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ON FIBREWISE RETRACTION AND EXTENSION *)

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Abstract. We study fibrewise retracts and extensions. In section 2, we introduce notions of absolute (nbd) retracts (or extensor) over B relative to a fibrewise class Q_B. In section 3, we introduce a notion of fibrewise adjunction spaces and study the relations of fibrewise ANR and ANE. In sections 4 and 5, we study fibrewise contractibility and fibrewise ANE.

1. Preliminaries

Fibrewise General Topology or General Topology of Continuous Maps is concerned most of all in extending the main notions and results concerning spaces to continuous maps. Most of the results obtained so far in this field can be found in [2],[3],[4],[9],[10], [11] and [12], where one can also find an extensive bibliography on the subject.

Unless otherwise stated, B is a fixed topological space with topology τ . The collection of all neighborhoods (nbd(s)) of a point $b \in B$ is denoted by $N(b)$. For continuous maps $f : X \to B$ and $g : Y \to B$, a continuous map $\lambda : X \to Y$ satisfying the property $f = g \circ \lambda$, is called a morphism of f into g and is denoted by $\lambda : f \to g$. These are the morphisms in the category Top_B, the objects of which are continuous maps into the space B. A morphism $\lambda : f \to g$ is called surjective, closed, perfect, etc, if respectively, such is the map $\lambda : X \to Y$. If $[\lambda X]= Y$ then the morphism λ is said to be dense and if $\lambda : X \to Y$ is a homeomorphism then the morphism λ is said to be an isomorphism. Here by $[\bullet]$ or $[\bullet]_X$ we mean the closure operator in the respective space. We note that this situation is a generalization of the category Top (of topological spaces and continuous maps as

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morphisms), since the category *Top* is isomorphic to the particular case of Top_B in which the space B is a singleton set.

In this paper, we assume that all maps are continuous and all spaces are topological spaces. We use the following terminology and notations: B is the fixed base space, a map $f : X \to B$ is said to be a *projection*, X is said to be a *fibrewise space* over B and we denote it by (X, f). A morphism $\lambda : X \to Y$ is said to be a *fibrewise map*. $\mathbf{N}$ is the set of all positive integers and I the closed unit interval $[0, 1]$.

We now give some definitions concerning maps. For more details one can consult [10] and [12].

Definition 1.1. A map $f : X \to B$ is called a T_i*-map*, $i = 0, 1, 2$, if for all $x, x^{'} \in X$ such that $x \neq x^{'}, fx = fx^{'}$ the following condition is respectively satisfied:

$i = 0$: at least one of the points $x, x^{'}$ has a nbd in X not containing the other point;

$i = 1$: each of the points $x, x^{'}$ has a nbd in X not containing the other point;

$i = 2$: the points x and $x^{'}$ have disjoint nbds in X.

A T_2-map is also called Hausdorff. We note that for $i = 0, 1$ the property for a map $f : X \to B$ to be a T_i-map, is equivalent to the property that all the fibres $f^{-1}b, b \in B$, are T_i-spaces. This is not the case for T_2-maps.

Definition 1.2. The subsets M and N of the space X are said to be respectively:

(a) *nbd separated* in $U \subset X$,

(b) *functionally separated* in $U \subset X$,

if the sets $M \cap U$ and $N \cap U$

(a) have disjoint nbds in U,

(b) are functionally separated in U (i.e. there is a map $\phi : U \to I$ such that $M \cap U \subset \phi^{-1}(0)$ and $N \cap U \subset \phi^{-1}(1)$).

Definition 1.3. (a) A map $f : X \to B$ is called *functionally Hausdorff*, if for every two points x and x' in X such that $x \neq x', fx = fx'$, there is a nbd $O \in N(fx)$, such that the sets $\{x\}$ and $\{x'\}$ are functionally separated in $f^{-1}O$.

(b) A map $f : X \to B$ is called *completely regular* (resp. *regular*), if for every point $x \in X$ and every closed set F in X such that $x \notin F$, there is a nbd $O \in N(fx)$, such that the sets x and F are functionally separated (resp. nbd separated) in $f^{-1}O$. A completely regular (resp. regular) T_0-map is called *Tychonoff* or $T_{3\frac{1}{2}}$-(resp. T_3-)*map*.

It can be easily verified that every T_j-map is a T_i-map for j, i =0,1,2,3,3$\frac{1}{2}$ and $i \leq j$. We also have that every Tychonoff map is functionally Hausdorff, and every functionally Hausdorff map is Hausdorff. In Example 5.1, we shall give an example of a normal compact map which is not functionally Hausdorff.

Definition 1.4. A map $f : X \to B$ is called *functionally prenormal* (resp. *prenormal*) if for every $b \in B$ and every two disjoint, closed sets F and H in X, there is $O \in N(b)$ such that F and H are functionally separated (resp. nbd separated) in $f^{-1}O$. If for every $O \in \tau$, the map $f|f^{-1}O : f^{-1}O \to O$ is functionally prenormal(resp. prenormal) then f is called *functionally normal*(resp.*normal*). A normal T_3-map is called a *T_4-map*.

Definition 1.5. By a *compact map* is meant a perfect (i.e. continuous, closed and fibrewise compact) T_2-map.

Definition 1.6. Let $f : X \to B$ be a map.

(1) For $b \in B$, a collection of subsets of X is said to be *b-locally finite* if for every $x \in f^{-1}b$, there is a nbd O_x of x in X, such that O_x meets finitely many elements of the collection. A T_2-map $f : X \to B$ is said to be *paracompact* if for every point $b \in B$ and every open (in X) cover $\mathcal{U}$=$\{U_\alpha : \alpha \in \Lambda\}$ of the fibre $f^{-1}b$ (i.e. $f^{-1}b \subset \cup\{U_\alpha : \alpha \in \Lambda\}$), there is a nbd O_b of b such that $f^{-1}O_b$ is covered by $\mathcal{U}$ and $(f^{-1}O_b \wedge \mathcal{U})$ has an open (in X) locally finite refinement in $f^{-1}O_b$.

(2) For $b \in B$, let $\mathcal{U}$ be an open cover of $f^{-1}b$. A collection $\mathcal{V}$ of subsets of X is said to be a *star refinement* of $\mathcal{U}$ if $V \cap f^{-1}b \neq \emptyset$ for every $V \in \mathcal{V}$, $f^{-1}b \subset \cup\{V : V \in \mathcal{V}\}$ and there is a nbd O_b of b such that $\mathcal{U}$ covers $f^{-1}O_b$ and $\{St(V, \mathcal{V})|V \in \mathcal{V}\}$ refines $f^{-1}O_b \wedge \mathcal{U}$.

Note that in Definitions 1.5 and 1.6, we assume that compact (and paracompact) maps are T_2, which is different from that in [9](and [2]). Some characterizations of paracompact maps can be found in [2] and [3]. Note that if f is paracompact then it is a closed map. The following is used in the later.

Theorem 1.1. *([2;Theorem 3.12])For a T_1-map $f : X \to B$, the following are equivalent.*

(1) The map f is paracompact.

(2) For every $b \in B$ and every open (in X) cover $\mathcal{U}$ of the fibre $f^{-1}b$, there is an open star refinement $\mathcal{V}$ of $\mathcal{U}$.

The following Definitions 1.7 and 1.8 can be seen in [2] and [4] respectively.

Definition 1.7. A map $f : X \to B$ is *finally compact* if f is closed and for every $b \in B$ the fibre $f^{-1}b$ is finally compact. A finally compact T_3-map is called a *Lindelöf* map.

Definition 1.8. Let $f : X \to B$ be a T_1-map.

(1) f is said to be *collectionwise prenormal* if for every discrete collection $\{F_\alpha : \alpha \in \Lambda\}$ of closed subsets of X and for every $b \in B$, there are $O_b \in N(b)$ and a collection of open subsets $\{U_\alpha : \alpha \in \Lambda\}$, such that $F_\alpha \cap f^{-1}O_b \subset U_\alpha$ and $\{U_\alpha\}$ are discrete in $f^{-1}O_b$. The map f is said to be *collectionwise normal* if for every open set O of B, $f|f^{-1}O : f^{-1}O \to O$ is collectionwise prenormal.

