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ALGEBRAIC CURVATURE TENSORS FOR INDEFINITE METRICS WHOSE SKEW-SYMMETRIC CURVATURE OPERATOR HAS CONSTANT JORDAN NORMAL FORM

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Abstract. We classify the connected pseudo-Riemannian manifolds of signature (p,q) with $q \geq 5$ so that at each point of M the skew-symmetric curvature operator has constant rank 2 and constant Jordan normal form on the set of spacelike 2 planes and so that the skew-symmetric curvature operator is not nilpotent for at least one point of M.

1. Introduction

Let (M,g) be a smooth connected pseudo-Riemannian manifold of signature (p,q). Let ∇ be the Levi-Civita connection on TM. The *Riemann curvature operator* and the associated *curvature tensor* are defined by:

$$
\begin{aligned}
{}^gR(x,y) &:= \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]} \text{ and} \\
{}^gR(x,y,z,w) &:= g({}^gR(x,y)z,w).
\end{aligned}
$$

We have the following symmetries

$$
\begin{aligned}
&{}^gR(x,y,z,w) = {}^gR(z,w,x,y) = -{}^gR(y,x,z,w), \text{ and} \\
&{}^gR(x,y,z,w) + {}^gR(y,z,x,w) + {}^gR(z,x,y,w) = 0.
\end{aligned}
$$

Let V be a vector space with an inner product $(\cdot,\cdot)$ of signature (p,q). We say that a 4 tensor $R \in \otimes^4 V^*$ is an *algebraic curvature tensor* if it satisfies the symmetries given above; the associated curvature operator is then defined by $R(x,y,z,w) = (R(x,y)z,w)$. We say that (M,g) is a *geometrical realization* of an

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algebraic curvature tensor R at a point P of M if there exists an isometry Θ from T_PM to V so that ${}^gR(x,y,z,w) = R(\Theta x, \Theta y, \Theta z, \Theta w)$ for all $x,y,z,w \in T_PM$. Every algebraic curvature tensor has such a geometrical realization. Conversely, it is often useful to study problems in geometry by first passing to the purely algebraic setting and then drawing geometrical conclusions from results obtained algebraically.

Let R be an algebraic curvature tensor on a vector space V of signature (p,q). Let $\{e_1, e_2\}$ be an oriented basis for a non-degenerate oriented 2 plane $\pi \subset V$. Let

$$R(\pi) := |(e_1,e_1)(e_2,e_2) - (e_1,e_2)^2|^{-1/2} R(e_1,e_2)$$

be the associated *skew-symmetric curvature operator*; $R(\pi)$ depends on the orientation of π but is independent of the particular oriented basis chosen. In this paper, we will examine the geometric consequences which follow from assuming that the skew-symmetric curvature operator has certain algebraic properties.

We say that a vector $v \in V$ is *spacelike* if $(v,v) > 0$, *timelike* if $(v,v) < 0$, and *null* if $(v,v) = 0$. We say that a 2 plane π is *spacelike* if the induced metric on π has signature $(0,2)$, *timelike* if the induced metric has signature $(2,0)$, and *mixed* if the induced metric has signature $(1,1)$. Otherwise π is said to be *degenerate*.

The simplest invariant of a linear map is the rank. We say that an algebraic curvature tensor R has *spacelike rank* r if $\operatorname{rank}(R(\pi)) = r$ for every oriented spacelike 2 plane π. The notions of *timelike* and *mixed rank* r are defined similarly. If $\operatorname{rank}(R(\pi))$ is not constant on the set of oriented spacelike 2 planes, then we say R does not have constant spacelike rank. In §2, we shall construct algebraic curvature tensors which have spacelike rank 2 but which do not have constant timelike or constant mixed rank.

