ or $g^{-1}(q)$ is degenerate (which proves the theorem).

Let $r, s \in S^1$ such that $\{r, s\}$ separates p from q in S^1. By Lemma 5.1, there is an arc, A, in M from a point of $g^{-1}(r)$ to a point of $g^{-1}(s)$. Since $r, s \in g(A)$, clearly $g(A) \supset rps$ or $g(A) \supset rqs$, say

$$g(A) \supset rqs.$$

Let $L = g^{-1}(rps)$. By Theorem 3.1, g is monotone. Hence, L is a continuum. Thus, since $A \cap L \neq \varnothing$, $A \cup L$ is a continuum; furthermore, $g(A \cup L) = S^1$ since $g(A) \supset rqs$ and $g(L) = rps$. Therefore, since g is irreducible, we have that

$$A \cup L = M.$$

By Theorem 3.1, $g^{-1}(q)$ is a continuum that is nowhere dense in M. By the definition of L, $g^{-1}(q) \subset M - L$ and $M - L$ is open in M; hence, $g^{-1}(q)$ is nowhere dense in $M - L$. Now, recall that $A \cup L = M$ and, hence, $M - L \subset A$. Thus, $g^{-1}(q)$ is nowhere dense in the arc A. Therefore, since $g^{-1}(q)$ is a continuum, $g^{-1}(q)$ must be degenerate. $\square$

Our next theorem describes the structure of the continuum M in Theorem 5.2. The theorem leads to the characterization in Theorem 5.6.

We use the following terminology: An *end of a compactification*, (Z, e), *of* $\mathbb{R}^1$ is either one of the sets $\cap_{t>0}\overline{e([t, +\infty))}$ and $\cap_{t<0}\overline{e((-\infty, t])}$; the *remainder of* (Z, e) is $Z - e(\mathbb{R}^1)$. It is easy to see that the ends and the remainder do not depend on the choice of e; thus, we speak of an *end of* Z and the *remainder of* Z (without mentioning e).

Obviously, the remainder is the union of the two ends, and each end is a continuum (by 1.8 of [N2, p. 6]). As an example, one end of $\overline{U}$ in Figure 3 is C and the other end is degenerate (the point at the top of the vertical line segment).

Theorem 5.3. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible. Then, M is a compactification of $\mathbb{R}^1$ with a continuum as the remainder such that at least one end of the compactification is degenerate.*

Proof. Let $g = f|M$. Then, by Theorem 5.2, there is a point $p \in S^1$ such that $g|(M - g^{-1}(p))$ is one-to-one. An easy argument with sequences shows that $g^{-1}|(S^1 - \{p\})$ is continuous. Hence, $g|(M - g^{-1}(p))$ is a homeomorphism of $M - g^{-1}(p)$ onto $S^1 - \{p\}$. Thus, $M - g^{-1}(p)$ is homeomorphic to $\mathbb{R}^1$. Also, by Theorem 3.1, $M - g^{-1}(p)$ is dense in M and $g^{-1}(p)$ is a continuum. This proves that M is a compactification of $\mathbb{R}^1$ with the continuum $g^{-1}(p)$ as the remainder. Finally, by Lemma 5.1, there is an arc in M from a point of $M - g^{-1}(p)$ to a point of $g^{-1}(p)$; therefore, it follows that at least one end of the compactification is degenerate. $\square$

The proof of Theorem 5.3 shows how the map $f|M$ is related to M as a compactification of $\mathbb{R}^1$. This information is important, so we state it in the corollary below; we will use the corollary in this section and in section 7.

Corollary 5.4 (to proof Theorem 5.3)**.** *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such* *that* $f|M : M \twoheadrightarrow S^1$ *is irreducible. Then there is a point* $p \in S^1$ *such that* $f|(M-f^{-1}(p))$ *is a homeomorphism of* $M-f^{-1}(p)$ *onto* $S^1-\{p\}$*, and* $(f|M)^{-1}(p)$ *is a nowhere dense continuum in* M*. Thus,* M *is a compactification of* $M-f^{-1}(p)$ *with remainder* $(f|M)^{-1}(p)$*, where* $M - f^{-1}(p)$ *is homeomorphic to* $\mathbb{R}^1$ *and* $(f|M)^{-1}(p)$ *is a continuum.*

