Houston Journal of Mathematics

Volume 28, No. 2, 2002

CURVATURE AND GLOBAL RIGIDITY IN FINSLER MANIFOLDS

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Dedicated to S. S. Chern

Abstract. We present some strong global rigidity results for reversible Finsler manifolds. Following É Cartan's definition (1926), a locally symmetric Finsler metric is one whose curvature is parallel. These spaces strictly contain the spaces such that the geodesic reflections are local isometries and also constant curvature manifolds. In the case of negative curvature, we prove that the locally symmetric Finsler metrics on compact manifolds are Riemannian and this, therefore, extends A. Zadeh's rigidity result. Our approach uses dynamical properties of the flag curvature. We also give a full generalization of the Ossermann Sarnak minoration of the metric entropy of the geodesic flow. In positive curvature, we just announce some partial results and remarks concerning Finsler metrics of curvature +1 on the 2-sphere. We show that in the reversible case the geodesic flow is conjugate to the standard one. We also observe that a condition of integral geometry (of Radon type) forces such a metric to be Riemannian. This indicates a deep link with (exotic) projective structures.

Contents

1. Introduction

Our general aim in this paper is to investigate global properties linked with assumptions on the flag curvature. Local invariants that one may associate to a smooth Finsler metric are now familiar objects, thanks to the work of many mathematicians such as Cartan, Berwald, Chern [15], Funk , Busemann and others. Very rough observations are that one may consider several relevant connections and curvatures; a local assumption on curvature is much more flexible than in the Riemannian case. However, all approaches are concerned with the so-called flag curvature, which governs the Jacobi equation and the second variation of length. Due to this simple remark, one may easily understand that there exists a description of the flag curvature in terms of the dynamics of the geodesic flow. In fact, the generator of the geodesic flow of a Finsler metric is a second order differential equation. It may be observed ([22]) that, in the much more general context of second order differential equations, there is a natural operator which plays the role of the flag curvature. (In an appendix at the end of this paper, I briefly recall the construction.) In this text, we shall see that, even if there are much larger local moduli of deformations in Finsler geometry, the interplay between dynamics and geometry forces some global rigidity and in some cases we obtain results, as beautiful as in Riemannian geometry, but we have to follow a more delicate path. Before going into the details, we need to say that in this paper we will only focus on reversible metrics (for which unit tangent balls are symmetric). The non reversible case is also very interesting, but several signs indicate that very different situations may occur. For instance we know from the work of A. Katok the existence of non reversible metrics on the two-sphere having only two closed geodesics. We also know from the work of P. Funk and R. Bryant, ([10] and his contribution to this volume) the existence of non reversible metrics with constant positive curvature and which are projectively flat (this is much more stringent). As we shall see, the space form equipped with reversible metrics will appear rather well understood in the negative curvature case. But, just little is known in the non compact case. In positive curvature, the situation is far from being completely understood. This is why we shall indicate some partial results.

1.1. **Results.** A reversible C^k Finsler space is a manifold, M, equipped with a field of norms, $F : TM \mapsto \mathbb{R}$, such that, the unit bundle UM is a C^k submanifold whose fibers are the boundaries of strictly convex positive symmetric sets. As in the case of Riemannian geometry, one may introduce (see for instance [4]) some useful local tools such as a "connection". Here we will focus (see the Appendix and [22]) on a canonical first order operator $\mathcal{D}_X$ which we call the dynamical derivative and which plays the role of a covariant derivative but only along the geodesics. It also provides us with a part of a curvature operator that we call R later on and known as the Jacobi endomorphism. Via the Sasaki metric, this operator gives rise to the now well-known flag curvature. The Appendix at the end of this paper contains miscellaneous comments in this respect.

We will adopt a new terminology needed in Finsler geometry. A smooth reversible Finsler metric is said to be **parallel** if and only if $\mathcal{D}_X R = 0$, i.e. if the curvature is parallel along the flow lines. Using the Chern connection (for instance) one could also declare that a manifold is parallel if and only if the flag curvature is parallel with respect to the connection. We did not use here this stronger assumption but it deserves further study. We reserve the word **locally symmetric** to metrics such that for any chosen point the geodesic reflection is a local isometry.

There are many examples of non-Riemannian parallel Finsler spaces; for instance, $\mathbb{R}^n$, equipped with a Banach norm satisfying the positivity condition. But these cases are flat Finsler spaces. The most famous negatively curved symmetric Finsler space was invented by Hilbert in 1894. It is known as the Hilbert geometry of bounded convex sets in $\mathbb{R}^n$. Given a bounded convex open set $\mathcal{C}$ with boundary $\partial\mathcal{C}$, the distance between two different points a, b in $\mathcal{C}$ is given by the formula $d(a, b) = 1/2 \; log[a, b, x, y]$, where $[a, b, x, y]$ denotes the cross ratio of the quadruple of ordered points $\{y, a, b, x\}$, y and x being the ends of the affine line (a, b) on the boundary $\partial\mathcal{C}$. The real hyperbolic space, $\mathbb{R}H^n$, corresponds to the case where $\mathcal{C}$ is the unit open ball in $\mathbb{R}^n$. We first observe the following

Proposition 1. *A reversible, locally symmetric, C^3 Finsler metric is parallel.*

In contrast to the Riemannian case, the converse is not true in general. For instance, in (D. Egloff [17]) it is shown that a Hilbert geometry is locally symmetric if and only if it is Riemannian. There are several important works concerning the locally symmetric metrics that we will not investigate or use here. Let us just mention that Busemann and Phadke ([12]) proved (without the differentiability assumptions) that on the universal cover, geodesic reflections extend to global isometries.

- Negative flag curvature -

We can now state the main rigidity theorem. The results of this part have been previously announced in [25]. Here we give a complete proof.

Theorem 1. *A compact Finsler space with parallel negative curvature is isometric to a Riemannian locally symmetric negatively curved space.*

Combining Theorem 1 with Proposition 1 immediately implies the following.

Corollary 1. *A locally symmetric compact Finsler space with negative curvature is isometric to a negatively curved Riemannian locally symmetric space.*

Theorem 1 contains, as a particular case compact manifolds with constant curvature, for which this result was known by a theorem of Akbar Zadeh [37]. (The proof he gives is very different from ours.)

Let us remark that there exist compact continuous locally symmetric Finsler spaces (see the nice examples by Verovic ([36]) of higher rank which are not isometric to Riemannian models).

This theorem also brings a complete answer to a result that I obtained in [23] and which extends to the Finsler setting a result of R. Osserman and P. Sarnak [33]. This result provides an estimate from below for the metric entropy of a closed negatively curved Finsler manifold. I can now state it in a complete form which includes the analysis of the equality case.

Theorem 2. *The Liouville entropy h_μ of the geodesic flow of a closed reversible Finsler space (M,F) with negative Jacobi endomorphism, R, satisfies the following inequality*

$$h_\mu \geq \int_{HM} Tr((-R)^{1/2}).\ d\mu \ , \tag{1.1}$$

with equality if and only if (M,F) is isometric to a Riemannian locally symmetric space of negative curvature.

Another nice direct application is a new proof of a result of Benzecri ([7]) .

Corollary 2. *Let $\mathcal{C}$ be a convex bounded set in $\mathbb{R}^n$ with smooth strictly positive boundary (positive second-fundamental form) $\partial\mathcal{C}$. Assume furthermore that*
i) there exists a subgroup $\Gamma \in PGl(\mathbb{R}^n)$ which preserves $\mathcal{C}$.
ii) $\Gamma \setminus \mathcal{C}$ is a closed manifold.
Then $\mathcal{C}$ is the interior of an ellipsoid.

For these good smooth Hilbert geometries, Edith Socié obtains a complete result [35] which goes beyond Corollary 2.

Theorem (E. Socié). *Let $\mathcal{C}$ be a convex bounded set in $\mathbb{R}^n$ with C^2 strictly positive boundary. Then the isometry group of the corresponding Hilbert geometry is compact except if ∂C is an ellipsoid.*

This proves in particular that the only non-trivial quotients of a Hilbert geometry which are manifolds without boundary are Riemannian.

- Positive flag curvature -

In this part, I will only consider Finsler metrics on the 2-sphere whose flag curvature is $+1$. The results that I announce here, are joint work with A. Reissman [21]. There are very good references on the subject. One is Robert Bryant's text in this issue and his other texts on the same topic for instance [10]. I also recommend the texts of Zhongmin Shen (in particular[34]) and the recent work of D. Bao and Z. Shen ([5]). What we have observed is that the dynamics of the geodesic flow is very simple. We obtain a result comparable in essence to the one of A. Weinstein for the Zoll metrics.