(2) The sequence $\mathcal{W}_1, \mathcal{W}_2, \dots$ of open (in X) covers of $f^{-1}b, b \in B$, is said to be a *b-development* if for every $x \in f^{-1}b$ and every $U(x)$ of x in X, there exist $i < \omega$ and $O \in N(b)$ such that $x \in St(x, \mathcal{W}_i) \cap f^{-1}O \subset U(x)$. The map f is said to have an *f-development* if it has a b-development for every $b \in B$.

(3) The map f is *metrizable type* (abbreviated *MT*) if it is closed and collectionwise normal and has an f-development.

If f is collectionwise normal, it is easy to see that f is regular, therefore f is normal. Thus f is T_4. We use the following theorem in the later.

Proposition 1.2. *([4;Proposition 2.7])Every paracompact map is collectionwise normal.*

Definition 1.9. For a fibrewise space (X, f), a collection $\mathcal{B}_b$ of open sets of X is said to be a *base at b for the map* $f, b \in B$, if for every $x \in f^{-1}b$ and every open nbd $U(x)$ of x there are $O \in N(b)$ and $V \in \mathcal{B}_b$ such that $x \in V \cap f^{-1}O \subset U(x)$. One can assume that for every $V \in \mathcal{B}_b$ we have $V \cap f^{-1}b \neq \emptyset$.

Thus $\mathcal{B}_f = \cup\{\mathcal{B}_b | b \in B\}$, where $\mathcal{B}_b$ is a base at b for f, will give a base for the map f. Conversely, if $\mathcal{B}_f$ is a base for f, by taking $\mathcal{B}_f(b) = \{V \in \mathcal{B}_f | V \cap f^{-1}b \neq \emptyset\}$ one gets a base at $b \in B$ for the map f.

Definition 1.10. Let $f : X \to B$ be a map and $b \in B$. A collection $\mathcal{U}$ of subsets of X is said to be *b-discrete* if there is a nbd $O_b \in N(b)$ such that $\mathcal{U} \wedge f^{-1}O_b$ is discrete in $f^{-1}O_b$. Further $\mathcal{U}$ is said to be *b-σ-discrete* if $\mathcal{U} = \cup\{\mathcal{U}_n | n \in \mathbf{N}\}$ and each $\mathcal{U}_n$ is b-discrete for each $n \in \mathbf{N}$.

The following is used in the later.

Theorem 1.3. *([4;Theorem 2.12]) For a map $f : X \to B$, the following are equivalent:*

(1) f is an MT-map.

(2) f is a closed T_3-map with a b-σ-discrete b-base for every $b \in B$.

Definition 1.11. A compact map $f : X \to B$ is *compact metrizable type* (abbreviated *MT-*)map if it is compact and MT-map.

For CMT-map, we have the following:

Theorem 1.4. *([4;Theorem 2.18]) A compact map $f : X \to B$ is a MT-map if and only if it has a countable b-base for every $b \in B$.*

Definition 1.12. For the collection of maps $f_\alpha : X_\alpha \to B$, $\alpha \in \Lambda$, the subspace $X = \{x = \{x_\alpha\} \in \prod\{X_\alpha : \alpha \in \Lambda\} : f_\alpha x_\alpha = f_\beta x_\beta, \forall \alpha, \beta \in \Lambda\}$ of the Tychonoff product $\prod = \prod\{X_\alpha : \alpha \in \Lambda\}$ is called the *fan product* of the spaces X_α with respect to the maps $f_\alpha, \alpha \in \Lambda$ and is denoted by $\{X_\alpha rel f_\alpha : \alpha \in \Lambda\}$.

For the projection $pr_\alpha : \prod \to X_\alpha$, the restriction π_α on X is called the *projection* of the fan product onto the factor $X_\alpha, \alpha \in \Lambda$. From the definition of fan product we have that, $f_\alpha \circ \pi_\alpha = f_\beta \circ \pi_\beta$ for every α and β in Λ. Thus one can define a map $f : X \to B$, called the projection of the fan product, by

$$f = f_\alpha \circ \pi_\alpha, \alpha \in \Lambda$$

Obviously, the projections f and $\pi_\alpha, \alpha \in \Lambda$, are continuous.

The projection f is also called the *fibrewise product* of the maps $f_\alpha, \alpha \in \Lambda$ (since for every point $b \in B$, the inverse image $f^{-1}b$ is homeomorphic to the Tychonoff product of the fibres $f_\alpha^{-1}b, \alpha \in \Lambda$). The fact that f is the fibrewise product of the maps $f_\alpha, \alpha \in \Lambda$, will be denoted by $f = \prod\{f_\alpha : \alpha \in \Lambda\}$.

In particular the fan product P of the spaces X and Y with respect to the maps $f : X \to B$ and $g : Y \to B$ will be denoted by $X_f \times_g Y$ and the projections π_α by π_X and π_Y.

2. Fibrewise absolute (nbd) retracts and extensors

In this section, we shall introduce the fundamental notions in fibrewise retract theory; i.e., fibrewise retraction, fibrewise (nbd) retraction of maps, fibrewise (nbd) extension of maps, mapping class Q_B of fibrewise maps with the property Q over a base space B, fibrewise absolute (nbd) retract for a class Q_B and fibrewise absolute (nbd) extensor for a class Q_B. Further we will study of some properties of these notions. For retract theory in general topology, one can consult [1],[8] and [6].

We shall begin with the following definitions.

Definition 2.1. Let X be a space and A a subspace of X. Let (X, f) and (A, g) be two fibrewise spaces.

(1) A map $r : X \to A$ is said to be a *fibrewise retraction* of f to g if it is a retraction and a fibrewise map i.e., $f = g \circ r$. In this case, g is said to be a *retract* of f.

(2) If there are a nbd Y of A in X and a fibrewise retraction $r : Y \to A$, g is said to be a *nbd retract* of f.

Definition 2.2. Let X be a fibrewise space over B. A fibrewise map $e : X \to X$ is said to be *fibrewise idempotent* if $e \circ e = e$.

Proposition 2.1. *Let X be a fibrewise space over B, A a subspace of X and $r : X \to A$ a fibrewise retraction. For the inclusion $h : A \to X$, the composition $e = h \circ r : X \to X$ is fibrewise idempotent.*

Proof. Since r and h are fibrewise maps, e is obviously a fibrewise map. Since r is a fibrewise retraction, the composed map $r \circ h$ is the identity map i on A. Hence, we have

$e \circ e = (h \circ r) \circ (h \circ r) = h \circ (r \circ h) \circ r = h \circ i \circ r = h \circ r = e.$

This proves that e is idempotent. □

Conversely, we have the following.

Proposition 2.2. *Let (X, f) be a fibrewise space, $e : X \to X$ a fibrewise idempotent, $e(X) = A$ and $r : X \to A$ a map defined by $r(x) = e(x)$. Then r is a fibrewise retraction of f to $f|A$.*

Proof. Since the given map e is fibrewise idempotent, we have $e = e \circ e$ and r is a fibrewise map. Let a be an arbitrary point of A. Since $A = e(X)$, there is a point $x \in X$ with $a = e(x)$. Hence we have

$r(a) = e(a) = e(e(x)) = (e \circ e)(x) = e(x) = a.$

This proves that r is a retraction. □

Proposition 2.3. *Let X be a space and A a subspace of X. For two fibrewise spaces (X, f) and (A, g), if f is a Hausdorff map and g is a retract of f, A is closed in X.*

Proof. Let $r : X \to A$ be a fibrewise retraction. We shall prove that the complement $M = X - A$ is an open set of X. Let x be an arbitrary point in M. Then $r(x)$ is a point in A. Since r is fibrewise, $g(r(x)) = f(x)$. Denote $a = r(x)$ and $b = g(a) = f(x)$. Since $a \in A$, $x \in M$ and $f(x) = f(a)$, we have $a \neq x$ and

$a \in f^{-1}b, x \in f^{-1}b$. Since f is a Hausdorff map, there are two open set U and V of X such that $a \in U, x \in V, U \cap V = \emptyset$. By the continuity of r, $r^{-1}(U \cap A)$ is an open set containing the point x. It follows that $W = r^{-1}U \cap V$ is an open set containing x. It remains to prove that $W \subset M$. For this purpose, let z be an arbitrary point in W. Then $z \in V$ and $z \in r^{-1}U$. The latter implies $r(z) \in U$. Since U and V are disjoint, it follows that $r(z) \neq z$. Hence $z \in M$. This proves $W \subset M$. □

Definition 2.3. Let $(X,f),(A,g)$ and (Z,h) be fibrewise spaces, $A \subset X$ and $g = f|A$. For fibrewise maps $\psi : X \to Z$ and $\varphi : A \to Z$,

(1) ψ is called to be a *fibrewise extension* of φ if $\psi|A = \varphi$.