The continuum M in Theorem 5.3 need not be arcwise connected, as the following example shows:

Example 5.5. Let X be the continuum in Figure 3: X consists of the open subset U homeomorphic to $\mathbb{R}^1$, the $\sin(1/x)$-continuum $C = \overline{U} - U$, and the arc A joining two points in different arc components of C. Let $M = \overline{U} = U \cup C$. Let Y and $f : X \twoheadrightarrow Y$ be as in Example 2.5. Then, by Example 2.5, $f : X \twoheadrightarrow Y$ is HWC and Y is homeomorphic to S^1; also, it is easy to see that $f|M : M \twoheadrightarrow Y$ is irreducible. Therefore, the assumptions in Theorem 5.3 are satisfied; however, M is not arcwise connected.

We have the following characterization theorem:

Theorem 5.6. *Let* X *be an arcwise connected continuum. Then*

(1) *there is an HWC, irreducible map of* X *onto* S^1

if and only if

(2) X *is a compactification of* $\mathbb{R}^1$ *with a continuum as the remainder.*

Furthermore, (2) implies (1) without assuming that X *is arcwise connected.*

Proof. That (1) implies (2) is by Theorem 5.3. Conversely, assume that X is as in (2) (we do not assume that X is arcwise connected). By (2), there is an embedded copy, U, of $\mathbb{R}^1$ in X such that U is dense in X and $X - U$ is a continuum. For the purpose of using Example 2.5, note that U is open in X and that $X - U = \overline{U} - U$. Hence, letting Y and $f : X \twoheadrightarrow Y$ be as in Example 2.5, we see from Example 2.5 that f is HWC and Y is homeomorphic to S^1. Finally, note from the way f is defined that for each $x \in U$, $f^{-1}(f(x)) = \{x\}$; thus, since U is dense in X, it follows easily that $f : X \twoheadrightarrow Y$ is irreducible. Therefore, we have proved that f (when followed by a homeomorphism of Y onto S^1) is a map with the properties in (1). □

We have the following uniqueness theorem:

Theorem 5.7. *Let* X *be an arcwise connected continuum, and let* $f : X \twoheadrightarrow S^1$ *be HWC. Then there is only one subcontinuum,* M*, of* X *such that* $f|M : M \twoheadrightarrow S^1$ *is irreducible.*

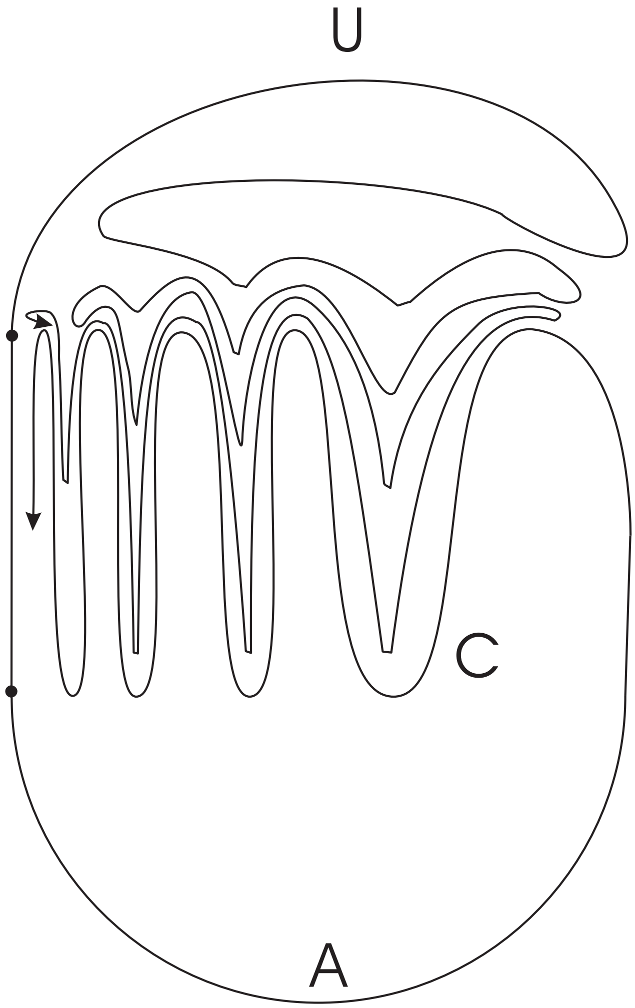

FIGURE 3. (For Example 5.5.)