Theorem 3. *The geodesic flow on the homogeneous bundle of a reversible complete smooth Finsler structure of curvature $+1$ on the two-sphere is smoothly conjugate to the geodesic flow of the standard sphere.*

It is not clear whether this conjugacy result still holds true for non reversible metrics. The idea behind the proof of this theorem is (following Bryant in [10]) to transport all flow invariant structural objects of the Finsler metric to the space of geodesics Λ, which is a sphere. Then the geodesic field can be viewed as the vertical field of a Riemannian metric on Λ, and we conclude by using a conformal change of this metric.

The geometry of such a metric is far from being understood if we do not add complementary assumptions on the nature of the geodesics. For instance, if one assumes that the metric is locally projectively flat, then in the reversible case it is known for a long time by a work of Funk that the metric has to be Riemannian [26]. The general case (non reversible) is completely handled by R. Bryant in [11]. As previously said, in our case, the space Λ of oriented geodesics is a also a 2-sphere. We observe then that the fibers of the homogeneous bundle project on, Λ, to a family of curves, that we call Y-curves. They have the same incidence properties as geodesics of the Euclidean sphere (see Proposition 8 for a precise statement). Due to the reversibility, we obtain an abstract projective two-plane which is not a priori Desarguian. Now we may consider a completely different question.

Definition 1. Given a smooth projective structure on a manifold M we shall say that it satisfies a **Radon-Gelfand** condition if for any 1-form α on M, the following conditions are equivalent :
i) α is closed
ii) for any projective line l in M

$$\int_l \alpha = 0$$

For instance this condition is satisfied by the standard $\mathbb{R}P^2$.
If this condition is also satisfied for a family of Y-curves, we will say by abuse, that the fibers satisfy a Radon-Gelfand condition. (The best technique to deal with these questions is the so-called Gelfand double fibration. For more information see [1])

Theorem 4. *A Finsler metric F on S^2 with constant flag curvature 1 whose fibers satisfy a Radon-Gelfand condition is isometric to the Euclidean 2-sphere.*

In particular this gives another proof of the rigidity of the Finsler metrics on the 2-sphere with flag curvature $+1$ and whose geodesics are great circles. There are many examples of exotic projective planes. It would be nice to see if the fibers of a smooth reversible Finsler metric with constant positive flag curvature, may provide an exotic projective structure such that the Radon-Gelfand condition is not satisfied.

1.2. **Dedication.** It is my pleasure to dedicate this article to Professor Shiing Shen Chern. Among all his contributions to mathematics, he has been one of the pioneers of Finsler geometry, bringing with him a unifying point of view such as in [15] and [14]. Later in the nineties, he has brought new ideas ([13]) which have been the signal of a revival of Finsler geometry. Let me thank him for having largely contributed to have given a new life to this domain and attracted talented young mathematicians.

2. Preliminaries and notations

I just want to give a very short summary of the local Finsler geometry needed for this paper. The point of view that we will adopt can is developed in [22], [17]. The geodesic flow ϕ_t of a smooth Finsler metric F on a manifold M is a one parameter group of diffeomorphisms of the tangent bundle TM. One can restrict it

to the F-unit bundle or equivalently to the homogeneous bundle $HM = \overset{\circ}{T}M/\mathbb{R}^{+*}$ of tangent half lines. Then, its generator $X : HM \mapsto THM$ is a very special vector field on HM. It is a second order differential equation on M. This implies that there exists a 1-order differential operator $\mathcal{D}_X$ acting on τHM the space of smooth sections of the bundle $\pi_{HM} : THM \mapsto HM$.
It satisfies

$$\forall \zeta \in \tau HM \ , \ f \in C^1(HM, \mathbb{R}) \ , \ D_X(f\zeta) = fD_X(\zeta) + (X.f).\zeta$$

There exists a $\mathcal{D}_X$ - parallel splitting of the tangent bundle THM, in

$$THM = \mathbb{R}X \oplus V \oplus H,$$

where V is the vertical bundle and $\mathbb{R}X \oplus H$ the horizontal bundle and a pseudo complex D_X-parallel structure $J^X : V \mapsto H \mapsto V$.
The lack of integrability of the horizontal distribution is measured by curvature. The Jacobi endomorphism or flag curvature (see also D. Bao and S.S. Chern [4]) is defined for any horizontal vector as

$$R(h) = J^X(p_v[X, \widetilde{h}]) \ ,$$

where p_v is the vertical projection and $\widetilde{h}$ is any local horizontal extension of the vector h. Using the pseudo-complex structure one may define the Jacobi endomorphism for any vector tangent to HM by requiring that R commutes with J^X and that $R(X) = 0$.
The second main feature of the vector field X is that it is the Reeb field of the contact 1-form $A \in \Omega^1(HM)$ given by the restriction to HM of the vertical derivative of the Finsler metric, i.e. $dA(X, .) = 0 \ , \ A(X) = 1$.

2.1. **Locally symmetric Finsler spaces.** A space is locally symmetric if, for any $p \in M$, the geodesic reflection s_p is a local isometry of the Finsler metric. Then, by a classical argument of calculus of variation, the induced map $\tilde{s}_p$ on HM commutes with the geodesic flow

$$d\tilde{s}_p \cdot X = X \cdot \tilde{s}_p. \tag{2.1}$$

The Dynamical derivative and the Jacobi endomorphism are only defined in terms of the second order differential equation, X, hence they are invariant by $\tilde{s}_p$, i.e.

$$\tilde{s_p}^* R^X = R^X \ , \ \tilde{s_p}^* \mathcal{D}^X = \mathcal{D}^X.$$

We now come to the proof of Proposition 1.

Proof. The reversibility condition means that the fibered antipody σ of the tangent bundle, defined for any $p \in M$ by $\sigma|T_pM = -Id|T_pM$, is an isometry of the Finsler metric. It is immediate to see that the antipody reverses the geodesic vector field thus producing a conjugacy between the geodesic flow and its inverse (hence the name reversibility). If $\tilde{\sigma}$ is the induced map on HM, this can be written as

$$\tilde{\sigma} \circ \varphi_t = \varphi_{-t} \circ \tilde{\sigma}. \tag{2.2}$$

Thus we obtain $\tilde{\sigma}^* X = -X$. By looking at the definitions of $\mathcal{D}^X$ and R^X, we immediately observe that

$$\mathcal{D}^{-X} = -\mathcal{D}^X \quad , \quad R^{-X} = R^X \tag{2.3}$$

from which we deduce,

$$\tilde{\sigma}^* \mathcal{D}^X = -\mathcal{D}^X \quad , \quad \tilde{\sigma}^* R^X = R^X. \tag{2.4}$$

The proposition will result in the combination of the two types of isometries. Give $p \in M$, and set, $\delta_p = \tilde{S}_p \circ \tilde{\sigma}$. From the previous properties, we obtain

$$\delta_p^* \mathcal{D}^X R^X = \tilde{\sigma}^* \tilde{S}_p^* \mathcal{D}^X R^X = \tilde{\sigma}^* \mathcal{D}^X R^X = -\mathcal{D}^X R^X \ .$$

On the other hand one observes that $\tilde{S}_p \circ \tilde{\sigma}$ induces the identity map on the fiber over p. Then, for the restriction to the tangent bundle of the fiber over p, we also have

$$\delta_p^* \big(\mathcal{D}^X R_z^X(Y)\big) = \mathcal{D}^X R_{\delta_p(z)}^X \big(T\delta_p(Y) = \mathcal{D}^X R_p^X(Y)\big) \tag{2.5}$$

for any $z \in \pi^{-1}(p)$, $Y \in V_z HM$.
The comparison of the two previous relations proves that the restriction of $\mathcal{D}^X R$ to vertical vectors satisfies

$$\mathcal{D}^X R^X_{|V_z HM} = -\mathcal{D}^X R^X_{|V_z HM} = 0 \tag{2.6}$$

But this also implies (see the appendix) that $\mathcal{D}^X R^X_{|T_z HM} = 0$.