(2) ψ is called to be a *fibrewise nbd extension* of φ if there is a nbd U of A in X with $\psi|U = \varphi$.

Proposition 2.4. *Let (X,f) and (A,g) be fibrewise spaces and $A \subset X$, $g = f|A$. Then g is a retract of f if and only if, for every fibrewise space (Z,h) and every fibrewise map $\varphi : A \to Z$, φ has a fibrewise extension of X to Z.*

Proof. "Only if" part: Since g is a retract of f, there is a fibrewise retraction $r : X \to A$. Then it is easy to see that $\varphi \circ r$ is a fibrewise extension of φ.

"If" part: Since the identity i of A is a fibrewise map for (A,g), then by taking (Z,h) to be (A,g), i has a fibrewise extension r from the condition. Therefore r is a fibrewise retraction of f to g. □

Let Q_B be a class of projections $f : X \to B$ satisfying a topological property Q; i.e. $Q_B=\{(X,f)|f : X \to B$ is a projection satisfying $Q\}$. We call Q_B is a *fibrewise class* satisfying Q. For example, we consider as Q, perfect maps, paracompact maps, normal maps, metrizable type (MT-)maps, etc. Further, we require that "If $(X,f) \in Q_B$ and A is a closed subset of X, then $(A, f|A) \in Q_B$". For $(X,f) \in Q_B$ and a closed subset A of X, a pair $((X,f),(A,f|A))$ will be called a *Q_B-pair*. In this case, we will simply write that (X,A) is a Q_B-pair.

Definition 2.4. Let (X,f) be a fibrewise space. (X,f) is said to be an *absolute* (resp. *nbd) retract* over B relative to a fibrewise class Q_B if, for every $(Z,h) \in Q_B$ satisfying (Z,X) is a Q_B-pair, f is a (resp. nbd) retract of h. By $AR_B(Q_B)$ (resp. $ANR_B(Q_B)$) we denote all absolute (resp. nbd) retracts over B relative to Q_B.

Definition 2.5. Let (Z,h) be a fibrewise space. (Z,h) is said to be an *absolute* (resp. *nbd) extensor* over B relative to a fibrewise class Q_B if, for every fibrewise map $\varphi : A \to Z$, where A is a closed subspace of X with $(X,f) \in Q_B$, φ has a fibrewise (resp. nbd) extension to X(resp. a nbd U of A). By $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$) we denote all absolute (resp. nbd) extensor over B relative to Q_B.

We shall prove the following propositions for any fibrewise class Q_B. The first two propositions are obvious.

Proposition 2.5. *Every $(X,f) \in AR_B(Q_B)$ is also in $ANR_B(Q_B)$.*

Proposition 2.6. *Every $(E,g) \in AE_B(Q_B)$ is also in $ANE_B(Q_B)$.*

Proposition 2.7. *Let $(E,g) \in Q_B$. If (E,g) is in $AE_B(Q_B)$, then it is also in $AR_B(Q_B)$. If it is in $ANE_B(Q_B)$, then it is also in $ANR_B(Q_B)$.*

Proof. For any $(X,f) \in Q_B$ such that (X,E) is a Q_B-pair, since (E,g) is in $AE_B(Q_B)$, the identity $i : E \to E$ has a fibrewise extension φ of i to X. Since φ is also a fibrewise retraction of f to g, g is a retract of f. Hence $(E,g) \in AR_B(Q_B)$. The case $ANE_B(Q_B)$ follows by the same steps. □

The following two propositions are obvious.

Proposition 2.8. *Let Q_B and Q'_B be fibrewise classes such that $Q_B \subset Q'_B$. If $(E,g) \in AE_B(Q'_B)$ (resp. $ANE_B(Q'_B)$), then it is also in $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$).*

Proposition 2.9. *Let Q_B and Q'_B be fibrewise classes such that $Q_B \subset Q'_B$. If $(X,f) \in AR_B(Q'_B)$ (resp. $ANR_B(Q'_B)$) and it is in Q_B, then it is also in $AR_B(Q_B)$ (resp. $ANR_B(Q_B)$).*

Proposition 2.10. *Let $(E,f) \in AE_B(Q_B)$ (resp. $ANE_B(Q_B)$). If, for a fibrewise space (Y,g), g is a retract (resp. nbd retract) of f, then (Y,g) is in $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$).*

Proof. For any $(X,h) \in Q_B$ and any closed subset A of X, let $\varphi : A \to Y$ be a fibrewise map. The map φ can also be considered as a map of A into E. Since $(E,f) \in AE_B(Q_B)$, φ has an extension $\psi : X \to E$. On the other hand, since g is a retract of f, there is a fibrewise retraction $r : E \to Y$. Then it is easily seen that $r \circ \psi : X \to Y$ is a fibrewise extension of φ. Hence, (Y,g) is in $AE_B(Q_B)$. The case $ANE_B(Q_B)$ can be similarly proved. □

Proposition 2.11. *Let $(E,g) \in ANE_B(Q_B)$. If O is an open set of E, $(O, g|O)$ is in $ANE_B(Q_B)$.*

PROOF. Let $(X.f) \in Q_B$ and A a closed subset of X. For a fibrewise map $\varphi : A \to O$, this φ can also be considered as a map of A into E. Since (E, g) is in $ANE_B(Q_B)$, there are a nbd U of A in X and a map $\psi : U \to X$ which is a fibrewise extension of $\varphi : A \to E$. Then $\psi|\psi^{-1}(O) : \psi^{-1}(O) \to O$ is a fibrewise extension of φ, which shows that $(O, g|O)$ is in $ANE_B(Q_B)$. □

Proposition 2.12. *Any fan product of $AE_B(Q_B)$ is in $AE_B(Q_B)$.*

PROOF. Let $(X_\alpha, f_\alpha) \in AE_B(Q_B)$, $\alpha \in \mathcal{A}$, X the fan product of the spaces X_α, $\alpha \in \mathcal{A}$, and f the projection of X to B. For any $(Z, h) \in Q_B$ and any closed subset A of Z, let $\varphi : A \to X$ be a fibrewise map. Since $(X_\alpha, f_\alpha) \in AE_B(Q_B)$, for each $\alpha \in \mathcal{A}$, $\pi_\alpha \circ \varphi : A \to X_\alpha$ has a fibrewise extension $\psi_\alpha : Z \to X_\alpha$ of $\pi_\alpha \circ \varphi$. We define $\psi : Z \to X$ by $\psi(z) = \prod \psi_\alpha(z)$. Then it is easily verified that ψ is a fibrewise extension of φ. □

The following proposition can be proved by the same steps as in the above.

Proposition 2.13. *Any fan product of a finite number of $ANE_B(Q_B)$ is in $ANE_B(Q_B)$.*

3. Fibrewise adjunction spaces

In this section, we define the concept of fibrewise adjunction spaces, and consider some properties of this concept. We shall prove that some fibrewise classes are preserved under the fibrewise adjunction operation, and for such a class Q_B, a fibrewise space in Q_B is in $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$) if and only if it is in $AR_B(Q_B)$ (resp. $ANR_B(Q_B)$).