PROOF. There is a subcontinuum, M, of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible (4.36(b) of [N2, p. 68]). Let M' be a subcontinuum of X such that $f|M' : M' \twoheadrightarrow S^1$ is irreducible. We show that $M = M'$.

Suppose, to the contrary, that $M \neq M'$. Then, since $f|M' : M' \twoheadrightarrow S^1$ is irreducible, $M \not\subset M'$. Hence, $M - M'$ is a nonempty open subset of M. Let

$U = M - f^{-1}(p)$ in Corollary 5.4; by Corollary 5.4, U is a dense open set in M such that $f|U$ is one-to-one. Since U is a dense open set in M, $U - M'$ is a nonempty open set in M; hence, there is a nondegenerate subcontinuum, K, of $U - M'$ (5.5 of [N2, p. 74]).

Since $f|U$ is one-to-one, $f(K)$ is nondegenerate. Since $f|M' : M' \twoheadrightarrow S^1$ is WC, there is a subcontinuum, L, of M' such that $f(L) = f(K)$. Since $K \cap M' = \varnothing$, $K \cap L = \varnothing$. These properties of K and L contradict Corollary 4.5. Therefore, $M = M'$. □

The example below shows the necessity of assuming that X is arcwise connected in Theorem 5.7; in addition, the continuum in the example is rational and has only three arc components.

Example 5.8. Let H and $f : \overline{H} \twoheadrightarrow S^1$ be as in the Example 3.4 (see Figure 2). Let X_0 and X_1 denote the copies of $\overline{H}$ in $\mathbb{R}^2 \times \mathbb{R}^1$ given by $X_0 = \overline{H} \times \{0\}$ and $X_1 = \overline{H} \times \{1\}$. Let e denote the end point of H, and let A be the line segment in $\mathbb{R}^2 \times \mathbb{R}^1$ from $(e, 0)$ to $(e, 1)$. Let

$$X = X_0 \cup X_1 \cup A.$$

Let π denote the projection map of X onto X_0. Now, considering X_0 to be the same as $\overline{H}$, let $g = f \circ \pi$. It is easy to check that π is HWC. Therefore, since f is HWC by Example 3.4, we see that $g : X \twoheadrightarrow S^1$ is HWC. Clearly, g maps each of $S^1 \times \{0\}$ and $S^1 \times \{1\}$ irreducibly onto S^1.

6. Nonexistence of HWC Maps of Cartesian Products onto S^1

We prove a nonexistence theorem (Theorem 6.2). We also give an example related to the theorem.

Note the following definition: Let X be a continuum, and let Y be a subcontinuum of X. A point, y, in Y is said to be *continuumwise accessible* from a point, x, in $X - Y$ provided that there is a subcontinuum, K, of X such that $x \in K$ and $K \cap Y = \{y\}$.

We prove the following general lemma:

Lemma 6.1. *Let X be a continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. If Y is a proper subcontinuum of X such that $f(Y) = S^1$, then some point of Y is not continuumwise accessible from some point of $X - Y$.*

Proof. If $\operatorname{int}(Y) \neq \varnothing$, then any point of $\operatorname{int}(Y)$ obviously has the desired property. Hence, for the purpose of proof, assume that $\operatorname{int}(Y) = \varnothing$. Since $f(Y) = S^1$, there is a subcontinuum, M, of Y such that $f|M : M \twoheadrightarrow S^1$ is irreducible (4.36(b) of [N2, p. 68]). We show that any point of M has the desired property.

Let $p \in M$. Since $f(M) = S^1$, there exists $q \in M$ such that $f(q) \neq f(p)$. Hence, by the continuity of f, there is an open subset, V, of X such that $q \in V$ and $f(p) \notin f(V)$. Since we have assumed that $\mathrm{int}(Y) = \varnothing$, there exists $r \in V - Y$.