The following result of Gilkey, Leahy and Sadofsky [5] and of Zhang [9] uses results from Adams [1] and Borel [2] to bound the rank:

Theorem 1.1. *Let R be an algebraic curvature tensor on a vector space V of signature (p,q) which has spacelike rank r.*

(1) *Let $p = 0$. Let $q \geq 5$ and $q \neq 7, 8$. Then $r \leq 2$.*

(2) *Let $p = 1$. Let $q = 5$ or $q \geq 9$. Then $r \leq 2$.*

(3) *Let $p = 2$. Let $q \geq 10$. Then $r \leq 4$. Furthermore, if neither q nor $q+2$ are powers of 2, then $r \leq 2$.*

Let V be a vector space of signature (p, q) and let ϕ be a self-adjoint linear map of V. We define a 4 tensor and associated endomorphism:

$$R_\phi(x, y, z, w) := (\phi y, z)(\phi x, w) - (\phi x, z)(\phi y, w) \text{ and}$$
$$R_\phi(x, y)z := (\phi y, z)\phi x - (\phi x, z)\phi y.$$

The tensor R_ϕ is an algebraic curvature tensor and the set of tensors arising in this fashion spans the space of all algebraic curvature tensors [4]. Such tensors arise as the curvature tensors of hypersurfaces in flat space - see Lemma 3.1.

These tensors play a crucial role in the classification of the algebraic curvature tensors of rank 2. We refer to [5, 6] for the proof of the following result:

Theorem 1.2. *Let R be an algebraic curvature tensor on a vector space V of signature (p, q) where $q \geq 5$. Then R has spacelike rank 2 if and only if $R = \pm R_\phi$ where ϕ is a self-adjoint map of V whose kernel contains no spacelike vectors.*

The spectrum (i.e. the eigenvalues counted with multiplicity) is also a useful invariant of a linear map. We say that an algebraic curvature tensor is *spacelike IP* if the eigenvalues of $R(\pi)$ are the same for any two oriented spacelike 2 planes; the notions of *timelike* and *mixed IP* are defined similarly. (The notation 'IP' is chosen as the fundamental classification results in dimension 4 are due to Ivanov and Petrova [7] - see also related work in [8]). One can use analytic continuation to see that the notions of spacelike, timelike, and mixed IP coincide so we shall simply say that R is IP if any of these three equivalent conditions holds. Two linear maps T and $\tilde{T}$ of V are said to be *Jordan equivalent* if any of the following three equivalent conditions are satisfied:

(1) There exist bases $\mathcal{B} = \{e_1, ..., e_m\}$ and $\tilde{\mathcal{B}} = \{\tilde{e}_1, ..., \tilde{e}_m\}$ for V so that the matrix representation of T with respect to the basis $\mathcal{B}$ is equal to the matrix representation of $\tilde{T}$ with respect to the basis $\tilde{\mathcal{B}}$.
(2) There exists an isomorphism Θ of V so $T = \Theta\tilde{T}\Theta^{-1}$, i.e. T and $\tilde{T}$ are *conjugate*.
(3) The real Jordan normal forms of T and $\tilde{T}$ are equal.

In the positive definite setting, the spectrum of a skew-symmetric linear map determines the conjugacy class of the map. This is not, however, the case in the indefinite setting. We say that R is *spacelike Jordan IP* if the Jordan normal form of $R(\cdot)$ is the same for any two oriented spacelike 2 planes; the notions of *timelike* and *mixed Jordan IP* are defined similarly. We note that if R is spacelike Jordan IP, then R has spacelike rank r for some r.

The Jordan form of a linear map determines the spectrum. Thus if R is spacelike Jordan IP, then R is spacelike IP and hence IP. In §2, we construct algebraic curvature tensors which are IP but which are not spacelike, timelike, or mixed Jordan IP. We will also construct algebraic curvature tensors which are spacelike Jordan IP but not timelike or mixed Jordan IP; thus these notions are distinct.

Let ϕ be a linear map of a vector space V of signature (p,q). If $(\phi v,\phi w)=(v,w)$ for all $v,w\in V$, then ϕ is an *isometry*. If $(\phi v,\phi w)=-(v,w)$ for all $v,w\in V$, then ϕ is a *para-isometry*; para-isometries exist if and only if $p=q$, i.e. if we are in the *balanced* setting. Suppose that ϕ is self-adjoint. Then ϕ is an isometry if and only if $\phi^2=\mathrm{id}$; ϕ is a para-isometry if and only if $\phi^2=-\mathrm{id}$.

