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ON THE KRASINKIEWICZ - MINC THEOREM CONCERNING COUNTABLE FANS

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Abstract. A strengthening of a remarkable theorem of Krasinkiewicz and Minc is discussed to the effect that there are planar fans D_α, $\alpha < \omega_1$, such that if X is completely metrizable separable and each D_α is a continuous (homeomorphic) image of a continuum in X, then so is every chainable continuum. We shall also give an analogous strengthening of a theorem of Maćkowiak concerning hereditarily decomposable chainable continua.

1. Introduction

Our terminology follows mainly [14] and [15]. We consider only separable metrizable spaces, and by mappings we mean continuous functions.

A continuum X is a countable fan if there is $v \in X$ (the vertex of X) and arcs J_i, $i = 1, 2, \ldots$, such that $X = \bigcup_{i=1}^{\infty} J_i$, v is an end point of each J_i, and $J_i \setminus \{v\}$ are pairwise disjoint, cf. [7].

A continuum X is indecomposable if it is not the union of any two proper subcontinua, and X is hereditarily decomposable (indecomposable) if no nontrivial (each) subcontinuum of X is indecomposable. Chainable continua are the ones that, for each $\epsilon > 0$, can be mapped onto an interval by a mapping with fibers of diameter $< \epsilon$ (cf. [15], where the term snake - like continua is used). The pseudoarc is the hereditarily indecomposable chainable continuum (unique, up to a homeomorphism), cf. [15], §48, X.

Krasinkiewicz and Minc [13] defined a collection D_α, $\alpha < \omega_1$, of planar countable fans such that for any hereditarily decomposable continuum X there is D_α that is not a continuous image of any subcontinuum of X. In particular, the class

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of hereditarily decomposable continua has no universal elements (cf. [20], sec. 20, 21 (b)).

In this note we present a strengthening of the remarkable result of Krasinkiewicz and Minc, using a different approach.

Theorem 1.1. *There is a collection D_α, $\alpha < \omega_1$, of planar countable fans such that for any completely metrizable separable X, if each D_α is a continuous (homeomorphic) image of a continuum in X then so is every chainable continuum.*

Since no hereditarily decomposable continuum can be mapped onto the pseudoarc (cf. [18], 9.A and 6.F) the fans D_α in the theorem have the property considered by Krasinkiewicz and Minc. The bracket version of the assertion, with X being the plane, implies the Bing's theorem [6] that chainable continua are planar. We use, however, in our proof Oversteegen's result [22] that countable fans locally connected at the vertex are planar.

In a similar way we shall refine also some results of Maćkowiak [19], Corollary (3.11) and (3.12) concerning hereditarily decomposable chainable continua.

Theorem 1.2. *There is a collection E_α, $\alpha < \omega_1$, of hereditarily decomposable chainable continua such that for any completely metrizable separable X, if each E_α is a continuous (homeomorphic) image of a continuum in X, then so is every chainable continuum.*

The topic has close links with a result of Darji [8] (cf. [4]) that hereditarily decomposable continua form a non - analytic set in the hyperspace of the plane, cf. sec.5. Darji's proof is based on a classical Hurewicz's theorem that the collection of compact subsets of the rationals is not analytic in the hyperspace of the interval, and the Hurewicz's theorem is also in the core of our reasoning.

2. Collections of planar fans associated with chainable continuum

Let K be a chainable continuum. We shall fix a metric compactification L of the half - line $[0,\infty)$ with K being the remainder, i.e., the ray wraps around K in L (cf. [1], [5]). Let R_i be the copy of $[i,\infty)$ in L and let e be the origin of the ray R_0.

Since K is chainable, so is L (cf. [5]). For each $n = 1, 2, \ldots$, let us fix a continuous map $p_n : L \longrightarrow [0,1]$ with fibers of diameter $\leq 1/n$ taking $R_1 \cup K$ to $[1/2, 1]$ and mapping $R_0 \setminus R_1$ homeomorphically onto $[0, 1/2)$.