Now, suppose there is a subcontinuum, N, of X such that $r \in N$ and $N \cap Y = \{p\}$. Then, since $p \in M \subset Y$, $N \cap M = \{p\}$. Thus, since r, $p \in N$, we have by Proposition 4.1 that $f(r) = f(p)$; however, since $r \in V$, $f(r) \neq f(p)$. This contradiction shows that there is no N with the properties assumed. Therefore, the point p, which is a point of Y, is not continuumwise accessible from the point r of $X - Y$. □

Theorem 6.2. *There is no HWC map of the cartesian product of any two nondegenerate continua onto S^1.*

Proof. Suppose that there are nondegenerate continua, Z_1 and Z_2, such that there is an HWC map, f, of their cartesian product, $Z_1 \times Z_2 = X$, onto S^1. By Theorem 2.1, $f : X \twoheadrightarrow S^1$ is essential. Hence, by 5.6 of [W, p. 223], there exists $p_1 \in Z_1$ or $p_2 \in Z_2$, say $p_1 \in Z_1$, such that

$$f|(\{p_1\} \times Z_2) : \{p_1\} \times Z_2 \to S^1 \text{ is essential.}$$

Thus, letting $Y = \{p_1\} \times Z_2$, we see that $f(Y) = S^1$. Now, note that any point, (p_1, q_2), of Y is continuumwise accessible from any point, (z_1, z_2), of $X - Y$ (as is seen by considering the union of $\{z_1\} \times Z_2$ and $Z_1 \times \{q_2\}$). Therefore, since $f : X \twoheadrightarrow S^1$ is HWC and $f(Y) = S^1$, we have a contradiction to Lemma 6.1. □

Regarding Theorem 6.2, it is easy to give examples of HWC maps of continua of any dimension onto S^1. For example, see Example 2.5 or Theorem 8.4; however, note that the continua in Example 2.5 and Theorem 8.4 are one-dimensional at some points. As is more pertinent to Theorem 6.2, Wlodzimierz J. Charatonik and Januasz R. Prajs have constructed n-dimensional Cantor manifolds for each $n \leq \infty$ that admit HWC maps onto S^1; their construction is done by replacing the layers of the Knaster $V - \Lambda$ continuum [K, p. 191] with a given n-dimensional Cantor manifold and making some identifications. (Their construction was done after a lecture that the second author of this paper gave at Professor J. J. Charatonik's seminar at the Instituto de Matemáticas of the Universidad Nacional Autónoma de Mexico). The following example also shows that there are n-dimensional Cantor manifolds for each $n \leq \infty$ that admit HWC maps onto S^1:

Example 6.3. Let $Y = H \cup S^1$ be as in Figure 2. Let Z be any continuum. Let $\sim$ denote the following equivalence relation on $Y \times Z : (p, z_1) \sim (p, z_2)$ for each point $p \in S^1$, and $(y, z) \sim (y, z)$ for each $(y, z) \notin (S^1 \times Z)$. Let X denote the quotient space $(Y \times Z)/_{\sim}$. We define an HWC map $f : X \twoheadrightarrow S^1$. First, let $q : Y \times Z \twoheadrightarrow X$ be the quotient map, and let $\pi_Y : Y \times Z \twoheadrightarrow Y$ be the projection map. Note that π_Y is constant on each point inverse of q, and let $k = \pi_Y \circ q^{-1} : X \twoheadrightarrow Y$; k is continuous by 3.22 of [N2, p. 45]. Now, letting r be as in Example 3.4, we let

$$f = r \circ k.$$

We see that $f : X \twoheadrightarrow S^1$ is HWC by using Theorem 2.1 as we did in Example 3.4. Furthermore, if Z is a Cantor manifold [HW, p. 93], then X is a Cantor manifold. Therefore, there are n-dimensional Cantor manifolds for each $n \leq \infty$ that admit HWC maps onto S^1.

7. The Cyclic Element Retraction

Let X be a semi-locally-connected (slc) continuum [W, p. 19], and let E be a cyclic element of X. Then (3.31 of [W, p. 69]) the components of $X - E$ form a null sequence, and if L is one of these components, then $Bd(L)$ consists of a single point x_L (i.e., $\overline{L} \cap E = \{x_L\}$). Thus, on defining $r : X \twoheadrightarrow E$ by

$$r(x) = \begin{cases} x, & \text{if } x \in E \\ x_L, & \text{if } x \in L = \text{a component of } X - E \end{cases}$$

it follows easily that r is a (continuous) retraction of X onto E (compare with 3.9 of [W, p. 70]). We call r the *cyclic element retraction of X onto E*. It is unique in that it is the only retraction of X onto E that is constant on each component of $X - E$.