The existence of a reflection for any $p \in M$, gives Proposition 1. $\square$

2.2. **A smooth conjugacy.** In the compact case, it is proven in ([23]) that the geodesic flow ϕ_t of a Finsler metric with negative Jacobi endomorphism is a contact Anosov Flow. Let us recall that it implies the existence of a complete flow-invariant splitting, called the Anosov splitting

$$TM = \mathbb{R}X \oplus E^s \oplus E^u, \tag{2.7}$$

where E^s is the tangent distribution to the stable foliation $\mathcal{E}^s$ whose leave at the point $x \in HM$ is the set, $\mathcal{E}^s_x = \{y \in HM \ , \ \lim_{t\to\infty} d(\phi_t(y), \phi_t(x)) = 0\}$. One also considers the weak stable distribution $\mathbb{R}X \oplus E^s$ tangent to the weak stable foliation whose leave at the point x is $\mathcal{W}^s_x = \cup_{t\in\mathbb{R}} \ \phi_t(\mathcal{E}^s_x)$. Similarly, E^u is the unstable distribution, $\mathbb{R}X \oplus E^u$ the weak unstable distribution. They are also both integrable with foliations of the same name. The first important remark is the following

Lemma 1.
(i) The Jacobi endomorphism is parallel if it is flow invariant.
(ii) If the Jacobi endomorphism is flow invariant, then the Anosov Splitting is smooth.

This is an immediate application of the commutation formulas that one can find in the appendix. Let us check it for any vertical vector field on an open set $\mathcal{U} \subset HM$. (The case of horizontal vector fields is similar.)

$$\begin{aligned} L_X\big(R(Y)\big) &= [X, R(Y)] = J\big(R(Y)\big) + \mathcal{D}\big(R(Y)\big) = R\big(J(Y) + \mathcal{D}(Y)\big) + (\mathcal{D}R)(Y) \\ &= R\big([X,Y]\big) + (\mathcal{D}R)(Y) \end{aligned}$$

and from the definition of the Lie derivative one gets that

$$L_X\big(R(Y)\big) = (L_X R)(Y) + R\big([X,Y]\big),$$

and therefore $L_X R = \mathcal{D}R$.

Given any orbit, $\gamma : \mathbb{R} \to US$, of the flow, such that, $\gamma(0) = z \in US$, and any tangent vector, $\xi \in T_z US$, the flow-invariant vector field defined along the orbit by, $\xi_t = d\varphi_t(\xi)$, can be decomposed according to, $TUS = \mathbb{R}X \oplus V \oplus H$. Here, H denotes the orthogonal part to the vector field X in the horizontal distribution. It is well known (see Appendix) that $\xi_t = \lambda_t \cdot X + J(v_t) + \mathcal{D}v_t$ and that the vertical vector field v_t is a solution of the Jacobi equation

$$\mathcal{D}^2 v_t + R(v_t) = 0.$$

ii) A useful tool to study solutions of the Jacobi equation is to consider the associated Ricatti equation. To do so, one deals with fields of endomorphisms U, i.e. for all $z \in HM$, $U_z \in End(V_z HM)$, solution of the Ricatti equation

$$\mathcal{D}^X U + U^2 + R^X = 0. \tag{2.8}$$

In negative curvature it is well known ([16], [28], see also [23]), that there exists only two global bounded solutions, U^s (resp U^u), and that, for any $z \in HM$,

$$E^s_z = graph\{U^s_z : V_z HM \to V_z HM\} \tag{2.9}$$

(resp for U^u).
In our case, the Jacobi endomorphism is parallel, and therefore these two solutions are explicit

$$U^s = -\sqrt{-R}, \qquad U^u = \sqrt{-R} \tag{2.10}$$

and of course C^∞.

I shall now use a rigidity result for contact Anosov flows (see [8])

Theorem 5. [8] . *Let $\phi_t, t \in \mathbb{R}$ be a smooth Anosov flow on a closed manifold N of dimension $2n-1$ such that*
(i) the Anosov splitting is smooth,
(ii) its canonical 1-form is a contact form.
Then up to finite covers and to time change, the flow is conjugate to the geodesic flow of a negatively curved locally symmetric space.

More precisely there exists a triple $(\tilde{S}, \Gamma, [\alpha])$
where $\tilde{S}$ is a simply connected symmetric space of non compact type and Γ is a discrete subgroup of the isometry group G such that $N_0 = \Gamma \setminus U\tilde{S}$ is compact.
The class $[\alpha]$ in $H^1(N_0, \mathbb{R}) = Hom(\Gamma, \mathbb{R})$ is such that there exists a closed 1-form α representing $[\alpha]$ and satisfying at each point $\alpha(X^0)+1 > 0$ for X^0 the generator of the corresponding geodesic flow.
The group $G' = G \times \mathbb{R}$ acts transitively on $U\tilde{S}$ by

$$\forall (g,t) \in G \times \mathbb{R},\ \ v \in U\tilde{S}, \qquad (g,t).v = \phi^0_t(g.v) \tag{2.11}$$

One can show that for the corresponding subgroup $\Gamma_1 = (\Gamma, [\alpha](\Gamma))$ the quotient $N_1 = \Gamma_1 \setminus U\tilde{S}$ is a compact manifold and that the associated flow ϕ^1_t is again an Anosov flow. Finally we can completely state the theorem.
There exists a smooth diffeomorphism $\Theta : N \mapsto N_1$ such that $\Theta \circ \phi_t = \phi^1_t \circ \Theta$.

Furthermore, as mentioned in ([8] Lemma 7.1.2), there exists a smooth diffeomorphism $\chi : N_0 \mapsto N_1$ such that

$$\frac{T\chi(X^0)}{(1+\alpha(X^0))} = X^1 \circ \chi\ .$$

Therefore there exists a smooth diffeomorphism, $\Omega = \chi^{-1} \circ \Theta : N \mapsto N_0$, such that if one set $\beta = -\Omega_* \alpha$, then

$$\frac{T\Omega(X^)}{(1 + \beta(X))} = X^0 \circ \Omega \ . \tag{2.12}$$

This rigidity result relies on the existence for contact Anosov flows with smooth horospherical foliations of a very well adapted connection. I will briefly recall the construction later. It uses the techniques of rigid geometric structures as introduced by M. Gromov ([29]), (see also [6], [38]).

In order to use the reversibility of the Finsler metric, one proves the following Lemma.

Lemma 2. *Let ϕ_t be as in Thm 5 and furthermore assume that there exists a smooth involution Σ of N diffeotopic to the identity and such that*

$$\phi_t \circ \Sigma = \Sigma \circ \phi_{-t} \ . \tag{2.13}$$

Then $[\alpha] = 0$.

In the sequel, we suppose that E^u (and then E^s) are orientable. Otherwise we take a cover.

Proof. To the smooth Anosov splitting (4.2) and the canonical contact form, it is possible to associate a Kanai connection ∇ (see section 2.4) on N which preserves all the structure. In ([8]), it is proved that the connection induced by ∇ on the bundle of volume forms on E^u (or E^s) are flat. Thus the holonomy on this bundle is described by a cohomology class $[\alpha^u]$.

This is the cohomology class labeled $[\alpha]$ in the statement of Theorem 5.
Because the volume form $A \Lambda dA^{n-1}$ is parallel with respect to ∇, we also have $[\alpha^u] + [\alpha^s] = 0$.

Now Relation (2.13) means that the flow and its inverse are $C^\infty-$conjugate. The inverse is also an Anosov flow but its canonical contact form is the opposite, and one has to exchange the stable and unstable bundles, $\sigma^* E^u = E^s$. Hence one may also associate a Kanai connection to ϕ_{-t}. From the construction the two connections coincide. This means that $\sigma^* \alpha^u = \alpha^s$, but σ^* acts trivially on $H^1(N, \mathbb{R})$. $\square$

It is an easy fact that the antipody σ of the unit-bundle of a Finsler metric is diffeotopic to the identity, because any cycle C is homologous to $\sigma \circ C$ along the fibers.

From Lemma 2 and by inspecting possible subgroups Γ we deduce.

Corollary 3. *There exists a diffeomorphism.* $\Im : N \mapsto N_0$ *which conjugates the flow* ϕ_t *to* ϕ_t^0
If $\mathbf{n} \geq \mathbf{3}$ *, there exists a closed locally symmetric n-manifold* S *such that* $\rho : US \mapsto N_0$ *is a finite Riemannian cover.*
If $\mathbf{n} = \mathbf{2}$*, there exists a finite cover* N_2 *of* N *and a closed negatively curved Riemann surface* S *such that the corresponding lifted flows are conjugate.*

Remark. : $\Gamma \setminus S$ is not necessarily a manifold.