Let C be the Cantor set and let $Q = \{q_1, q_2, \dots\}$ be a countable set dense in C. We shall consider the upper semi - continuous decomposition of the product $C \times L$ whose elements are the sets $\{q_n\} \times p_n^{-1}(a)$, for $a \in (0, 1]$, the set $C \times \{e\}$, and the singletons $\{(t, x)\}$ with $t \notin Q$ and $x \neq e$ (cf. [2], proof of Theorem 1).

Let M be the decomposition space and $\pi : C \times L \longrightarrow M$ the quotient map. Let us consider the "sections"

$$M_t = \pi(\{t\} \times L) \quad \text{for} \quad t \in C. \tag{1}$$

Then

$$M_q \quad \text{is an arc for} \quad q \in Q, \tag{2}$$

$$M_t \quad \text{is a copy of} \quad L \quad \text{for} \quad t \in C \setminus Q. \tag{3}$$

For each compact set $A \subset C$ we let

$$M(A) = \pi(A \times L) = \bigcup \{M_t : t \in A\}. \tag{4}$$

If A is a compact subset of Q, then $M(A)$ is a countable fan (cf.(2)) with the vertex $v = \pi(t, e)$ (t -arbitrary). The neighbourhood $\pi(C \times (R_0 \setminus R_1))$ of the vertex is an open cone over A, hence $M(A)$ is locally connected at v, and by Oversteegen's theorem [22], Th.5.2, the fan $M(A)$ is planar (cf. [17]).

For each countable ordinal α, let us choose a compact set $A_\alpha \subset Q$ with the Cantor - Bendixson index $\geq \alpha$ (see [14], §24, IV or [12], 6.C), and let us define the countable fans by

$$D_\alpha = M(A_\alpha), \quad \alpha < \omega_1. \tag{5}$$

We shall close this section with the following remark (cf. [23], sec.8). Given a space E we denote by $\mathcal{K}(E)$ the hyperspace of compact subsets of E equipped with the Vietoris topology, cf. [15], §42.I. A classical theorem of Hurewicz [11], sec.5, implies that any analytic set in the hyperspace $\mathcal{K}(C)$ containing all compacta A_α, $\alpha < \omega_1$, must contain also some A with $A \setminus Q \neq \emptyset$. Since the function $A \longrightarrow M(A)$ from $\mathcal{K}(C)$ to $\mathcal{K}(M)$ is Borel (cf. [15], §43.I,VII) we conclude that if $\mathcal{E} \subset \mathcal{K}(M)$ is analytic and $D_\alpha \in \mathcal{E}$ for $\alpha < \omega_1$, then there is $M(A) \in \mathcal{E}$ with $A \setminus Q \neq \emptyset$.

3. Proof of Theorem 1.1.

(A) We shall prove first the part of the assertion dealing with mappings.

Let K be the pseudoarc and let M and $D_\alpha = M(A_\alpha)$ be the compactum and the planar countable fans associated with K in sec. 2, cf. (4) and (5).

Let X be a completely metrizable separable space, and let $\mathcal{E}$ be the collection of continua $f(Z)$, where Z is a continuum in X and $f : Z \longrightarrow M$ is a mapping. Then $\mathcal{E}$ is an analytic set in the hyperspace $\mathcal{K}(M)$, cf. [15], §44, II, Th.2. Let us assume that $D_\alpha \in \mathcal{E}$ for all $\alpha < \omega_1$. Then, by the remark at the end of sec. 2, there exists a compactum A in the Cantor set C such that $M(A) \in \mathcal{E}$ and some $t \in A \setminus Q$. Let $f : Z \longrightarrow M(A)$ be a continuous surjection from a continuum Z in X and let us consider the section M_t, cf.(1). By (3), the vertex v of $M(A)$ is the origin of the ray R_0 wrapping around a copy of K (which we shall identify with K) in M_t. Let us choose inductively continua $Z_1 \supset Z_2 \supset \dots$ in Z such that $K \subset f(Z_n) \subset R_n \cup K$, where R_n are the terminal rays in R_0, cf. sec. 2. Then $T = \bigcap_{n=1}^{\infty} Z_n$ is a continuum in X with $K = f(T)$, cf. [10], Th.3.4.