We prove the proposition below for use in the proofs of our main theorems in the next section.

Proposition 7.1. *If X is an slc continuum and E is a cyclic element of X, then the cyclic element retraction of X onto E is HWC.*

Proof. Let r denote the cyclic element retraction of X onto E. Let Y be a subcontinuum of X. We wish to show that $r|Y : Y \twoheadrightarrow r(Y)$ is WC. This is evident if $Y \subset X - E$ (since Y is then contained in some component of $X - E$ and, hence, $r|Y$ is constant). Therefore, we assume for the rest of the proof that $Y \cap E \neq \varnothing$. We prove that

$$(*) \quad r(Y) = Y \cap E.$$

Proof (of $(*)$). Assume that L is any component of $X - E$ such that $Y \cap L \neq \varnothing$; then, since $Y \cap E \neq \varnothing$ and $Bd(L) = \{x_L\}$, we see that $x_L \in Y$; thus, since $r(L) = \{x_L\}$, we have that $r(Y \cap L) \subset Y \cap E$. Therefore, since L was any component of $X - E$ such that $Y \cap L \neq \varnothing$, we have proved that

$$r(Y - E) \subset Y \cap E.$$

Thus, since $r|E$ is the identity map of E, it follows that

$$r(Y) = r(Y - E) \cup r(Y \cap E) = Y \cap E,$$

which proves $(*)$.

Now, let B be a subcontinuum of $r(Y)$. Then, by $(*)$, $B \subset Y \cap E$. Therefore, B is a subcontinuum of Y and $r(B) = B$ (since $r|E$ is the identity map of E). This proves that $r|Y : Y \twoheadrightarrow r(Y)$ is WC. □

Remark 7.2. We only used the hypotheses in Proposition 7.1 to guarantee a retraction as defined preceding Proposition 7.1 (in particular, the proof of Proposition 7.1 did not use that X is slc or that E is a cyclic element of X). Thus, retractions that are so defined in general are always HWC. It would be of interest to have other results concerning when retractions are HWC.

8. HWC Maps of Arcwise Connected Semi-locally-connected Continua onto S^1

Recall from the preceding section that semi-locally-connected is abbreviated slc. We refer the reader to [W] for definitions and basic results about slc continua. We only mention the following terminology from [W] since failure to do so may cause confusion: A *cut point* of a continuum X is a point, p, of X such that $X - \{p\}$ is not connected [W, p. 41]; a continuum is said to be *cyclic* provided that it has no cut point [W, p. 107]. Thus, the term cut point is what nowadays is often called a separating point (to avoid confusion with points that weakly cut).

In this section we obtain two definitive results: The first result shows how any HWC map of an arcwise connected slc continuum onto S^1 must be defined (Theorem 8.4); the second result shows that there is a simple characterization of the arcwise connected slc continua that admit an HWC map onto S^1 (Corollary 8.5).

Lemma 8.1. *Let X be an arcwise connected slc cyclic continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Then there is a point $p \in S^1$ such that $f|(X - f^{-1}(p))$ is one-to-one.*

Proof. There is a subcontinuum, M, of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible (4.36(b) of [N2, p. 68]). By Corollary 5.4, there is a point $p \in S^1$ such that

(1) $f|(M - f^{-1}(p))$ is a homeomorphism of $M - f^{-1}(p)$ onto $S^1 - \{p\}$.

We show that the point p just chosen satisfies the conclusion of our lemma.

Suppose, to the contrary, that $f|(X - f^{-1}(p))$ is not one-to-one. Then, since $f(M) = S^1$, there is a point $m \in M - f^{-1}(p)$ such that $f^{-1}(f(m))$ is nondegenerate. Hence, by (1), there is a point $x \in f^{-1}(f(m)) - M$. Thus, by Theorem 4.7, there is an arc, A, in $f^{-1}(f(m))$ from x to m such that $A \cap M = \{m\}$. Since X is slc and cyclic, there is an open neighborhood, U, in X of m such that $x \notin U$, $p \notin f(U)$, and $X - U$ is a continuum (4.14 of [W, p. 50]).