For this presentation we will only consider the case $\mathbf{n} \geq \mathbf{3}$. The case of dimension 2 was known to several authors ([37] and others).

The case of dimension 2 may be also obtained by using another approach. It is an easy application of the structure equations. One of the coefficients is a bounded solution of the Jacobi equation and hence vanishes. The rigidity occurs also in the non reversible case.

When specialized to our settings, Corollary 3 gives us the commutative diagram

$$\begin{array}{ccccc} US & \xrightarrow{\Phi_t} & US & \xrightarrow{Id} & US \\ \Psi\downarrow & & \Psi\downarrow & & \rho\downarrow \\ HM & \xrightarrow{\phi_t} & HM & \xrightarrow{\Im} & M_0 \end{array}$$

Where $\Psi = \Im^{-1} \circ \rho$ is a smooth finite covering.

In the sequel we will in fact consider a finite cover of HM so that Ψ will denote a diffeomorphism.

2.3. **The image of the fibers.** We now want to show that Ψ is the tangent map of a local isometry.
We will adopt the following convention. If E is an integrable distribution in the tangent bundle of US, then $\mathcal{E}$ will denote the corresponding foliation and $\mathcal{E}_z$ the leaf containing the point z. For instance, $\mathcal{V}_z$ is the fiber over $\pi(z)$. In order to deal with conjugacy, Finsler objects will appear with a complementary prime, pullbacked objects with a complementary $*$, Riemannian objects will be written without further distinctive signs.

One can easily observe that

Lemma 3. $\Psi_* A' = A$ *where* A *is the canonical contact form on* US*.*

Because Ψ is a conjugacy the 1-form $A_* = \Psi_* A'$ is flow invariant. Using the Anosov splitting we see that by duality one may decompose the 1-form in

$$A_* = \lambda \cdot A + \alpha^u + \alpha^s,$$

each of these forms being flow invariant. We observe that

$$A'(X') = A_*(X) = \lambda \cdot A(X) = \lambda.$$

The two 1-forms, α^u and α^s have to vanish. Let us show this for α^s.
For any $Z \in E^s_x$, the flow invariance implies $\alpha^s_{\phi_t(x)}\big(T\phi_t(Z)\big) = \alpha^s_x(Z)$.
By compactness; the form α^s being bounded, the second term goes to zero as t goes to infinity.

Before considering the effect of the conjugacy let us recall that the Jacobi endomorphism R', is symmetric with respect to the Sasaki metric associated to the Finsler metric. The flow invariance of R', implies that its characteristic polynomial is also flow invariant. By ergodicity its coefficients are thus everywhere the same. Therefore the eigenvalues of the Jacobi endomorphism and their multiplicities do not depend of the point in the manifold. Furthermore the eigenspaces are flow invariant and $\mathcal{D}'$-parallel. From the expression of the stable and unstable distributions when R' is parallel, they admit an R'-invariant splitting

$$E'^s = \sum_{1 \le i \le r} E'^s_i \ , \ E'^u = \sum_{1 \le i \le r} E'^u_i \ ,$$

each E'^s_i or E'^u_i is a distribution of eigenspaces of the Jacobi endomorphism R corresponding to one of the eigenvalues and r is the number of different eigenvalues.
We also observe that for any $1 \le i \le r$, $\xi^s_i \in E'^s_i$, the Lyapunov exponent

$$\gamma_i = \lim_{t \to \pm\infty} \frac{1}{t} Ln \| d\phi_t(\xi^s_i) \| \tag{2.14}$$

where $\| \quad \|$ stands for any Riemannian metric on HM, is such that $-\gamma_i^2$ is the corresponding R'-eigenvalue. We can use this to prove

Lemma 4. *The Jacobi endomorphisms coincide through the conjugacy* $\Psi_* R' = R$.

Proof. Through a conjugacy, the Liapunov exponents are preserved. Then the two operators R' and R have the same eigenvalues counted with multiplicities. Furthermore $\Psi^* \cdot E'^s_i = E^s_i$, and the same holds for the unstable distribution, thus proving the lemma. $\square$

Remark: We know that (S, g) is a negatively curved Riemannian symmetric space of rank one. Thus, after possibly changing g by a scalar factor, its universal cover $\tilde{S}$ is isometric either to
$\mathbb{R}H_n-$ the real hyperbolic space, in which case $R = -Id$.
or to

$\mathbb{C}H_n$, $\mathbb{H}H_n$, CaH_2 - the Complex, Quaternion and Cayley hyperbolic spaces, in which case the Jacobi endomorphism has two distinct eigenvalues.

Lemma 5. *The distributions V_* and H_* are $\mathcal{D}-$parallel.*

Proof. We begin by a property of parallel spaces. Let $\gamma : \mathbb{R} \to HM$ be an orbit of the flow ϕ_t. Select an eigenvalue λ_i of R'. If at the point $\gamma(0)$ we choose a basis $\{e_\mu\}, 1 \leq \mu \leq p_i$ of the distribution ${E'}^s_i$, then the vector fields $\{e_\mu(t)\}$ defined along the curve by $e_\mu(t) = d\phi_t(e_\mu)$ are flow invariant and then the fields $\xi_\mu(t) = e^{+(-\lambda_i)^{1/2}t} \cdot e_\mu(t)$ are $\mathcal{D}'$-parallel, as seen from the Jacobi equation. If we consider such a basis, its image by Ψ is again a basis along $\Psi \circ \gamma$, which is flow invariant. Therefore the vector fields $\Psi^*\big(\xi_\mu(t)\big)$ are now $\mathcal{D}-$parallel. The same argument holds for all the distributions ${E'}^s_i$ or ${E'}^u_j$.

This proves that the conjugacy maps $\mathcal{D}'-$parallel vector fields on $\mathcal{D}-$parallel vector fields. The two $\mathcal{D}'-$parallel distributions V' and H' are mapped onto V_* and H_*. Hence V_* and H_* are parallel. □

Lemma 6. *The distribution V_* is everywhere transversal to H.*

Proof. As observed in Lemma 3, the pullback of the Anosov splitting of the flow ϕ_t coincides with the one of the geodesic flow Φ_t. Because $\Psi^* A' = A$, the conjugacy preserves the symplectic structure induced by dA' on $kerA'$, and dA on $kerA$. It is a classical fact that $V', H', {E'}^s, {E'}^u$ are Lagrangian subdistributions of $kerA'$ (The same property holds on the Riemannian side.). Thus V_*, H_* are Lagrangian subdistributions of $kerA$.

We are now going to develop an argument using the Maslov index.

For the sake of the reader, let us recall that, if we consider a symplectic vector space (F, ω), to any triple of Lagrangian spaces (E_1, E_2, E_3) we can associate an integer called the Maslov index of the triple. This is done by considering the quadratic form $q(x_1, x_2, x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$, on the vector space $E_1 \times E_2 \times E_3$, one then sets
$Index(E_1, E_2, E_3) = \text{signature}(q)$.
The construction may of course be done on a contact manifold for bundles whose fiber is, in each point, a Lagrangian subspace of the kernel of the contact form.

Using the fact that the conjugacy Ψ preserves the contact structures we immediately deduce that

$$Index\big(E^u, V_*, E^s\big)(z) = Index\big({E'}^u, V', {E'}^s\big)\big(\Psi(z)\big).$$

There is a useful trick to compute the Maslov index of (E_1, E_2, E_3), in the case where $E_2 = E_1 \oplus E_3$. If p_1, p_3 denote the associated projectors, we may equip E_2

with a quadratic form defined for any $x \in E_2$, by

$$Q(x) = \omega(p_1 x, p_3 x).$$

The basic fact is that, $Index(E_1, E_2, E_3) = signature(Q)$.

From the study of the Ricatti equation, we know that for any Y in V', the vectors
$\xi^s = \sqrt{-R'} \cdot Y + I^{X'}(Y)$, $\big($ resp $\xi^u = -\sqrt{-R'} \cdot Y + I^{X'}(Y)\big)$ are in E^s (respE^u). It is then easy to compute the Maslov index of $({E'}^u, V', {E'}^s)$. Using the fact that $\sqrt{-R'}$ is positive definite we obtain,

$$Ind({E'}^s, V', {E'}^u) = n - 1 = dim(V').$$

The same trick may be used to compute on US, the Maslov index of the triple $\{E^s, H, E^u\}$, one obtains

$$Ind(E^s, H, E^u) = -(n-1).$$

This implies the transversality of V_* and H at each point.