We say that R is *spacelike rank 2 Jordan IP* if R has spacelike rank 2 and if R is spacelike Jordan IP, the notions of *timelike rank 2 Jordan IP* and *mixed rank 2 Jordan IP* are defined similarly. We have the following classification result for such tensors [6]:

Theorem 1.3. *Let R be an algebraic curvature tensor on a vector space of signature (p,q) where $q\geq 5$. Then R is a spacelike rank 2 Jordan IP algebraic curvature tensor if and only if exactly one of the following three conditions is satisfied:*

(1) *$R=CR_\phi$ where ϕ is a self-adjoint isometry, and where $C\neq 0$.*
(2) *$R=CR_\phi$ where ϕ is a self-adjoint para-isometry, and where $C\neq 0$.*
(3) *$R=\pm R_\phi$ where ϕ is self-adjoint, where $\phi^2=0$, and where $\ker\phi$ contains no spacelike vectors.*

Remark 1.4. The map ϕ in Theorem 1.3 is uniquely defined up to sign; the constant C in assertions (1) and (2) is uniquely determined. The tensors CR_ϕ in assertions (1) and (2) where $\phi^2=\pm\mathrm{id}$ are also timelike and mixed rank 2 Jordan IP. The tensor in assertion (3) is *nilpotent*; if $\phi^2=0$, then $R_\phi(x,y)^2=0$ for all (x,y). This tensor is timelike rank 2 Jordan IP if and only if $\ker\phi$ contains no timelike vectors; it is not constant mixed rank.

We say that a pseudo-Riemannian manifold (M,g) is *spacelike rank r Jordan IP* if ${}^gR$ is spacelike rank r Jordan IP at every point of M; the Jordan form is allowed to vary with the point but the rank is assumed to be constant. The notions of *timelike rank r Jordan IP* and *mixed rank r Jordan IP* are defined similarly. In §3, we will construct two families of spacelike, timelike, and mixed rank 2 Jordan IP pseudo-Riemannian manifolds. Lemma 3.2 deals with the pseudo-spheres and Lemma 3.3 deals with warped products of a manifold with constant sectional curvature with an interval $I\subset\mathbb{R}$.

The following is the main result of this paper; it generalizes previously known results [3, 5, 7] from the Riemannian to the pseudo-Riemannian setting.

Theorem 1.5. *Let (M,g) be a connected spacelike rank 2 Jordan IP pseudo-Riemannian manifold of signature (p,q) where $q \geq 5$. Assume that ${}^gR$ is not nilpotent for at least one point P of M.*

(1) *For each point $P \in M$, we have ${}^gR_P = C(P)R_{\phi(P)}$ where $\phi(P)$ is self-adjoint map of T_PM so that $\phi(P)^2 = \mathrm{id}$; ${}^gR_P$ is never nilpotent.*
(2) *If $\phi = \pm\mathrm{id}$, then (M,g) has constant sectional curvature and is locally isometric to one of the manifolds constructed in Lemma* 3.2.
(3) *If $\phi \neq \pm\mathrm{id}$, then (M,g) is locally isometric to one of the warped product manifolds constructed in Lemma* 3.3.

We shall prove Theorem 1.5 in §4. The classification of spacelike rank 2 Jordan IP pseudo-Riemannian manifolds with nilpotent algebraic curvature tensors is incomplete and in §5, we present some preliminary results on this case.

2. Examples of algebraic curvature tensors

We begin this section with a technical observation.

