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ON DOUBLE-DERIVED SETS IN TOPOLOGICAL SPACES

ONDŘEJ F.K. KALENDA
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Abstract. We characterize topological spaces which have a subset with non-closed double-derived set. As a corollary we obtain that the double-derived set of an arbitrary subset of a T_0 topological space is closed. This answers in the negative a question asked by A.Lelek in Houston Problem Book (1995).

In [1, Problem 174] A.Lelek asked whether there is a T_0 topological space X and a subset $A \subset X$ such that the double-derived set $(A^d)^d$ is not closed. This problem also appeared in the Problem book of the Open Problem Seminar held at Department of Mathematical Analysis at Charles University ([2, Problem 28]). We present here a short negative solution.

Let us first recall some definitions.

Definition 1. A topological space X is called a T_0 *space*, if for any pair of distinct points $x, y \in X$ there exists an open set containing exactly one of the points x, y.

Definition 2. If X is a topological space and $A \subset X$ an arbitrary subset, then the *derived set* A^d is the set of all $x \in X$ which belong to $\overline{A \setminus \{x\}}$.

Definition 3. A topological space X is called *indiscrete* if $\emptyset$ and X are the only open sets in X.

Let us remark that in any T_1-space X (X is T_1, if for any two distinct points $x, y \in X$ there is an open set U such that $x \in U$ and $y \notin U$) the derived set of an arbitrary set is necessarily closed. For T_0 spaces the first derived set need not be closed. Consider, for example, the set $\mathbf{R}$ of real numbers with the topology generated by $\{(-\infty, a) : a \in \mathbf{R}\}$. Then the derived set of $A = \{0\}$ is $A^d = (0, \infty)$ which is not closed.

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On the other hand, if X is the two-point indiscrete space and A a one-point subset of X, then clearly none of the iterated derived sets of A is closed. In view of these remarks, the question of A.Lelek seems to be natural. The answer is a consequence of the following theorem which characterizes topological spaces in which there is a subset with non-closed double derived set.

Theorem. *Let X be topological space. The following assertions are equivalent.*

1. *There is a subset $A \subset X$ such that the double-derived set $(A^d)^d$ is not closed in X.*
2. *There is a subset $A \subset X$ such that none of the iterated derived sets A^d, $(A^d)^d$, $((A^d)^d)^d$, $\dots$ is closed.*
3. *X contains a copy of two-point indiscrete space, as a subset of the form $F \cap G$ with F closed and G open in X.*

As a corollary we get the answer to the mentioned question of A.Lelek.

Corollary. *Let X be a T_0 topological space and $A \subset X$ an arbitrary subset. Then the double derived set $(A^d)^d$ is closed in X.*

Proof of Theorem. $3 \Rightarrow 2$ Let $\{a, b\} = F \cap G$ be a copy of two-point indiscrete space with F closed and G open in X. Put $A = \{a\}$. Then it is clear that in the sequence $A^d \cap G$, $(A^d)^d \cap G$, $((A^d)^d)^d \cap G$, $\dots$ the odd members are equal to $\{b\}$, and the even ones to $\{a\}$. Hence none of the iterated derived sets is closed in X.

$2 \Rightarrow 1$ This is trivial.

$1 \Rightarrow 3$ Suppose that $(A^d)^d$ is not closed. Pick a point $a \in \overline{(A^d)^d} \setminus (A^d)^d$. As $a \notin (A^d)^d$, there is an open set U containing a such that

$$U \cap A^d \subset \{a\} \tag{1}$$

Since a belongs to the closure of $(A^d)^d$, there is a point $b \in U \cap (A^d)^d$. Clearly $b \neq a$. As U is a neighborhood of b and $b \in (A^d)^d$, we obtain that $U \cap A^d \neq \emptyset$. By (1) we get

$$a \in A^d \tag{2}$$

Now suppose that $b \in U \cap (A^d)^d$ is arbitrary. If there is an open set V containing b and not containing a, then $U \cap V \cap A^d \neq \emptyset$, as $U \cap V$ is a neighborhood of b. But it follows that $(U \cap A^d) \setminus \{a\} \neq \emptyset$, which contradicts (1). Hence we get

$$(\forall b \in U \cap (A^d)^d)(\forall V \ni b,\ V \text{ open})(a \in V) \tag{3}$$

As $b \notin A^d$ (by (1)), we obtain the following.

$$(\forall b \in U \cap (A^d)^d)(\exists V_b \text{ open})(b \in V_b \ \&\ V_b \cap A \subset \{b\}) \tag{4}$$

Due to (3), the set V_b is a neighborhood of a, and hence, by (2), $V_b \cap A \neq \emptyset$. Hence, by (4) we deduce

$$U \cap (A^d)^d \subset A \tag{5}$$

If there are two distinct points $b, c \in (A^d)^d$, then $V_b \cap V_c$ is a neighborhood of a (by (3)), and hence there is some $d \in A \cap V_b \cap V_c$, due to (2). But it follows from the choice of V_b and V_c (cf. (4)) that $d = b = c$, a contradiction. Therefore

$$\text{there is exactly one } b \in (A^d)^d \cap U \tag{6}$$

If there is an open set W such that $a \in W$ and $b \notin W$, then $W \cap U$ is a neighborhood of a, and hence there is some $c \in (A^d)^d \cap U \cap W$. This c is necessarily different from b, which contradicts (6). So such a W does not exist. Together with (3) this yields that

$$\{a, b\} \text{ is a copy of the two-point indiscrete space.} \tag{7}$$

It remains to prove that the set $\{a, b\}$ is of the form $F \cap G$ with F closed and G open in X. Put $G = U \cap V_b$ and $F = \overline{\{a, b\}}$. Then clearly $\{a, b\} \subset F \cap G$. Let us prove the inverse inclusion. Let $x \in (F \cap G) \setminus \{a, b\}$. By (1) we have $x \notin A^d$, and so there is an open set W containing x such that $A \cap W \subset \{x\}$, in particular $b \notin W$ by (5). It follows from (7) that $a \notin W$ as well, hence $W \cap \{a, b\} = \emptyset$. In particular, $x \notin F$, a contradiction. This completes the proof. □

References

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Department of Math. Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail address: kalenda@karlin.mff.cuni.cz