□

Lemma 7. *The distribution V_* is the graph of a section L of the bundle $\mathcal{E}nd(V)$. that is, $\forall z \in US$, $\forall v \in V_{*z}$, $\exists! Y \in V_z$, $v = Y + I^X(L(Y))$*

The section L satisfies the following properties

i) $\mathcal{D}L = 0$

ii) $[L, R] = 0$

iii) L is symmetric with respect to the vertical metric g_v.

Proof. The first statement is a direct consequence of Lemma 6.

i) The distributions V, H and V_* are $\mathcal{D}-$parallel (by the Lemma 5) and the pseudo-complex structure J^X commutes with $\mathcal{D}$ thus L is also parallel.

ii) As already observed the conjugacy preserves the Lyapunov exponents and the Jacobi endomorphism. Let us consider the $R'-$splitting into eigenspaces of the distribution $V' = \bigoplus_{i=1}^{r} V'_i$. From (4.4) we deduce that ${E'}^s_i \oplus {E'}^u_i = V'_i \oplus J'(V_i)$. The same relation is also true for US. Thus $V_{i_*} \subset V_i \oplus J(Vi)$. This means that $J^X(LV_i) \subset J^X(V_i)$. Thus $[L, R] = 0$.

iii) This is immediate from the fact that V_* is Lagrangian and from the definition of the vertical metric in term of the symplectic form (see the Appendix). □

¿From the parallelism of the section L and the ergodicity of the flow one deduces that the eigenvalues $\lambda_1 < \cdots < \lambda_p$ are independent of the point and that the corresponding L-eigenspace splitting

$$V = \bigoplus_{j=1}^{j=r} W_j \ , \ L_{\lfloor W_j} = \lambda_j . Id_{\lfloor W_j} \tag{2.15}$$

is parallel.
For rank one symmetric spaces, a Jacobi endomorphism has at most two eigenvalues. They may be written in the form $r_i = -(ia)^2, i = 1, 2$. (The multiplicity m_2 of r_2 is zero in the constant negative curvature case.) We are now going to use the parallel splitting of the vertical bundle V associated to this pair of commuting operators

$$V = \bigoplus_{j=1,i=1}^{j=r,i=2} V_{i,j} \ , \ L_{\lfloor V_{i,j}} = \lambda_j . Id_{\lfloor V_{i,j}}, \ R_{\lfloor V_{i,j}} = r_i . Id_{\lfloor V_{i,j}} \tag{2.16}$$

to produce a sub-foliation of the strong stable foliation. The main feature is that this foliation will be invariant under the holonomy of the weak unstable foliation. We will finally observe that such a foliation has to be trivial. This will force L to be a multiple of the identity.

Lemma 8. *Let $V_{i,j} \subset V$ be the distribution of common eigenspace of $\sqrt{-R}$ and L with respective eigenvalues ia and λ_j, and $\Delta = \{ \ \tau_{i,j} \ , \ 1 \le i \le 2 \ , \ 1 \le j \le p \ , \ tanh(a.i.\tau_{i,j}) = \lambda_j \}$*
i) the distribution $V_{*i,j} \stackrel{def}{=} V_{i,j} + J^X(L(V_{i,j})) \subset V_*$ is such that $\Phi^*_{\tau_{i,j}} V_{*i,j}$ is vertical.
ii) for any $\tau \in \Delta$, the distribution $V_\tau = \bigoplus_{i,j} V_{i,j}$, with i, j such that $\tau_{i,j} = \tau$ is integrable.

Proof. i) The Jacobi Equation is in our case a linear second order equation with constant coefficients. Let fix a point $z \in US$, and consider a tangent vector ξ in the distribution $V_{*i,j}$. To express its image by the geodesic flow by we know that $\xi = Y + \lambda_j \cdot h$, with $Y \in V_{i,j}$ and $h = J^X(Y)$. The splitting in eigenspaces being $\mathcal{D}-$parallel one may introduce along the flow orbit of z the parallel vector fields Y_t and $h_t = J^X(Y_t)$ obtained by parallel transport of Y and h.
Then we have

$$d\varphi_t(\xi) = ia(\cosh tia + \lambda_j \sinh tia) Y_t + \big(\sinh tia + \lambda_j \cosh tia\big) h_t$$

From this we deduce that $d\varphi_{-\tau}(\xi)$ is vertical provided that τ is a solution of the equation $\tanh(ai\tau) = \lambda_j$.

ii) From the definition, we see that a vector field ξ in V_* takes its values in the smooth bundle V_τ if and only if $\Phi_\tau^*(\xi)$ is a vertical vector field. If we consider now two vector fields ξ and η in V_τ we first notice that $[\xi,\eta]$ is in V_* by the integrability of the distribution tangent to the foliation $\mathcal{V}_*$. By transport along the flow we also have

$$\Phi_\tau^*[\xi,\eta] = \big[\Phi_\tau^*(\xi),\Phi_\tau^*(\eta)\big].$$

But now, the two vector fields on the right handside of this equation are vertical vector fields. The integrability of the vertical distribution forces their Lie bracket to be vertical. Hence $[\xi,\eta]$ is also valued in V_τ. This proves the integrability.

□

Notation. One finally may consider the flow invariant subdistributions $E^s{}_{i,j} \stackrel{def}{=} \{v \in E^s \ ; \ \exists Y \in V_{i,j} \ ; \ v = -\sqrt{(-R)}(Y) + J^X(Y)\}$ of the stable distribution.

2.4. **An adapted connection.** Let us recall that the weak unstable foliation and the stable foliation are transversal. Thus given two points $z \in US$, $z' \in \mathcal{W}_z^u$ and a path $c : [0,1] \mapsto \mathcal{W}_z^u$ such that $c(0) = z, c(1) = z'$, one may consider the holonomy map $\mathcal{H}^u_{z,z',c} : \mathcal{E}^s_z \mapsto \mathcal{E}^s_{z'}$ which is smooth, due to the smoothness of the foliations. For any vector $Z \in E^s_z$, a parallel transport along the path c, is defined by setting $\mathcal{T}_c(Z) = d\mathcal{H}_{z,z',c}(Z) \in E^s_{z'}$. Furthermore, one may also define a parallel transport along c for any vector $Z \in \mathbb{R}X \oplus E^u_z$ by deciding that both E^u_z, X and dA are parallel.

This means that the parallel transport of a vector in E^u_z is given by the symplectic duality. That is, for any $\xi^u \in E^u_z, \xi^s \in E^u_z$, the parallel transport is defined by the relation $dA\big(\mathcal{T}_c(\xi^u),\mathcal{T}_c(\xi^s)\big) = dA(\xi^u,\xi^s)$.

We have thus introduced a covariant derivative along the weak unstable leaves. It is useful to remark that the parallel transport along an orbit of the flow coincides with the flow transport. By also considering the holonomy of the weak stable foliation, we introduce thus a covariant derivative ∇ known as a Kanai [32] connection.

We may sum up its main properties :

$$\nabla X = 0, \qquad \nabla E^s \subset E^s, \qquad \nabla E^u \subset E^u, \qquad \nabla dA = 0. \tag{2.17}$$

And for any vector fields ξ^u in E^u and ξ^s in E^s we have from the definition of the holonomy maps

$$\nabla_X \xi^u = \ [X,\xi^u], \quad \nabla_X \xi^s = \ [X,\xi^s] \ , \tag{2.18}$$

$$\nabla_{\xi^u} \xi^s = \ p^s[\xi^u,\xi^s], \quad \nabla_{\xi^s} \xi^u = \ p^u[\xi^s,\xi^u] \ . \tag{2.19}$$

From which we may check that the torsion is given by : $T = dA \otimes X$. (I have used this geometric approach because I will use it in the sequel. But it is also possible to introduce it in a more axiomatic way.)