Definition 3.1. Let (X, f) and (Y, g) be fibrewise spaces, A a closed subspace of X, $\varphi : A \to Y$ a fibrewise map with $g \circ \varphi = f|A$ and $X \cup_\varphi Y (= Z)$ the adjunction space. A map $h : Z \to B$ defined by

$$h(x) = \begin{cases} f(x) & (x \in X - A) \\ g(x) & (x \in Y) \end{cases}$$

is a fibrewise map. Then (Z, h) is called a *fibrewise adjunction space* determined by $(X, f), (Y, g)$ and φ, and h an *adjunction map* determined by f, g and φ, which is denoted by $f \cup_\varphi g$.

In this definition, there are three natural maps $j : X \to Z$, $k : Y \to Z$ and $p : X \cup Y \to Z$. We use these j, k and p in this section, and $X - A$ and Y are identified with $p(X - A)$ and $p(Y)$, respectively.

The fibrewise retraction problem, as defined in the previous section, is a special case of the fibrewise extension problem. Also, we can see from Proposition 2.4 that the notion of retracts of maps gives a necessary and sufficient condition for the fibrewise extension problem to be trivial. On the other hand, the fibrewise extension problem for a given map $\varphi : A \to Y$, defined on a closed subspace A of a fibrewise space X into another fibrewise space Y can be reduced to a fibrewise retraction problem by means of the fibrewise adjunction space. We have the following proposition.

Proposition 3.1. *Using the notation of Definition 3.1, g is a retract of h if and only if φ has a fibrewise extension to a map of X to Y.*

Proof. "Only if" part: Let $r : Z \to Y$ be a fibrewise retraction. Define a map $\psi : X \to Y$ be taking $\psi(x) = r(p(x))$ for every point $x \in X$. For each $x \in A$, we have $f(x) = r(p(x)) = r(\varphi(x)) = \varphi(x)$. This proves that ψ is a fibrewise extension of φ.

"If" part: Let $\psi : X \to Y$ be a fibrewise extension of the given map φ. Define a function $r : Z \to Y$ as follows. Let z be an arbitrary point of Z. If $z \in Y$, let $r(z) = z$. If $z \notin Y$, then there is a unique point x in $X - A$ with $p(x) = z$. Define $r(z) = \psi(x)$. Let $s = r \circ p : X \cup Y \to Y$. It follows that $s|X = \psi$ and that $s|Y$ is the identity map. Therefore s is continuous. Since p is the natural projection, it follows that r is continuous. Since $r|Y$ is the identity map by definition and $g \circ r = h$, we conclude that r is a fibrewise retraction. □

We use the following notations of fibrewise classes in this paper.

$\mathcal{N}_B$ = All fibrewise spaces with normal T_3 maps

$\mathcal{FN}_B$ = All fibrewise spaces with functionally normal maps

$\mathcal{CN}_B$ = All fibrewise spaces with collectionwise normal maps

$\mathcal{P}_B$ = All fibrewise spaces with paracompact maps

$\mathcal{L}_B$ = All fibrewise spaces with Lindelöf maps

$\mathcal{C}_B$ = All fibrewise spaces with compact maps

$\mathcal{M}_B$ = All fibrewise spaces with MT-maps

$\mathcal{CM}_B$ = All fibrewise spaces with CMT-maps

We shall show in the following propositions that each of $\mathcal{N}_B$, $\mathcal{CN}_B$, $\mathcal{P}_B$, $\mathcal{L}_B$, $\mathcal{C}_B$ and $\mathcal{CM}_B$ is closed under the fibrewise adjunction operation.

Proposition 3.2. *Let (X, f) and (Y, g) be in $\mathcal{N}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{N}_B$.*

PROOF. Let $X \cup_\varphi Y = Z$ and $f \cup_\varphi g = h$. First, we shall show that h is a T_0-map. For all $z, z' \in Z$ such that $z \neq z', hz = hz'$, we must consider three cases: (a) $z, z' \in Y$, (b) $z \in X - A, z' \in Y$, and (c) $z, z' \in X - A$. For the case (a), since g is a T_0-map, there is a nbd U of z in Y with $z' \notin U$. Then $U \cup (X - A)$ is a nbd of z in Z which does not contain z'. For the other cases (b) and (c), since f is a T_0-map and $X - A$ is open in Z, it is easy to see that at least one of the points z, z' has a nbd in Z not containing the other point. Thus, h is a T_0-map.

Next, we shall show that h is a regular map. For a point $z \in Z$ and a closed set F of Z such that $z \notin F$, we must consider two cases: (1) $z \in X - A$, and (2) $z \in Y$. For the case (1), since f is regular, there is $O \in N(fz)(= N(hz))$ such that the sets z and $A \cup j^{-1}F$ have disjoint nbds U and V in $f^{-1}O$, respectively. Then $U \cap (X - A) \cap h^{-1}O$ and $p(V \cup Y) \cap h^{-1}O$ are disjoint nbds of z and $F \cap h^{-1}O$ in $h^{-1}O$, respectively. For the case (2), since g is a regular map, there is $O \in N(gz)(= N(hz))$ such that the sets z and $k^{-1}F \cap g^{-1}O$ have nbds U and V in $g^{-1}O$, respectively, with $[U]_Y \cap [V]_Y \cap g^{-1}O = \emptyset$. For disjoint closed (in $f^{-1}O$) sets $\varphi^{-1}[U]_Y \cap f^{-1}O$ and $(j^{-1}F \cup \varphi^{-1}[V]_Y) \cap f^{-1}O$, since f is normal, there is $O_1 \in N(hz)$ such that $O_1 \subset O$ and there are disjoint nbds U_1 and V_1 in $f^{-1}O_1$ of the above sets, respectively. Then we shall show that $(j(U_1 \cap (X - A)) \cup kU) \cap h^{-1}O_1 = U'$ and $(j(V_1 \cap (X - A)) \cup kV) \cap h^{-1}O_1 = V'$ are disjoint nbds of the sets z and $F \cap h^{-1}O_1$, respectively. Since it is easy to see that $z \in U', F \cap h^{-1}O_1 \subset V'$ and $U' \cap V' = \emptyset$, we show that U' and V' are open in Z. For U', it is sufficient to show that $j^{-1}U'$ is open in X because $k^{-1}U' = U \cap g^{-1}O_1$ is open in Y. Since $\varphi^{-1}(U \cap g^{-1}O_1)$ is open in A, there is an open set W of X such that $W \cap A = \varphi^{-1}(U \cap g^{-1}O_1)$. Since $U_1 \cap ((X - A) \cup W) = j^{-1}U'$, $j^{-1}U'$ is open in X. For V', we can prove in a similar way. Thus h is a regular map.

Finally, we shall show that h is a normal map. It is enough to show that h is prenormal, because it is easy to see, in the same way, that $h|h^{-1}O$ is prenormal for every open set O of Z. For every $b \in B$ and any two disjoint closed sets F and H of Z, since g is normal, there is $O_1 \in N(b)$ such that $F \cap Y$ and $H \cap Y$ are separated in $g^{-1}O_1$. Therefore, there are two open sets U_1 and U_2 of Y such that $k^{-1}F \cap g^{-1}O_1 \subset U_1, k^{-1}H \cap g^{-1}O_1 \subset U_2$ and $[U_1]_Y \cap [U_2]_Y \cap g^{-1}O_1 = \emptyset$. Since $\varphi^{-1}([U_1]_Y \cap g^{-1}O_1) \cup (j^{-1}F \cap (X - A) \cap f^{-1}O_1)$ and $\varphi^{-1}([U_2]_Y \cap g^{-1}O_1) \cup (j^{-1}H \cap (X - A) \cap f^{-1}O_1)$ are disjoint closed sets of $f^{-1}O_1$ and f is normal, there is $O_2 \in N(b)$ with $O_2 \subset O_1$ such that the above disjoint closed sets have disjoint nbds V_1 and V_2 in $f^{-1}O_2$, respectively. Then $j(V_1 \cap (X - A)) \cup k(U_1 \cap g^{-1}O_2)$ and $j(V_2 \cap (X - A)) \cup k(U_2 \cap g^{-1}O_2)$ are disjoint nbds of $F \cap h^{-1}O_2$ and $H \cap h^{-1}O_2$ in $h^{-1}O_2$. Thus h is prenormal. $\square$

Proposition 3.3. *Let (X, f) and (Y, g) be in $\mathcal{CN}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{CN}_B$.*

Proof. Let $f \cup_\varphi g = h$. Since (X, f) and (Y, g) are in $\mathcal{N}_B$, note by Proposition 3.2 that h is a T_4-map. Further, it is easy to see by the same steps in the proof of Proposition 3.2 that h is collectionwise normal. □

Proposition 3.4. *Let (X, f) and (Y, g) be in $\mathcal{P}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{P}_B$.*

To prove this proposition, we need the following lemma. Let $X \cup_\varphi Y = Z$ and $f \cup_\varphi g = h$.