Since $U \cap f^{-1}(p) = \varnothing$, we see from (1) that $f|(U \cap M)$ is an open embedding of $U \cap M$ in S^1. Thus, since $m \in U \cap M$, there is an arc, α, in $f(U \cap M)$ such that $m \in \text{int}(\alpha)$. Therefore, since $f(x) = f(m)$, $f(x) \in \text{int}(\alpha)$. Note that $p \notin \alpha$.

Next, recall that $X - U$ is a continuum, $x \in X - U$, and $f^{-1}(p) \subset X - U$. Hence, $f(X - U)$ is a subcontinuum of S^1 containing the points $f(x)$ and p. Thus, since $f(x) \in \text{int}(\alpha)$ and $p \notin \alpha$, we see that $f(X - U)$ must contain a subarc, β, of α. Therefore, since f is HWC and $X - U$ is a continuum, there is a subcontinuum, K, of $X - U$ such that $f(K) = \beta$.

Let $L = [f|(U \cap M)]^{-1}(\beta)$. Then, since $f|(U \cap M)$ is an embedding (by (1)) and since $\beta \subset \alpha \subset f(U \cap M)$, we see that L is an arc in $U \cap M$ such that $f(L) = \beta$. Since $L \subset U$, $K \cap L = \varnothing$.

We have shown that K and L are disjoint subcontinua of X such that $f(K) = f(L) = \beta$.

This contradicts Corollary 4.5. Therefore, $f|(X - f^{-1}(p))$ is one-to-one. □

Lemma 8.2. *Let X be an arcwise connected slc cyclic continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible. Then, M is a simple closed curve.*

Proof. Suppose, to the contrary, that M is not a simple closed curve.

By Theorem 5.3, M is a compactification of $\mathbb{R}^1$ with a continuum, L, as the remainder. If L is degenerate, then M is the one-point compactification of $\mathbb{R}^1$ and, therefore, M is a simple closed curve. Hence, by our initial supposition, we have that

(1) L is nondegenerate.

Since L is a nowhere dense subcontinuum of M, the last part of Theorem 3.1 gives us that $f|L$ is a constant map, say $f(L) = \{p\}$. Moreover, by Corollary 5.4,

(2) $L = (f|M)^{-1}(p)$.

By (1) and (2), we see that the point p in (2) must be the point p guaranteed by Lemma 8.1. Hence, $f|(X - f^{-1}(p))$ is one-to-one. Therefore, since $f(M) = S^1$, it follows easily that

(3) $X = M \cup f^{-1}(p)$.

Now, fix a point $y \in L$. By (1), y has arbitrarily small neighborhoods in X that do not contain L. Thus, since X is slc and cyclic, there is an open neighborhood, U, in X of y such that $U \not\supset L$ and $X - U$ is a continuum (4.14 of [W, p. 50]). Now, recall that M is a compactification of $\mathbb{R}^1$ with remainder L. Therefore, since $U \cap L \neq \varnothing$ and $U \not\supset L$, there is a component, A, of $M - U$ such that $A \cap L = \varnothing$.

Since $A \cap L = \varnothing$ and $L \cap (X - U) \neq \varnothing$, clearly $A \neq X - U$. Hence, A is a proper subcontinuum of the continuum $X - U$. Thus, since $A \cap L = \varnothing$, there is a sequence, $\{B_i\}_{i=1}^{\infty}$, of subcontinua of $(X - U) - L$ such that $A \subset B_i \neq A$ for each i and $\cap_{i=1}^{\infty} B_i = A$ (5.5 of [N2, p. 74]). Since A is a component of $M - U$, clearly $B_i \not\subset M$ for any i. Hence, by (3), $B_i \cap f^{-1}(p) \neq \varnothing$ for each i. Thus, since $\cap_{i=1}^{\infty} B_i = A$, we have that $A \cap f^{-1}(p) \neq \varnothing$. Hence, by (2), $A \cap L \neq \varnothing$. This contradicts the fact that $A \cap L = \varnothing$. Therefore, our supposition at the beginning of the proof is false; in other words, M is a simple closed curve. □

The following theorem is a key step in the proof of our main theorem:

Theorem 8.3. *Let X be an arcwise connected slc cyclic continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Then, X is a simple closed curve and f is a homeomorphism.*

Proof. There is a subcontinuum, M, of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible (4.36(b) of [N2, p. 68]). By Lemma 8.2, M is a simple closed curve. Hence, by Corollary 2.2, $f|M : M \twoheadrightarrow S^1$ is a homeomorphism. We show that $M = X$, which will complete the proof of the theorem.