Since each chainable continuum is a continuous image of the pseudoarc (cf. Lelek [16], Mioduszewski [21] and Fearnley [9]), this completes the proof of the non-bracket part of the assertion.

(B) To check the remaining part of the Theorem let us consider any chainable continuum K containing topologically each chainable continuum (cf. Schori [25]) and let M and D_α be the compacta defined for K in sec.2.

Let X be a completely metrizable separable space and let $\mathcal{E}$ be the set of compacta in M that embed in X. Then $\mathcal{E}$ is an analytic set in the hyperspace $\mathcal{K}(M)$, cf. [15], §44, II. Assume that $\mathcal{E}$ contains all fans D_α, $\alpha < \omega_1$. By the remark at the end of sec.2, $\mathcal{E}$ must contain also $M(A)$ with $t \in A \setminus Q$. But then, by (3), the compactum K embeds in X, and so does every chainable continuum.

4. Proof of Theorem 1.2

We shall use, with some adjustments, the key elements of the proof of Theorem 1.1.

(A) We begin with setting some background for the construction.

Let I be the unit interval, let I^∞ be the Hilbert cube and let $\Pi : I \times I^\infty \longrightarrow I$ be the projection. Let $\mathcal{G}$ be the collection of chainable continua in $I \times I^\infty$. One readily checks that

$$\mathcal{G} \text{ is a } G_\delta - \text{ set in the hyperspace } \mathcal{K}(I \times I^\infty). \tag{6}$$

Following Anderson [3] we shall say that the projection Π is atomic on a continuum Z in $I \times I^\infty$ if for every continuum T in Z, either $\Pi(T)$ is a singleton or $\Pi^{-1}(t) \cap Z \subset T$ for all $t \in \Pi(T)$.

We shall need the following fact, concerning the atomic maps cf. [19], (1.11). Let Z be a continuum in $I \times I^\infty$ on which Π is atomic. If all fibers $\Pi^{-1}(t) \cap Z$,

$t \in \Pi(Z)$, are chainable, then so is Z, i.e., $Z \in \mathcal{G}$. If moreover, the fibers are hereditarily decomposable, so is Z.

(B) Let us fix a chainable continuum L and let $p_n : L \longrightarrow I$ be a continuous map with fibers of diameter $\leq 1/n$. Let C and Q be as in sec. 2. We shall change slightly the construction from sec.2, considering in the product $C \times L$ the upper semi - continuous decomposition whose elements are the sets $\{q_n\} \times p_n^{-1}(a)$, $a \in I$, and the singletons $\{(t, x)\}$ with $t \notin Q$. Let $\pi : C \times L \longrightarrow M$ be the quotient map. Then, with the notation introduced in sec.2, we still have (2) and (3). However, the compacta $M(A)$, $A \subset \mathcal{K}(Q)$ are now disjoint unions of arcs. In the next step, the compacta $M(A)$ will be embedded in suitably chosen chainable continua $Z(A)$ and we shall declare $E_\alpha = Z(A_\alpha)$, cf. (5).

In the sequel we shall assume that $M \subset I \times I^\infty$ and

$$M_t = \Pi^{-1}(t) \cap M, \quad t \in C. \tag{7}$$

Indeed, we can consider the quotient space $\pi(C \times L)$ being embedded in I^∞, and the copy $\{(t, \pi(t, x)) :\ (t, x) \in C \times L\}$ of M satisfies (7).

We shall also need a continuous selection $\sigma : C \longrightarrow M$, $\sigma(t) \in M_t$. To get such σ, let us fix a point z in L and let $\sigma(t) = (t, \pi(t, z))$.