Lemma 3.5. *Under the same conditions of Proposition 3.4, for a point $b \in B$ and any nbd $O_b \in N(b)$, let $\{U_\lambda\}$ be any b-locally finite (in Y) cover of $g^{-1}b$. Then there are a nbd $O_b^{'} \in N(b)$ and an open (in Z) locally finite collection $\{W_\lambda\}$ in $h^{-1}O_b^{'}$ such that $O_b^{'} \subset O_b$ and $U_\lambda \cap g^{-1}O_b^{'} = W_\lambda \cap g^{-1}O_b^{'}$ for each λ.*

Proof. Let $\mathcal{U} = \{U_\lambda\}$. For $\mathcal{U}$, since (Y, g) is in $\mathcal{P}_B$, there is an open (in Y) cover $\mathcal{U}' = \{U_\mu^{'}\}$ of $g^{-1}b$ such that any $U_\mu^{'} \in \mathcal{U}'$ meets only a finite number of elements of $\mathcal{U}$. We may assume that $\mathcal{U}'$ is locally finite in $g^{-1}O_b^{'}$ for some $O_b^{'} \in N(b)$ with $O_b^{'} \subset O_b$ because (Y, g) is in $\mathcal{P}_B$. Further, there is an open (in Y) cover $\mathcal{U}^{"} = \{U_\nu^{"}\}$ of $g^{-1}O_b^{'}$ such that each $U_\nu^{"}$ meets only a finite number of elements of $\mathcal{U}'$. Since $\varphi^{-1}Y = A$, we have that $\{\varphi^{-1}U_\lambda\}, \{\varphi^{-1}U_\mu^{'}\}$ and $\{\varphi^{-1}U_\nu^{"}\}$ are open (in A) covers of $\varphi^{-1}g^{-1}O_b^{'}$. Put $V_\mu^{'} = \varphi^{-1}U_\mu^{'} \cup (X - A)$ for each μ. Then $\mathcal{V}' = \{V_\mu^{'}\}$ is an open (in X) cover of $f^{-1}b$. Since f is paracompact, there is $O_b^{"} \in N(b)$ such that $O_b^{"} \subset O_b^{'}$ and $\mathcal{V}' \wedge f^{-1}O_b^{"}$ has an open (in X) star refinement $\mathcal{G} = \{G_\kappa\}$. For each λ, let

$$V_\lambda = (\varphi^{-1}U_\lambda \cup (St(\varphi^{-1}U_\lambda, \mathcal{G}) - A)) \cap f^{-1}O_b^{"}.$$

Then V_λ is open in $f^{-1}O_b^{"}$. Next, for each λ, let

$$W_\lambda = (U_\lambda \cup pV_\lambda) \cap h^{-1}O_b^{"},$$

which is open in $h^{-1}O_b^{"}$ and $U_\lambda \cap g^{-1}O_b^{"} = W_\lambda \cap Y$. We shall show that $\{W_\lambda\}$ is locally finite in $h^{-1}O_b^{"}$. For an arbitrary point $z \in h^{-1}O_b^{"}$, we consider the following two cases: (1) $z \in Z - Y$, and (2) $z \in Y$.

Case (1): It is enough to show that $\{V_\lambda\}$ is locally finite in $f^{-1}O_b^{"}$. If $G_\kappa \cap V_\lambda \neq \emptyset$, then $G_\kappa \cap St(\varphi^{-1}U_\lambda, \mathcal{G}) \neq \emptyset$. So $St(G_\kappa, \mathcal{G}) \cap \varphi^{-1}U_\lambda \neq \emptyset$. Therefore, there is

$V_\mu^{'} \in \mathcal{V}'$ such that $St(G_\kappa, \mathcal{G}) \subset V_\mu^{'}$. From this we have $\varphi^{-1}U_\lambda \cap (\varphi^{-1}U_\mu^{'} \cup (X - A)) \neq \emptyset$, so that $\varphi^{-1}U_\lambda \cap \varphi^{-1}U_\mu^{'} \neq \emptyset$. But for fixed μ, this is possible only for a finite number of λ's. Hence, since $\{G_\kappa\}$ covers $f^{-1}O_b^{"}$, $\{V_\lambda\}$ is locally finite at $z \in j(X - A) = Z - Y$.

Case (2): We construct a nbd of z in $h^{-1}O_b^{"}$ as follows. Starting with some $U_\nu^{"}$ containing z, we put

$$V_\nu^{"} = (\varphi^{-1}U_\nu^{"} \cup (St(\varphi^{-1}U_\nu^{"}, \mathcal{G}) - A)) \cap f^{-1}O_b^{"},$$

and

$$W_\nu^{"} = (pV_\nu^{"} \cup U_\nu^{"}) \cap h^{-1}O_b^{"}.$$

Then it is easy to see that $W_\nu^{"}$ is a nbd of z in Z. Now let $W_\nu^{"} \cap W_\lambda \neq \emptyset$.

If $W_\nu^{"} \cap W_\lambda \subset Y$, we have $U_\nu^{"} \cap U_\lambda \neq \emptyset$. But $U_\nu^{"}$ only meets a finite number of sets $U_\mu^{'}$, each meeting only a finite number of sets U_λ. Hence, since ν is fixed, λ that satisfies $W_\nu^{"} \cap W_\lambda \neq \emptyset$ is restricted to a finite number of values.

If $W_\nu^{"} \cap W_\lambda \not\subset Y$, there is a point $z_1 \in (W_\nu^{"} \cap W_\lambda) \cap (Z - Y)$. Then

$$y_1 = j^{-1}(z_1) \in j^{-1}W_\nu^{"} \cap j^{-1}W_\lambda = V_\nu^{"} \cap V_\lambda \subset St(j^{-1}U_\nu^{"}, \mathcal{G}) \cap St(j^{-1}U_\lambda, \mathcal{G}),$$

where one can note that $j^{-1}U_\nu^{"} = \varphi^{-1}U_\nu^{"}, j^{-1}U_\lambda = \varphi^{-1}U_\lambda$. This implies that $St(y_1, \mathcal{G}) \cap j^{-1}U_\nu^{"} \neq \emptyset$ and $St(y_1, \mathcal{G}) \cap j^{-1}U_\lambda \neq \emptyset$. But $\mathcal{G}$ is an open star refinement of $\mathcal{V}' \wedge f^{-1}O_b^{"}$. Hence,

$$St(y_1, \mathcal{G}) \subset V_\mu^{'} \cap f^{-1}O_b^{"} = (\varphi^{-1}U_\mu^{'} \cup (X - A)) \cap f^{-1}O_b^{"}$$

for some μ. We obtain $j^{-1}U_\mu^{'} \cap j^{-1}U_\nu^{"} \neq \emptyset$ and $j^{-1}U_\mu^{'} \cap j^{-1}U_\lambda \neq \emptyset$. Again we see that, since ν is fixed, λ that satisfies $W_\nu^{"} \cap W_\lambda \neq \emptyset$ is restricted to a finite number of values. Hence $\{W_\lambda\}$ is locally finite in $h^{-1}O_b^{"}$. This completes the proof of Lemma 3.5. □

We can now prove Proposition 3.4.