In order to study the flow invariant distributions we introduce a labelling $\gamma : \{1, ..., 5\} \to \{-2a, -a, 0, a, 2a\}$ of the possible Liapunov exponents and decide that the flow invariant distribution E_i stands for the distribution of vectors whose Liapunov exponent is $\gamma(i)$. It is often useful to use different adaptations of the following result (lemma 2.5 in [8]) and specialize it to US.
Let B be a flow-invariant bounded p-tensor field valued in TUS.
Let Z_k, $1 \leq k \leq p$, be vector fields such that $Z_k \in E_{i_k}$, then

$$B(Z_1, \ldots, Z_p) \in E_{\gamma^{-1}\left(\gamma(i_1)+\cdots+\gamma(i_p)\right)}$$

¿From this we may deduce the following lemma

Lemma 9. *The curvature operator K of the Kanai connection is such that, for all smooth local sections $\xi^s \in E^s$, $\xi^u \in E^u$, $\eta^u \in E^u$, we have*
i) $K(X, \xi^u)\xi^s = 0$,
ii) $K(\xi^u, \eta^u)\xi^s = 0$.

Proof. i) A direct computation using formulas (2.17, 2.18) gives

$$\begin{aligned} K(X, \xi^u)\xi^s = & \quad \nabla_{\xi^u}[X, \xi^s] - [X, \nabla_{\xi^u}\xi^s] - \nabla_{[X,\xi^u]}\xi^s \\ = & \quad p^s\left[\xi^u, [X, \xi^s]\right] - \left[X, p^s[\xi^u, \xi^s]\right] - p^s\left[[X, \xi^u], \xi^s\right] \ , \end{aligned}$$

and one may remark that, due to the flow invariance of E^s, we have $\left[X, p^s[\xi^u, \xi^s]\right]$ $= p^s\left[X, [\xi^u, \xi^s]\right]$. Then i) comes from the Jacobi identity.

ii) We are in a situation to use the result recalled just above, then $K(\xi^u, \eta^u)\xi^s \in E_{\gamma(j)}$ and $j \in \{3, 4, 5\}$ with the notation introduced above, because we add a negative exponent to two positive ones.

But by the parallelism of the bundle E^s, we also have $K(\xi^u, \eta^u)\xi^s \in E^s$. $\square$

Lemma 10. *i) The "slow" distributions ${E^s}_{1,j}$ are invariant under the holonomy maps $\mathcal{H}^u_{z,z',c}$.*

ii) For any $\tau \in \Delta$, the distribution $S_\tau = \bigoplus_{i,j} {E^s}_{i,j}$, i, j such that, $\tau_{i,j} = \tau$ is integrable.

Proof. i) In this part, we only need to study the covariant derivative along the weak unstable leaves. Let us fix j and consider the flow invariant sub-distribution $E^s_{1,j}$ of the stable distribution. Any of the involved distributions being flow-invariant, the following zero-order differential operator $B : (\mathbb{R}X \oplus$

$E^u) \times E^s_{1,j} \rightarrow \underset{(i,k)\neq(1,j)}{\oplus} E^s_{i,k}$ defined on any pair of smooth vector fields by $B(Z,\xi) = \sum_{(i,k)\neq(1,j)} p^s_{i,k}(\nabla_Z \cdot \xi)$,
is also flow invariant, bounded and valued in $E^s = E_{\gamma(4)} \oplus E_{\gamma(5)}$.
By adapting Lemma 2.5 [8] we deduce that, for any $\xi^u \in E_k$, $4 \leq k \leq 5$ we have

$$B(\xi^u, \xi) \in E_{\gamma^{-1}}\big(\gamma(k) + \gamma(2)\big) \bigcap E^s = \{0\}.$$

It remains to consider terms of the form $B(X,\xi)$. These terms also vanish because the distribution $E^s_{1,j}$ is flow invariant and thus ∇-parallel by 2.17 (this means that $\nabla_X \xi$ is a section of $E^s_{1,j}$).

ii) For any $\tau \in \Delta$, the flow invariant bundle S_τ is a subdistribution of the stable distribution E^s. As one knows, the integrable distribution E^s ([16]) is the limit of the pullback flow of the vertical distribution, $E^s = \lim_{t\to\infty} d\phi_{-t} V_{\phi(t)}$. Furthermore, on a symmetric space, this convergence is C^∞. Then, S_τ is the $C^\infty-$limit of the Pullback of the integrable $\big($ by lemma 7 ii) $\big)$ subdistribution V_τ.

To explain why this implies the integrability of S_τ, let us recall some general facts. On a manifold M, if F is a C^1-subvector bundle of TM and G a complementary vector bundle i.e., $TM = F \oplus G$, one may define a bilinear operator $B : F \times F \to G$ such that for any $x \in M$, $(u,v) \in F^2_x$ and any C^1-germs of vector fields $\tilde{u}, \tilde{v}$ valued in F such that $\tilde{u}(x) = u$, $\tilde{v}(x) = v$, we have
$B_x(u,v) = p_G\big([\tilde{u},\tilde{v}]_x\big)$,
where p_G is the projection onto the distribution G. This operator measures the lack of integrability of the C^1-bundle. In particular, by the Frobenius theorem it vanishes if and only if F is integrable.
We know that, for any t, the bundle $\phi^*_t V_\tau$ is integrable. One may easily observe that the distribution, $G = \mathbb{R}X \oplus E^u \underset{\tau'\neq\tau}{\oplus} S_{\tau'}$ is complementary to $\phi^*_t V_\tau$ for all t and to S_τ. Hence the corresponding operators B^t as introduced above vanish. The convergence of the family $\phi^*_t V_\tau$ being at least C^1 the operator B^∞ corresponding to the limit-distribution S_τ satisfies, $B^\infty = \lim_{t\to\infty} B^t = 0$ with B^∞. This prove the integrability of S_τ $\square$

But in general neither the distributions ${E^s}_{1,j}$ need to be integrable nor the distributions S_τ be holonomy invariant. To produce a null holonomy invariant distribution, we refer to [20], [30], and consider the holonomy intersection of the distribution S_τ to be the continuous holonomy invariant distribution

$$(2.20) \qquad < S_\tau(z) > \overset{def}{=} \bigcap_{z',c} d\mathcal{H}_{z',z,c}(S_\tau(z')),$$

for any $z' \in \mathcal{W}^u_z$ and any path c joining z' to z.

Lemma 11. *For $\tau = \tau_{1,1}$, the distribution $< S_\tau >$ is not null, invariant under the holonomy of the weak unstable foliation and integrable.*

Proof. By construction $< S_\tau(z) >$ is holonomy invariant. Following the argument in (Lemma 2.7 of [30]) one readily proves that $< S_\tau(z) >$ is a smooth distribution. Lemma 10 i) implies that $< S_\tau(z) > \neq 0$ because it contains $E^s_{1,1}$. Hence for any $z \in US$, it is possible to choose smooth sections of the bundle $< S_\tau(z) >$ which are invariant by holonomy along the unstable leave $\mathcal{W}^u_z$.

Let us consider two such sections ξ and η, Lemma10 ii) implies that, $[\xi, \eta] \in S_\tau(z)$. Combining (2.18) and the Jacobi identity, we get $\nabla_X[\xi, \eta] = 0$.

Now for any vector field ξ^u defined in the neighborhood of $\mathcal{W}^u_z$ and tangent to the unstable distribution we have

$$\nabla_{\xi^u}[\xi, \eta] = p^s\big[\xi^u, [\xi, \eta]\big] = p^s\big[[\xi^u, \xi], \eta\big] + p^s\big[\xi, [\xi^u, \eta]\big].$$

By assumption, $[\xi^u, \xi] = \beta^u$ and $[\xi^u, \eta] = \delta^u$ are tangent to $\mathcal{W}^u$. Thus,

$$\nabla_{\xi^u}[\xi, \eta] = p^s[\beta^u, \eta] + p^s[\delta^u, \xi] = \nabla_{\beta^u}\eta + \nabla_{\delta^u}\xi = 0.$$

Hence the section $[\xi, \eta]$ is also holonomy invariant. Then, for any $z' \in \mathcal{W}^u_z$ and for any path c going from z' to z in $\mathcal{W}^u_z$,

$$d\mathcal{H}_{z,z',c}[\xi, \eta](z) = [\xi, \eta](z') \in S_\tau(z')$$

and therefore $[\xi, \eta] \in < S_\tau(z) >$, thus proving the integrability. □

2.5. **A structure on the boundary at infinity.** Let $\mathcal{F}$ be the foliation associated to $< S_\tau >$, for $\tau = \tau_{1,1}$, and $\tilde{\mathcal{F}}$ its lift to the universal cover. Let us recall that the boundary at infinity $\partial\tilde{S}$ of the hyperbolic space is the set of limits of geodesics. The boundary is in one to one correspondence with the set of weak unstable(stable) leaves. Let us briefly recall a useful way to understand that $\partial\tilde{S}$ is a canonical sphere (see also [19]). For any $z \in \widetilde{US}$, one may consider the two Stable leaves $\mathcal{E}^s_z$ and $\mathcal{E}^s_{\sigma(z)}$, where σ is the antipody.
One easily check that, $\mathcal{W}^u_z \bigcap \mathcal{E}^s_{\sigma(z)} = \emptyset$ and $\mathcal{W}^u_{\sigma(z)} \bigcap \mathcal{E}^s_{(z)} = \emptyset$.
Furthermore, any other weak unstable leaf intersects these two stable leaves in exactly one point. This defines a map $P : \mathcal{E}^s_z \backslash \{z\} \to \mathcal{E}^s_{\sigma(z)} \backslash \{\sigma(z)\}$ given by,
$P(y) = \mathcal{W}^u_y \bigcap \mathcal{E}^s_{\sigma(z)}$, which is by definition a holonomy map. Thus the manifold structure of the boundary is given by these two charts, which are two copies of $\mathbb{R}^{n-1}$ and the map P which describe the change of charts.