By Lemma 8.1, there is a point $p \in S^1$ such that $f|(X - f^{-1}(p))$ is one-to-one. Thus, since $f(M) = S^1$, it follows easily that $X = M \cup f^{-1}(p)$ and that $M \cap f^{-1}(p) \neq \varnothing$. Since $M \cap f^{-1}(p) \neq \varnothing$ and since $f|M$ is one-to-one, we have that $M \cap f^{-1}(p)$ consists of exactly one point, m. Since X is cyclic, m is not a cut point of X. Thus, since $X = M \cup f^{-1}(p)$ and $M \cap f^{-1}(p) = \{m\}$, it follows that $f^{-1}(p) \subset M$. Therefore, since $X = M \cup f^{-1}(p)$, we have that $X = M$. □

We are ready to prove the main theorem of the section. The theorem gives a formula for any HWC map of an arcwise connected slc continuum onto S^1.

Theorem 8.4. *Let X be an arcwise connected slc continuum, and let $f : X \twoheadrightarrow S^1$ be a map. Then*

(1) *f is HWC*

if and only if

(2) *there is a simple closed curve, C, in X such that C is a cyclic element of X and $f = h \circ r$, where r is the cyclic element retraction of X onto C (section 7) and h is a homeomorphism of C onto S^1.*

Proof. That (2) implies (1) follows easily using Proposition 7.1.

Conversely, assume that (1) holds. Then, by Theorem 2.1, $f : X \twoheadrightarrow S^1$ is essential. Hence, there is a cyclic element, C, of X such that $f|C : C \twoheadrightarrow S^1$ is essential (5.41 of [W, p. 222]). Since C is a cyclic element of X, it is easy to see that C has the properties assumed for X in the theorem; namely, C is an arcwise connected slc continuum. In addition, since X is slc, C is cyclic (1.7 of [W, p. 66]). Hence, we can apply Theorem 8.3 to C, which gives us that C is a simple closed curve and that $f|C : C \twoheadrightarrow S^1$ is a homeomorphism. Now, let $h = f|C$, and let r be the cyclic element retraction of X onto C (defined in the first paragraph of section 7). The proof of our theorem will be finished once we prove that $f = h \circ r$.

To prove that $f = h \circ r$, let $x \in X$. Assume first that $x \in C$. Then, $r(x) = x$; thus, since $h = f|C$,

$$(h \circ r)(x) = h(x) = f(x).$$

Therefore, to complete the proof, assume that $x \notin C$. Let L denote the component of $X - C$ containing x. Recall from the first paragraph of section 7 that $r(x) = x_L$, where $\overline{L} \cap C = \{x_L\}$. Since $\overline{L} \cap C = \{x_L\}$ and $f|C : C \twoheadrightarrow S^1$ is irreducible, we have by Proposition 4.1 that $f|\overline{L}$ is a constant map. Hence, $f(x) = f(x_L)$; thus, since $r(x) = x_L$ and $h = f|C$,

$$(h \circ r)(x) = h(x_L) = f(x_L) = f(x).$$

This completes the proof that $f = h \circ r$. Therefore, we have proved that (1) of Theorem 8.4 implies (2). □

The following simple corollary characterizes all the arcwise connected slc continua that can be mapped onto S^1 by an HWC map:

Corollary 8.5. *Let X be an arcwise connected slc continuum. Then there is an HWC map of X onto S^1 if and only if some simple closed curve in X is a cyclic element of X.*

Proof. Half of the corollary is immediate from the fact that (1) implies (2) in Theorem 8.4; the other half is due to Proposition 7.1. □

Remarks 8.6. (1) Arcwise connectivity is a necessary assumption in Theorem 8.3, Theorem 8.4 and Corollary 8.5: Pierce [P] has constructed an slc cyclic continuum that is not a simple closed curve and that admits an HWC map onto S^1 (in fact, Pierce's map is even hereditarily monotone). (2) Theorem 8.3, Theorem 8.4 and Corollary 8.5 are true for locally connected continua since locally connected continua are arcwise connected and slc (5.2 of [W, p. 38] and 13.21 of [W, p. 20], respectively).