Proof. We take a point $b \in B$ and any open (in Z) cover $\mathcal{R} = \{R_\kappa\}$ of $h^{-1}b$. Since (X, f) and (Y, g) are in $\mathcal{P}_B$, there is a nbd $O_b \in N(b)$ such that $f^{-1}O_b$ is covered by $j^{-1}\mathcal{R}$ and $f^{-1}O_b \wedge j^{-1}\mathcal{R}$ has an open (in X) locally finite refinement $\mathcal{V}_1$ in $f^{-1}O_b$, and $g^{-1}O_b$ is covered by $k^{-1}\mathcal{R}$ and $g^{-1}O_b \wedge k^{-1}\mathcal{R}$ has an open (in Y) locally finite refinement $\mathcal{U} = \{U_\lambda\}$ in $g^{-1}O_b$. For each U_λ, choose some $R_{\kappa_\lambda} \in \mathcal{R}$ such that $U_\lambda \subset R_{\kappa_\lambda}$. By Lemma 3.5, there are a nbd $O_b^{'} \in N(b)$ and an open (in Z) locally finite collection $\{W_\lambda\}$ in $h^{-1}O_b^{'}$ such that $O_b^{'} \subset O_b$ and $U_\lambda \cap g^{-1}O_b^{'} = W_\lambda \cap g^{-1}O_b^{'}$ for each λ. We may assume that $W_\lambda \subset R_{\kappa_\lambda}$ otherwise replacing W_λ by $W_\lambda \cap R_{\kappa_\lambda}$. Put $W = \bigcup_\lambda W_\lambda$. Then W is an open (in $h^{-1}O_b^{'}$)

set containing $Y \cap h^{-1}O_b^{'}$. Hence $j^{-1}W \cap f^{-1}O_b^{'}$ is open in $f^{-1}O_b^{'}$ and contains $A \cap f^{-1}O_b^{'}$. Since f is a paracompact map, $f_1 = f|f^{-1}O_b^{'} : f^{-1}O_b^{'} \to O_b^{'}$ is also paracompact, so f_1 is normal ([2] Proposition 3.2). The disjoint closed (in $f^{-1}O_b^{'}$) sets $A \cap f^{-1}O_b^{'}$ and $f^{-1}O_b^{'} - W$ are nbd separated in $f_1^{-1}O_b^{"}$ for some $O_b^{"} \in N(b)$ satisfying $O_b^{"} \subset O_b^{'}$. Therefore, there are disjoint open (in $f_1^{-1}O_b^{"} = f^{-1}O_b^{"}$) sets K and L such that $A \cap f^{-1}O_b^{"} \subset K$ and $f^{-1}O_b^{"} - W \subset L$. Then it is easily verified that $(\mathcal{V}_1 \wedge L) \cup (\{W_\lambda\} \wedge h^{-1}O_b^{"})$ is an open (in Z) locally finite refinement of $\mathcal{R}$ in $h^{-1}O_b^{"}$. This completes the proof of Proposition 3.4. □

Proposition 3.6. *Let (X, f) and (Y, g) be in $\mathcal{L}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{L}_B$.*

Proof. Let $X \cup_\varphi Y = Z$ and $f \cup_\varphi g = h$. By [2] Proposition 5.3 and Corollary 5.4, every Lindelöf map is a paracompact T_4-map. So (Z, h) is in $\mathcal{P}_B$ by Propositions 3.2 and 3.4. Thus h is closed. For any $b \in B$, it is easily verified that $h^{-1}b$ is finally compact. Thus (Z, h) is Lindelöf. □

Proposition 3.7. *Let (X, f) and (Y, g) be in $\mathcal{C}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{C}_B$.*

Proof. Let $f \cup_\varphi g = h$. Then it is not difficult to see that h is a closed map and $h^{-1}b$ is compact for each $b \in B$. □

Proposition 3.8. *Let (X, f) and (Y, g) be in $\mathcal{CM}_B$, A a closed subspace of X and $\varphi : A \to Y$ a fibrewise map. Then the fibrewise adjunction space $(X \cup_\varphi Y, f \cup_\varphi g)$ is in $\mathcal{CM}_B$.*

Proof. Let $X \cup_\varphi Y = Z$ and $f \cup_\varphi g = h$. By Proposition 3.7, h is a compact map. We shall show that, for each $b \in B$, $h^{-1}b$ has a countable b-base. By [4] Theorem 2.16, $f^{-1}b$ has a countable b-base $\mathcal{U}$ in X, and $g^{-1}b$ has a countable b-base $\mathcal{V}$ in Y. For every $V \in \mathcal{V}$, $\varphi^{-1}[V]_Y \cap f^{-1}b$ is compact. For each $V \in \mathcal{V}$, let

$$\mathcal{W}_V = \{Int_Z(V \cup (\bigcup_{i=1}^{n} U_i - A)) | \varphi^{-1}[V]_Y \cap f^{-1}b \subset \bigcup_{i=1}^{n} U_i, U_i \in \mathcal{U}, n \in \mathbf{N}\}$$

where Int_Z is the interior operator in Z. Then it is easy to see that $\mathcal{W}_V$ is countable, $\bigcup_{V \in \mathcal{V}} \mathcal{W}_V$ is countable, and $(\mathcal{U} \wedge (X - A)) \cup (\bigcup_{V \in \mathcal{V}} \mathcal{W}_V)$ is a countable b-base of $h^{-1}b$. Thus, by [4] Theorem 2.18, h is an MT-map. □

Finally, we obtain the following Theorem.

Theorem 3.9. *Let Q_B be one of the fibrewise classes $\mathcal{N}_B, \mathcal{CN}_B, \mathcal{P}_B, \mathcal{L}_B, \mathcal{C}_B$ and $\mathcal{CM}_B$. Then for any $(Y, g) \in Q_B$, it is in $AE_B(Q_B)$ (resp. $ANE_B(Q_B)$) if and only if it is in $AR_B(Q_B)$ (resp. $ANR_B(Q_B)$).*

Proof. We shall show only the case of $AE_B(Q_B)$ and $AR_B(Q_B)$, since the other case can be proved by a similar method.

"Only if" part: We assume that $(Y, g) \in Q_B$ is in $AE_B(Q_B)$. For every $(Z, h) \in Q_B$ satisfying (Z, Y) is a Q_B-pair, let $i : Y \to Y$ be the identity map. Since (Y, g) is in $AE_B(Q_B)$, i has a fibrewise extension $r : Z \to Y$. Then, it is obvious that r is a fibrewise retraction of h to g. Thus, (Y, g) is in $AR_B(Q_B)$.

"If" part: Let $(Y, g) \in Q_B$ be in $AR_B(Q_B)$. For any Q_B-pair $((X, f), (A, f|A))$ and for every fibrewise map $\varphi : A \to Y$, we take the fibrewise adjunction space $Z = X \cup_\varphi Y, h = f \cup_\varphi g$. Then from Propositions 3.2, 3.3, 3.4, 3.6, 3.7 and 3.8, (Z, h) is also in Q_B. Since (Z, Y) is a Q_B-pair and (Y, g) is in $AR_B(Q_B)$, there is a fibrewise retraction $r : Z \to Y$. Then, $r \circ p = X \to Y$ is a fibrewise extension of φ. Thus, (Y, g) is in $AE_B(Q_B)$. □

4. Fibrewise contractibility

In this section, we consider fibrewise contractibility. For a fibrewise space (X, f), by $(X \times I, q)$ we mean the fibrewise space $(X_f \times_{pr_B} (I \times B), f \times pr_B)$, where $pr_B : I \times B \to B$ is the (fibrewise) projection. We use the notation $(X \times I, q)$ for the rest of the paper.