Furthermore the action $\rho(\gamma)$ of each element $\gamma \in \pi_1(S)$ is a North-South dynamics. It means that $\rho(\gamma)$ is a homeomorphism with two fixed points $\{x^+, x^-\}$

and

$$\forall x \in \partial\tilde{S} - \{x^+, x^-\} \ , \lim_{n \mapsto \pm\infty} \rho(\gamma)^n(x) = x^\pm.$$

We are now at the key point.

Lemma 12. *There exists a real τ such that $\psi_{*\tau}V_*$ is vertical.*

Proof. This step is a direct consequence of [24] where it is proven that a C^0-foliation on a sphere which is invariant under the action of a North-South dynamics is trivial. There are two trivial foliations of a manifold : the foliation by points , the foliation with one leaf (the manifold itself).
The foliation $\tilde{\mathcal{F}}$ being invariant by the holonomy of the weak unstable foliation, it induces a foliation $\hat{\mathcal{F}}$ of $\partial\tilde{S}$ which is furthermore invariant under the action of the fundamental group. Thus the foliation $\hat{\mathcal{F}}$ is trivial. It cannot be the foliation by points because $< S_\tau > \neq \{0\}$. Hence $\hat{\mathcal{F}}$ is a foliation with only one leaf and thus the dimension of the distribution $< S_\tau >$ is equal to $n-1$. This implies that, $< S_\tau >= E^s$. Finally the set Δ contains only one element $\{\tau_{1,1}\}$ and Lemma 8 i) gives the conclusion. ☐

2.6. Proofs of Theorems 1 and 2.

Proof of Theorem 1. To end the proof of Theorem 1, we consider the map, $\eta : US \mapsto HM$, $\eta \overset{def}{=} \Psi \circ \Phi_\tau$, where τ is given by the previous Lemma. This is a covering and a local smooth diffeomorphism which maps fibers into fibers. It induces a map $f : S \mapsto M$ such that $\pi_M \circ \eta = f \circ \pi_S$. From Corollary 3, the map f sends geodesics onto geodesics. Then by uniqueness of the solutions of first order differential equations, the map $\delta : US \mapsto HM$, $\delta \overset{def}{=} Tf^{-1} \circ \eta$ is the identity. To see this, we just remark that the conjugacy relation , $\forall p \in US$, $t \in \mathbb{R}$, $\delta \circ \Phi_t(p) = \Phi_t \circ \delta(p)$, implies by projection $\pi_S((\Phi_t(\delta(p))) = \pi_S(\Phi_t(p))$. ☐

Proof of Theorem 2. The definition of the metric entropy of the geodesic flow is an example of an entropy associated to a triple (N, Θ, μ) where N is a manifold, Θ a homeomorphism and μ an invariant probability measure. I will not recall the first definition of entropy and its main properties. For our concern, the manifold is the homogeneous bundle $N = HM$, the diffeomorphism Θ is the time-one map of the flow and the invariant measure is the normalized Liouville measure, i.e the one associated to the contact volume. In ([23]), a generalization of the metric entropy formula given by Freire and Mane is proven for the geodesic flow of Finsler metrics without conjugate points. Inequality (1.1) is proven through and argument of Ossermann and Sarnak ([33]), from which one easily deduces that the equality case is equivalent to the flow invariance of the Jacobi endomorphism. But our first theorem precisely solves this question. □

Both theorems use the reversibility condition but this is not the case for Inequality (1.1) neither for Theorem 5 which may as well be used in the non reversible case. However the existence of non-reversible and locally symmetric or parallel Finsler spaces remains open even in the negative curvature case.

3. The dynamic on S^2 when $K = +1$

This part is just the continuation of the introduction and we give only a very short sketch of the proofs of Theorems 3 and 4. Akbar-Zadeh [37] proved that a complete Finsler manifold of constant positive flag curvature is a sphere. For reversible Finsler metrics, the following result due to Z. Shen [34] brings useful informations in any dimension.

Theorem 6. [34]. *Let (M, F) a complete simply connected Finsler manifold. Suppose that F has constant curvature +1. Then the following holds.*

i) *For every x in M, there exists a unique point $x* \in M$ with $d(x, x*) = \pi$.*
ii) *every geodesic issuing from x is a simple closed curve of length 2π, passing through $x*$.*
iii) *the map $\exp_x$ is a diffeomorphism from $\{v \in T_xM \mid 0 < F(v) < \pi\}$ to $M \setminus \{x, x*\}$.*

Remarks :

i) Shen's theorem uses the reversibility. If one removes this hypothesis, then the exponential map has no reason to be injective on this domain and one can probably have closed geodesics of different lengths.
ii) One may easily check that the map defined as $\sigma(x) = x*$ is an involution fixed-point free isometry of (M, F).

In dimension 2 we can improve this result with the following Lemma.

Lemma 3.1. *The intersection of two distinct geodesics consists of exactly two antipodal points.*

Let us now consider the fibration $\rho : HS^2 \to \Lambda$, whose basis is the space of oriented geodesics. We immediately obtain.

Lemma 3.2. *The manifold Λ is a 2-sphere.*

Then we produce a diffeomorphism $\Theta : HS^2 \to H\Lambda$ of the two bundles. The Sasaki metric associated to the Finsler metric being flow invariant can be projected to a Riemannian metric g on Λ. We prove that, the geodesic flow of the Finsler metric is conjugate to the angular flow (standard vertical flow of a Riemannian metric on an oriented surface). Such a flow only depends of the conformal class and thus is itself conjugate to the geodesic flow of a metric with curvature $+1$. This proves the Theorem 3.

To sketch the proof of Theorem 4, we need to introduce some more terminology. We say that a Y-curve on Λ is the projection by ρ of an integral curve of the unit vertical vector field Y in HS^2 (associated to a choice of orientation on S^2)(it is, the projection of a fiber). The family of Y-curves on Λ share the same incidence properties than the geodesics of the Euclidean sphere. More precisely:

Lemma 3.3. *The Y-curves have the following properties*

i) *Y-curves are simple and closed curves on Λ;*
ii) *for every direction v in $H\Lambda$, there exists one and only one Y-curve tangent to v;*
iii) *two distinct (up to orientation) Y-curves intersect at exactly two antipodal points;*
iv) *every two distinct points on Λ that are not antipodal can be joined by a unique (up to orientation) Y-curve.*

As observed above the antipody descends on Λ to an involution without fixed points and preserves each individual Y-curve. We can then consider the quotient $\mathcal{P}^2$ which from the previous Lemma inherits an axiomatic projective structure of dimension two. The projections of the Y-curves are the lines of this geometry. The Cartan structure equation for the Finsler metric of constant flag curvature on S^2 differs from the one of a Riemannian metric by a flow-invariant vector field. By the duality associated to the contact form, we obtain a flow-invariant 1-form, vanishing in the flow direction. This form vanishes if and only if the metric is Riemannian. It can be pushforwarded on the space of geodesics to a 1-form that we call β.