9. A Problem

It would be interesting to study HWC maps onto particular continua other than S^1. Perhaps a starting point would be some simple graphs — for example, a figure eight. The following result may be useful in this connection:

Proposition 9.1. *Let X be an arcwise connected continuum, and let $f : X \twoheadrightarrow S^1$ be HWC. Let M be a subcontinuum of X such that $f|M : M \twoheadrightarrow S^1$ is irreducible. If $M \neq X$ and $X - M$ is locally connected, then $(f|M)^{-1}(q)$ separates X for some $q \in S^1$ (in fact, for any $q \in f(X - M)$).*

Proof. Since $M \neq X$, there exists $q \in f(X - M)$. Let $M_q = (f|M)^{-1}(q)$, and let

$$H = f^{-1}(q) - M_q.$$

Note that $H \neq \varnothing$ (by our choice of q), H is a proper subset of $X - M_q$ (since $M - M_q \neq \varnothing$), and H is closed in $X - M_q$ (since $H = f^{-1}(q) \cap (X - M_q)$). Therefore, once we show that H is open in $X - M_q$, we will have proved that M_q separates X. Let $x \in H$. Then, by Theorem 4.7, there is an arc, A, in $f^{-1}(q)$ from x to a point $z \in M_q$ such that $A \cap M_q = \{z\}$ (note: $M_q \neq \varnothing$ since $f(M) = S^1$). Since $x \in X - M$ and since $X - M$ is locally connected and open in X, there is a connected open subset, U, of X such that $x \in U$ and $\overline{U} \subset X - M$. Now, let $N = A \cup \overline{U}$. Note that N is a continuum and that $M \cap N = \{z\}$ (since $M \cap N = M \cap A$). Therefore, we can apply Proposition 4.1 to obtain that $f|N$ is a constant map. Thus, since $A \subset N$ and $f(A) = \{q\}$, we have that $f(N) = \{q\}$.

Hence, $f(U) = \{q\}$. Thus, since $U \cap M = \varnothing$, $U \subset H$. Therefore, since $x \in U$, we have proved that H is open in $X - M_q$. □

References

[F] G. A. Feuerbacher, *Weakly chainable circle-like continua*, Fund. Math. **106** (1980), 1-12.

[HW] Witold Hurewicz and Henry Wallman, *Dimension Theory*, Princeton Univ. Press, Princeton, New Jersey, 1948.

[K] K. Kuratowski, *Topology*, vol II, Academic Press, New York, NY, 1968.

[M1] T. Maćkowiak, *The hereditary classes of mappings*, Fund. Math. **97** (1977), 123-150.

[M2] T. Maćkowiak, *Continuous Mappings on Continua*, Dissertationes Mathematicae No 158, PWN, Warsaw, 1979.

[N1] Sam B. Nadler, Jr., *Hyperspaces of Sets, Monographs and Textbooks in Pure and Applied Math.*, Vol.49, Marcel Dekker, Inc., New York and Basel, 1978.

[N2] Sam B. Nadler, Jr., *Continuum Theory, an Introduction, Monographs and Textbooks in Pure and Applied Math.*, Vol.158, Marcel Dekker, Inc., New York and Hong Kong, 1992.

[P] Robert Pierce, *Hereditarily monotone mappings onto S^1*, Top. Proceedings **21** (1996), 201-220.

[R] David R. Read, *Confluent and related mappings*, Colloq. Math. **29** (1974), 233-239.

[Wa] L. E. Ward, Jr., *Fixed point theorems for pseudo-monotone mappings*, Proc. Amer. Math. Soc. **13** (1962), 13-16.

[W] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloq. Publ., vol. 28, Amer. Math. Soc., Providence, Rhode Island, 1942 (1963 edition).

Received September 8, 1998

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