Proposition 4.1. *Let Q_B be one of the fibrewise classes $\mathcal{P}_B, \mathcal{L}_B, \mathcal{C}_B, \mathcal{M}_B$ and $\mathcal{CM}_B$. Then for any $(X, f) \in Q_B$, $(X \times I, q) \in Q_B$.*

Proof. $\mathcal{P}_B$ case: Since it is obvious that pr_B is a compact map, it is paracompact [2] Theorem 4.5.

$\mathcal{L}_B$ case: Since pr_B is compact, q is Lindelöf by [2] Corollary 5.8.

$\mathcal{C}_B$ case: Since f and pr_B are compact maps, it is easy to see that q is compact.

$\mathcal{M}_B$ case: Since f and pr_B is paracompact, q is paracompact by $\mathcal{P}_B$ case. So, it is a closed T_3-map by [2] Proposition 3.2. It is enough to show from Theorem 1.3(also see,[4] Theorem 2.12) that, for each point $b \in B$, q has a b-σ-discrete b-base. Let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}_n$ be a b-σ-discrete b-base for f, and $\mathcal{V} = \{V_n | n \in \mathbf{N}\}$ a countable base of I. Then we put for each $V_k \in \mathcal{V}$

$$\mathcal{W}_{nk} = \{(U \times (V_k \times B)) \cap X \times I | U \in \mathcal{U}_n\}$$

and

$$\mathcal{W} = \bigcup_{n,k=1}^{\infty} \mathcal{W}_{nk}.$$

Then we shall prove that $\mathcal{W}$ is a b-σ-discrete b-base for q. First, it is obvious that each $\mathcal{W}_{nk}$ is b-discrete. Next, for every $(x,t,b) \in q^{-1}b$ and any nbd U of (x,t,b), there are a nbd M of x in X, a nbd $V_k \in \mathcal{V}$ of t in I and $O \in N(b)$ such that $(M \times V_k \times O) \cap X \times I \subset U$. Since $\mathcal{U}$ is a b-σ-discrete b-base for f, there are an $n \in \mathbf{N}, N \in \mathcal{U}_n$ and $O_b \in N(b)$ such that $O_b \subset O$ and $x \in N \cap f^{-1}O_b \subset M$. Therefore, $V = (N \times V_k \times B) \cap X \times I \in \mathcal{W}_{nk}$ and $(x,t,b) \in V \cap q^{-1}O_b \subset U$.

$\mathcal{CM}_B$ case: Since f and pr_B is compact, q is compact. For every point $b \in B$, since f is a compact MT-map, there is a countable b-base $\mathcal{U}$ by Theorem 1.4(also see, [4] Theorem 2.18). Put $\mathcal{W} = \{(U \times V \times B) \cap X \times I | U \in \mathcal{U}, V \in \mathcal{V}\}$ where $\mathcal{V}$ is a countable base of I. Then it is easy to see that $\mathcal{W}$ is a countable b-base for q. Thus, q is an MT-map by Theorem 1.4. □

Definition 4.1. (1)([10]) By a *section* of a fibrewise space, we mean a continuous right inverse of the projection. A fibrewise map $\phi : X \to Y$ is *fibrewise constant* where (X,f) and (Y,g) are fibrewise spaces, if $\phi = t \circ f$ for some section $t : B \to Y$.

(2)([10]) Let $\theta, \phi : X \to Y$ be fibrewise maps. A *fibrewise homotopy* of θ into ϕ is a fibrewise map $H : X \times I \to Y$ such that $H(x,0) = \theta(x)$ and $H(x,1) = \phi(x)$. In this case, we say θ is *fibrewise homotopic* to ϕ and write $\theta \simeq_B \phi$. A fibrewise homotopy into a fibrewise constant map is called a *fibrewise nullhomotopy*.

(3)([10]) A fibrewise space X is *fibrewise contractible* if the identity on X is fibrewise nullhomotopic.

Proposition 4.2. *Let (X,f) be a fibrewise contractible space and A a subspace of X. If $f|A$ is a retract of f, then $(A, f|A)$ is fibrewise contractible.*

Proof. Since (X,f) is fibrewise contractible, there are a section $s : B \to X$ and a fibrewise map $H : X \times I \to X$ such that $s \circ f$ is fibrewise constant and H is a fibrewise homotopy of id_X to $s \circ f$. Since $f|A$ is a retract of f, there is a fibrewise retraction $r : X \to A$. We define a function $K : A \times I \to A$ as follows: $K = r \circ H \circ i$, where $i : A \times I \to X \times I$ is the inclusion map. Then it is easy to see that K is a fibrewise map and for each $x \in A$

$$K(x,0) = r \circ H \circ i(x,0) = r \circ H(x,0) = r \circ id_X(x) = r(x) = x = id_A(x),$$

$$K(x,1) = r \circ H \circ i(x,1) = r \circ H(x,1) = r \circ (s \circ f)(x) = (r \circ s) \circ (f|A)(x).$$

Since $r \circ s : B \to A$ is a section, K is a fibrewise homotopy of id_A to a fibrewise constant map $(r \circ s) \circ (f|A)$. Thus $f|A$ is fibrewise contractible. □

Theorem 4.3. *Let Q_B be one of the fibrewise classes $\mathcal{P}_B, \mathcal{L}_B, \mathcal{C}_B$ and $\mathcal{CM}_B$. Then any $(X, f) \in AR_B(Q_B)$ which has a section is fibrewise contractible.*

Proof. Since (X, f) is in Q_B, $(X \times I, q) \in Q_B$ from Proposition 4.1. Let s be a section of (X, f), i.e. $f \circ s = id_B$. Since $A = X \times \{0, 1\}$ is closed in $X \times I$, we can define a map $\varphi : A \to X$ by $\varphi(x, 0) = x$ and $\varphi(x, 1) = s \circ f(x)$. Since $(X, f) \in AE_B(Q_B)$ by Theorem 3.10, there is a fibrewise extension $\psi : X \times I \to X$ of φ. Thus (X, f) is contractible. □

Theorem 4.4. *Let Q_B be one of the fibrewise classes $\mathcal{N}_B$ and $\mathcal{CN}_B$. If $(X, f) \in \mathcal{P}_B \cap AR_B(Q_B)$ and it has a section, then it is fibrewise contractible.*

Proof. Since $\mathcal{P}_B \subset Q_B$ by [2] Proposition 3.2 and Proposition 1.2 of this paper, $(X, f) \in AR_B(\mathcal{P}_B)$ by Proposition 2.9. Since (X, f) has a section, it is fibrewise contractible by Theorem 4.3. □

5. Some special cases of fibrewise contractible spaces

In this section, we consider the case that for every fibrewise space (X, f), X is paracompact. Let $\mathcal{P}$ be the topological class of all paracompact Hausdorff spaces. For any fibrewise class Q_B, we use the notation $Q_B \cap \mathcal{P} = \{(X, f) | X \in \mathcal{P}$ and $(X, f) \in Q_B\}$. Then it is easy to see that any fibrewise space $(X, f) \in Q_B \cap \mathcal{P}$ is in $\mathcal{FN}_B$.