(C) Let $A \in \mathcal{K}(C)$ be infinite, and let us arrange the intervals in $I \setminus A$ with both end points belonging to A into a sequence (a_i, b_i), $i = 1, 2, \ldots$. Let J_i be segment in $I \times I^\infty$ joining the points $\sigma(a_i)$ and $\sigma(b_i)$ (cf. the end of (B)) and let $c_i = \frac{a_i + b_i}{2}$, $\epsilon_i = b_i - a_i$. One easily defines a continuous map $\phi_i : (a_i, b_i) \longrightarrow I^\infty$ whose graph $R_i = \{(t, \phi_i(t)) :\ t \in (a_i, b_i)\}$ is contained in the ϵ_i - neighbourhood of the continuum $M_{a_i} \cup J_i \cup M_{b_i}$, and the ray $R_i \cap \Pi^{-1}[c_i, b_i)$ wraps around M_{b_i}, while the ray $R_i \cap \Pi^{-1}(a_i, c_i]$ wraps around M_{a_i}, such that $M_{a_i} \cup R_i \cup M_{b_i}$ is a continuum (cf. the beginning of sec. 2). We define

$$Z(A) = M(A) \cup \bigcup_{i=1}^{\infty} R_i. \tag{8}$$

The definition of R_i's guarantees that the projection Π is atomic on the continuum $Z(A)$, cf. sec. (A). Moreover, the fibers $\Pi^{-1}(t) \cap Z(A)$ are chainable, and by the fact recalled at the end of (A) we get

$$Z(A) \in \mathcal{G}, \tag{9}$$

where $\mathcal{G}$ was introduced in (A).

Let A_α be the sets described in sec. 2, and let (cf. (5))

$$E_\alpha = Z(A_\alpha), \quad \alpha < \omega_1. \tag{10}$$

Since $A_\alpha \subset Q$, property (2) shows that all fibers $\Pi^{-1}(t) \cap E_\alpha$, $t \in \Pi(E_\alpha)$, are either arcs or singletons, and by the fact quoted at the end of (A), E_α is hereditarily decomposable.

(D) We are ready to justify Theorem 1.2 by a reasoning similar to that in sec. 3.

Let the continuum L in (B) be the pseudoarc, and let E_α be defined by (10).

Let X be a completely metrizable space such that the collection $\mathcal{E}$ of all continua in $I \times I^\infty$ which are continuous images of some continua in X, contains all E_α.

The set $\mathcal{E}$ is analytic and so is the set $\mathcal{A}$ of compact sets $A \subset C$ such that $M(A) = Z \cap \Pi^{-1}(A)$ for some $Z \in \mathcal{E} \cap \mathcal{G}$. This follows from the fact that the maps $A \to M(A)$ and $(Z, A) \to Z \cap \Pi^{-1}(A)$, where $A \in \mathcal{K}(C)$, $Z \in \mathcal{K}(I \times I^\infty)$, are Borel. By the assumption, $A_\alpha \in \mathcal{A}$ for all $\alpha < \omega_1$, cf. (8), (9), (10). Therefore, as was noticed at the end of sec. 2, there must be $A \in \mathcal{A}$ with some $t \in A \setminus Q$. It follows that there is a continuum H in X and a continuous map $f : H \longrightarrow I \times I^\infty$ such that $f(H) \in \mathcal{G}$ and $M_t = f(H) \cap \Pi^{-1}(t)$, $t \notin Q$. Since $f(H)$ is chainable, there is a continuum S in H with $M_t = f(S)$, cf. [19] (1.10) and [24], Theorem 4. It follows that S is mapped continuously onto the pseudoarc, and hence onto every chainable continuum.

The bracket version of Theorem 1.2 can be checked in a similar way, starting from any chainable universal continuum L, cf. sec. 3 (B).

5. Comments

Darji's construction [8] can also be used to demonstrate the Krasinkiewicz - Minc theorem. However the present approach seems more handy for the refinements.

The simplest instance of our construction, with K in sec. 2 being the unit interval, provides fans D_α, $\alpha < \omega_1$, such that no arcwise connected continuum maps onto every D_α. This gives another proof of Corollary 1 in [8].

As was mentioned in the Introduction, Darji proved that hereditarily decomposable continua in the plane form a coanalytic but not analytic set in the hyperspace. Theorem 1.2 shows that the same is true for hereditarily decomposable chainable continua in the plane.

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