Lemma 3.4. *The reversibility of the Finsler metric implies that the 1-form $\alpha = *\beta$, where the star operator $*$ is taken with respect to the metric g, is invariant under the action of the antipody.*

It can be easily observed that

$$\int_{Y-curve} \alpha = 0. \tag{3.1}$$

Therefore the projection to $\mathcal{P}^2$ of this form fulfills the conditions to apply the Radon-Gelfand property (that we assume here) and thus there exists a smooth function f on Λ such that $\alpha = df$. Now a conformal change of the metric g using the function f as density shows that the Y-curves are, up to the parameter, the geodesics of a Riemannian metric. But all the geodesics are closed and the new metric is also invariant by the involution. By Besse [9] (Blaschke theorem) we know that the modified metric is a standard metric. A curvature computation then shows that the 1-form β vanishes.

4. Appendix - About Finsler geometry

Variational problem

Let us consider (M,F) where $F : TM \to R$ is a Finsler metric, and $\sigma : HM \to M$, the bundle of oriented half-lines, $HM = \overset{\circ}{T}M/\mathbb{R}^{+*}$, where $\overset{\circ}{T}M$, is the tangent bundle without zero vectors. The associated variational problem is for given points x, y in M to find the critical values among C^1-piecewise paths $c : [0, 1] \to M$ with $c(0) = x, c(1) = y$ of the functional,

$$I(x, y, c) = \int_{[0,1]} F\big(c(t), c'(t)\big)dt\ . \tag{4.1}$$

The function F being assumed to be positively homogeneous in degree one along the fibers, it is well known that, there exists a unique [22] 1-form A on HM such that,

$$I(x, y, c) = \int_c A\ ,$$

where C is the canonical lifted path of c on HM (obtained by taking the tangent half-line at each point).

From [22] we have

Proposition 2. *The geodesic flow φ_t on HM with generator $X : HM \to THM$ must up to a normalization satisfy the following equations*

$$i_X dA = 0 \quad ; \quad A(X) = 1 \tag{4.2}$$

If F is regular and strictly convex, then A is a contact form and X is the associated Reeb field. Let VHM be the vertical bundle over HM. The 1-form A is semi-basic, that is,

$$A/VHM = 0 \ . \tag{4.3}$$

Second order differential equations

Let r be the map that sends each non zero vector into its corresponding half-line and $T\sigma$ the tangent map to the projection σ.

Definition 1. A second order differential equation on M is a vector field, X without singularities on HM such that

$$r \circ T\sigma \circ X = \text{Identity of } HM; \tag{4.4}$$

or equivalently: *the integral curves of X are the canonical lifted curves of M.*

Proposition 3. *The geodesic vector field X of a Finsler metric is a second order differential equation.*

Remark : The same construction can be done for the projectivized bundle PTM.

Dynamical approach

(see [22]) : At this step for any C^2 second order differential equation X we can construct a dynamical derivative and a Jacobi endomorphism without making any use of Finsler geometry.
(For instance, X will not be supposed to have an invariant volume form.)
The key lemma is the following.

Lemma 13. *Let $X : HM \to THM$ be a second order differential equation, Y_1 and Y_2 two vertical vector fields on HM. If at one point $z \in HM$, there exist two real numbers a, b, such that*

$$aX(z) + Y_1(z) + b[X, Y_2](z) = 0 \ ,$$

then $bY_2(z) = 0$ and $a = 0$.

Furthermore if $Y_2(z)$ is not zero, then $b = 0$ and $Y_1(z) = 0$.
A direct application is :

Corollary 4. *Let $X : HM \to THM$ be a second order differential equation. Then there exists on each fiber of THM a "vertical endomorphism" v_X such that* *for any vertical vector Y in V_zHM and Y^* any local C^1 vertical extension of Y,*

$$\begin{array}{lllll} \text{(i)} & v_X(X) & = & 0 & , \\ \text{(ii)} & v_X\big([X,Y^*]\big) & = & -Y & , \\ \text{(iii)} & v_X(Y) & = & 0 & . \end{array} \tag{4.5}$$

The horizontal distribution and the pseudo-complex structure (see, [22])

Lemma 14. *Let $z \in HM$ and $Y \in V_z$ HM, and let Y^* be a C^2 vertical extension of Y. The following operator*

$$H_X(Y) = -[X,Y^*]_{(z)} - 1/2v_X\big([X,[X,Y^*]]\big)_{(z)} \ , \tag{4.6}$$

acting on vertical vector fields is an injective 0-order differential operator mapping V_zHM into T_zHM.

The distribution

$$h_XHM = H_X(VHM) \ , \tag{4.7}$$

is transverse to $RX \oplus VHM$ and will be called the *"orthogonal horizontal"* distribution.

The definition formula (4.6) may seem rather mysterious, but the second term on the right is, thanks to (4.5), the good correction term to the ordinary Lie derivative. Therefore the first flow variation of vertical vector fields gives rise to the horizontal distribution. We will observe that the second will produce the Jacobi endomorphism.

Hence the tangent bundle splits as

$$THM = RX \oplus VHM \oplus h_XHM \tag{4.8}$$

with the corresponding projectors

$$P_X + P_v + P_h = Id \ . \tag{4.9}$$

From the definitions one can easily deduce the commutation relations,

$$V_X \circ H_X = Id/VHM, \quad H_X \circ V_X = Id/h_XHM \ . \tag{4.10}$$

It is equivalent to say that on the bundle $VHM \oplus h_XHM$, the linear operator J_X defined by

$$\begin{array}{lll} J_X(h) = -V_X(h) & \text{for} & h \in h_XHM \\ J_X(Y) = H_X(Y) & \text{for} & Y \in VHM \end{array} \tag{4.11}$$

is a pseudo-complex structure.

Dynamical derivative and Jacobi endomorphism

From now-on X is the flow generator, Y is a vertical vector field and h is horizontal.

Definition 2. The dynamical derivative D_X of a second-order differential equation is the first-order differential operator splitting-preserving and defined by the following equations :

$$\begin{aligned} \mathcal{D}^X(Y) &= P_V[X,Y] = -1/2V_X\big([X,[X,Y]]\big) \ , \\ \mathcal{D}^X(X) &= 0 \ , \\ \mathcal{D}^X(h) \ &= P_h[X,h] \ . \end{aligned} \tag{4.12}$$

From this we can deduce,

$$[\mathcal{D}^X, J_X] = 0. \tag{4.13}$$

Definition 3. The "Jacobi' endomorphism" at a point z in HM is the linear operator R^X on T_zHM commuting with J_X and preserving the splitting, such that

$$\begin{array}{rrcl} \text{(i)} & R^X(X) & = & 0, \\ \text{(ii)} & R^X(Y) & = & P_V[X,H_X(Y)]. \end{array} \tag{4.14}$$

The Jacobi equation

Proposition 4. *Let X be a second order differential equation on HM, φ_t the corresponding flow, $z \in HM, Z \in T_zHM, Z_t = T\varphi_t(Z) = a_tX + h_t + Y_t$.*
Then the following equations are satisfied

$$\begin{array}{rrcl} \text{(i)} & L_X a & = & P_X[X,h], \\ \text{(ii)} & \mathcal{D}^X(Y) & = & -R^X\big(v_X(h)\big) \ , \\ \text{(iii)} & \mathcal{D}^X(h) & = & H_X(Y) \ . \end{array} \tag{4.15}$$

Equations (ii) and (iii) can be combined into the "Jacobi equation"

$$\big(\mathcal{D}^X \circ \mathcal{D}^X + R^X\big)\big(v_X(h)\big) = 0 \ . \tag{4.16}$$

Finsler case

Now we can use the contact form A.

Lemma 15. *There exists a Riemannian metric g on HM called the Sasaki metric, such that*

(i) the splitting is orthogonal,

(ii) $g(X,X) = 1$,

(iii) on $ker A$, $g = dA(J^X, \ )$,
(iv) g *is* $\mathcal{D}^X$ *invariant,*
(v) R^X *is symmetric with respect to* g.

Remarks. For a Riemannian metric, $\mathcal{D}^X$ can be identified with the covariant derivative along the geodesics and R^X projects to the classical Jacobi endomorphism. In the Finsler case on HM one can consider the Levi-Civita connection of the metric g. The fibers are totally geodesic in the Riemannian case but not in the general Finsler case.

The dynamical approach is very efficient for explicit curvature computations. The important fact is that when one performs a time change on a geodesic flow it is not (in general) a geodesic flow any more but it is still a second-order differential equation. There exist nice and easy formulas (see [22]) expressing the new curvature after a time change. As an amusement I recommend to show using those formulas that the curvature of a Hilbert geometry is a negative constant.

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Received February 13, 2002

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