Theorem 5.1. *Let $Q_B = \mathcal{FN}_B \cap \mathcal{P}$. Then any fibrewise space (X, f) which is a fibrewise contractible space of $ANE_B(Q_B)$ is in $AE_B(Q_B)$.*

Proof. Since (X, f) is fibrewise contractible, there are a section $s : B \to X$ and a fibrewise homotopy $H : X \times I \to X$ such that $H(x, 0) = x, H(x, 1) = s \circ f(x)$. Let $(Y, g) \in Q_B, (Y, A)$ be a Q_B-pair and $\varphi : A \to X$ a fibrewise map. Since $(X, f) \in ANE_B(Q_B)$, there are an open (in Y) nbd U of A and a fibrewise map $\psi : U \to X$ such that $\psi|A = \varphi$. Since Y is paracompact, there is an open nbd V of $Y - U$ such that $A \cap [V]_Y = \emptyset$. Since $(Y, g) \in \mathcal{FN}_B$, for each $b \in B$, there is an open nbd $O_b \in N(b)$ such that A and $[V]_Y$ are functionally separated in $g^{-1}O_b$. Therefore, for each $b \in B$, there is a map $e_b : g^{-1}O_b \to I$ such that $A \cap g^{-1}O_b \subset e_b^{-1}(0)$ and $[V]_Y \cap g^{-1}O_b \subset e_b^{-1}(1)$. Since Y is paracompact and $\{g^{-1}O_b | b \in B\} = \mathcal{U}$ is an open cover of Y, there is a partition $\{\kappa_b | b \in B\}$ of unity subordinated to $\mathcal{U}$. Now we define a function $e : Y \to I$ by

$$e(y) = \sum_{b \in B} e_b(y)\kappa_b(y).$$

Then it is easy to see that e is well-defined, continuous and satisfies $A \subset e^{-1}(0)$ and $[V]_Y \subset e^{-1}(1)$. Finally, we can define a function $\phi : Y \to X$ as follows:

$$\phi(y) = \begin{cases} H(\psi(y), e(y)) & (y \in U) \\ s(g(y)) & (y \in V). \end{cases}$$

Then ϕ is easily proved to be continuous. Since $\phi|A = \varphi$, we conclude that $(X, f) \in AE_B(Q_B)$. □

Theorem 5.2. *(1) Let Q_B be one of the fibrewise classes $\mathcal{N}_B \cap \mathcal{P}$ and $\mathcal{CN}_B \cap \mathcal{P}$. Then a fibrewise space (X, f) which is in Q_B and has a section is in $AR_B(Q_B) \cap \mathcal{P}_B$ if and only if it is a fibrewise contractible space of $ANR_B(Q_B)$.*

(2) Let Q_B be one of the fibrewise classes $\mathcal{P}_B \cap \mathcal{P}$, $\mathcal{L}_B \cap \mathcal{P}$, $\mathcal{C}_B \cap \mathcal{P}$ and $\mathcal{CM}_B \cap \mathcal{P}$. Then a fibrewise space (X, f) which is in Q_B and has a section is in $AR_B(Q_B)$ if and only if it is a fibrewise contractible space of $ANR_B(Q_B)$.

Proof. (1) "Only if" part: This follows easily from Proposition 2.5 and Theorem 4.4.

"If" part: This follows easily from Theorems 3.9 and 5.1 and Proposition 2.9.

(2) "Only if" part: This follows easily from Theorem 4.3 and Propositions 2.5 and 2.9.

"If" part: This follows easily from Theorems 3.9 and 5.1 and Proposition 2.9. □

Finally, we shall give an example of a normal compact map which is not functionally Hausdorff. This example was constructed by David Buhagiar.

Example 5.1. We need a space Z which slightly differs from that of Tzannes([13]).

Let $[0, \Omega]$ be the closed ordinal space and let $[0, \Omega]_i, i \in J, |J| = |[0, \Omega]|$, be disjoint copies of $[0, \Omega]$, where Ω is the first uncountable ordinal. To the topological sum $X = \bigcup_{i \in J} [0, \Omega]_i$ we add another closed ordinal space $[0, \Omega]$ and we consider the set $Y = X \cup [0, \Omega]$. We define the bases of neighbourhoods of the points in $[0, \Omega]$ in Y as follows:

(a) If $x \in [0, \Omega[$, then let $V(x)$ be a neighbourhood of x in $[0, \Omega[$ and let $V(x_i)$ be the copy of $V(x)$ in $[0, \Omega]_i$. Then a base of neighbourhoods of x in Y is the collection of sets

$$V(x) \cup (\bigcup C),$$

where C is the set consisting of all but a finite number of $V(x_i)$;

(b) If $x = \Omega$, then let $V(\Omega)$ be a neighbourhood of Ω in $[0, \Omega]$ and let $W(x)$ be an open set in $[0, \Omega]$ satisfying (i) $W(x) \subset V(\Omega)$ and (ii) there exists an $\alpha < \Omega$ such that $[\alpha, \Omega] \cap W(x) = \emptyset$. Then a base of neighbourhoods of Ω in Y is the collection of sets

$$V(\Omega) \cup (\bigcup C),$$

where C is the set consisting of all but a finite number of $W(x_i)$.

As in Tzannes' example, it is easily seen that X is an open dense subspace of Y. Also, every sequence frequently in $Y \setminus \{\Omega_i : i \in J\}$ has an accumulation point (either in X or in $[0, \Omega]$). Let $L = \{\Omega_i : i \in J\}$ and $D = \{$ isolated points of $X\}$. Since $|L| = |D|$, there exists a 1–1 map g from L onto D. Consider the quotient space

$$Z = \{x, (\Omega_i, g(\Omega_i)) : x \in (X \setminus (L \cup D)) \cup [0, \Omega], i \in J\}$$

We now define a topology τ on Z weaker than the quotient topology. Let $U(\Omega)$ be a neighbourhood of Ω in $[0, \Omega]$ and let $U(\Omega_i)$ be a copy of $U(\Omega)$ in $[0, \Omega]_i$. Let $q : Y \to Z$ be the natural projection.

(A) For every $x_i \in X \setminus (L \cup D)$ a base of neighbourhoods is the collection of open sets $O(x_i)$ such that

$$q^{-1}(O(x_i)) \cap X = V(x_i) \cup \bigcup U(\Omega_k),$$

where k varies through all positive integers for which, for some finite sequence of positive integers $n_1, \dots, n_m$, $n_m = k$, $g(\Omega_{n_1}) \in V(x_i)$, and for all $1 \le j < m, g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$;

(B) For every $x \in [0, \Omega[$ a base of neighbourhoods is the collection of open sets $O(x)$ such that

$$q^{-1}(O(x)) = V(x) \cup \bigcup C \cup \bigcup U(\Omega_k),$$

where k varies through all positive integers for which, for some finite sequence of positive integers $n_1, \dots, n_m$, $n_m = k$, $g(\Omega_{n_1}) \in \bigcup C$, and for all $1 \le j < m, g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$;

(C) For every point $(\Omega_i, g(\Omega_i))$ a base of neighbourhoods is the collection of open sets $O((\Omega_i, g(\Omega_i)))$ such that

$$q^{-1}(O((\Omega_i, g(\Omega_i)))) \cap X = \{g(\Omega_i)\} \cup \bigcup U(\Omega_k),$$

where now for k and $n_1, \dots, n_m$, $n_m = k$, $g(\Omega_{n_1}) \in U(\Omega_i)$, and for all $1 \le j < m, g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$;

(D) Finally, for $\Omega \in [0, \Omega]$ a base of neighbourhoods is the collection of open sets $O(\Omega)$ such that

$$q^{-1}(O(\Omega)) = V(\Omega) \cup \bigcup C \cup \bigcup U(\Omega_k),$$

where now for k and $n_1, \ldots, n_m$, $n_m = k$, $g(\Omega_{n_1}) \in \bigcup C$, and for all $1 \leq j < m, g(\Omega_{n_{j+1}}) \in U(\Omega_{n_j})$. Note that here C is as in item (b) above in the definition of neighbourhoods of Ω in Y.

As in Tzannes, (Z, τ) is a countably compact Hausdorff space which is not functionally Hausdorff. Now consider the quotient space $Z/[0, \Omega]$. The map $p : Z \to Z/[0, \Omega]$ is a Hausdorff compact (perfect) map which is not functionally Hausdorff. Indeed, any two points in $[0, \Omega]$ cannot be separated by any function in any neighbourhood of $[0, \Omega]$ in Z. Consequently, the map p is a normal compact map which is not functionally normal.

In connection with this example, we don't know whether or not a CMT-map implies functionally Hausdorff. For other examples of normal compact Hausdorff maps which are not functionally Hausdorff, see [7,4.2] or [5, Example 3.4]. (These examples were pointed out by the referee. The author wishes to express his gratitude to the referee for his helpful comments.)

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