Lemma 2.1. *Let V be a vector space with an inner product of signature (p,q). Let $\{v_1,...,v_k\}$ be a set of linearly independent elements of V. Then there exist elements $\{w_1,...,w_k\}$ of V so that $(v_i,w_j) = \delta_{ij}$.*

Proof. We use the inner product to define a linear map $\psi : V \to V^*$ by the identity $\psi(w)(v) = (v,w)$. If $w \neq 0$, then there exists v so $(w,v) \neq 0$ and thus ψ is injective. Since $\dim V = \dim V^*$, ψ is a linear isomorphism. Let $\{v_1,...,v_k\}$ be a set of linearly independent elements of V. We extend this set to a basis $\{v_1,...,v_{p+q}\}$ for V to assume without loss of generality that $k = p+q$. Let $\{v^1,...,v^{p+q}\}$ be the associated dual basis for V^*; this means if $v \in V$, then $v = \sum_i v^i(v)v_i$. The desired elements w_j of V are then defined by $w_j = \psi^{-1}(v^j)$ since $v^i(v_j) = \delta_{ij}$. □

Let V be a vector space of signature (p,q). We can choose a *normalized* orthonormal basis $\mathcal{B}$ for V of the form:

$$\mathcal{B} := \{e_1^-,...,e_p^-,e_1^+,...,e_q^+\} \text{ so}$$
$$V^- := \mathrm{span}\{e_1^-,...,e_p^-\} \text{ and} \tag{1}$$
$$V^+ := \mathrm{span}\{e_1^+,...,e_q^+\} \tag{2}$$

are maximal orthogonal timelike and spacelike subspaces. Let $R = R_\phi$ where ϕ is a self-adjoint linear map. Let π be a non-degenerate 2 plane. By Lemma 2.1:

$$\begin{aligned} &\operatorname{rank}(R_\phi(\pi)) = 2 \text{ if } \ker\phi \cap \pi = \{0\} \text{ and} \\ &\operatorname{rank}(R_\phi(\pi)) = 0 \text{ if } \ker\phi \cap \pi \neq \{0\}. \end{aligned}$$

We use this observation to show that the notions of constant spacelike, timelike, and mixed rank are distinct notions.

Lemma 2.2. *Let V have signature (p, q) where $p \geq 2$ and $q \geq 2$.*

1. *There exists an algebraic curvature tensor R on V which has spacelike rank 2 but which does not have constant timelike or mixed rank.*
2. *There exists an algebraic curvature tensor R on V which has timelike rank 2 but which does not have constant spacelike or mixed rank.*
3. *There exists an algebraic curvature tensor R on V which has spacelike and timelike rank 2 but which does not have constant mixed rank.*

Proof. Let $\mathcal{B}$ be a normalized basis for V. Let ϕ be orthogonal projection on V^+;

$$\phi(e_i^-) = 0 \text{ and } \phi(e_i^+) = e_i^+.$$

As $\ker\phi = V^-$ is timelike, $\ker\phi$ contains no spacelike vectors and $\operatorname{rank} R_\phi(\pi) = 2$ for any spacelike 2 plane. Thus R has spacelike rank 2. We define:

$$\begin{aligned} &\pi_1 := \operatorname{span}\{2e_1^- + e_1^+, 2e_2^- + e_2^+\}, \ \tilde\pi_1 := \operatorname{span}\{e_1^-, e_2^-\}, \\ &\pi_2 := \operatorname{span}\{2e_1^- + e_1^+, e_2^+\}, \qquad\quad \tilde\pi_2 := \operatorname{span}\{e_1^-, e_2^+\}. \end{aligned}$$

The planes π_1 and $\tilde\pi_1$ are timelike and the planes π_2 and $\tilde\pi_2$ are mixed. We show that R does not have constant timelike or mixed rank by noting:

$$\ker\phi \cap \pi_i \neq \{0\} \text{ and } \ker\phi \cap \tilde\pi_i = \{0\} \text{ for } i = 1, 2.$$

Assertion (1) now follows; we interchange the roles of $+$ and $-$ to prove assertion (2) similarly.

To prove assertion (3), we define a self-adjoint linear map $\tilde\phi$ of V by setting:

$$\begin{aligned} &\tilde\phi(e_1^-) = e_1^- + e_1^+, \quad \tilde\phi(e_i^-) = e_i^- \text{ for } i > 1 \\ &\tilde\phi(e_1^+) = -e_1^- - e_1^+, \ \tilde\phi(e_j^+) = e_j^+ \text{ for } j > 1. \end{aligned}$$

Then $\ker\tilde\phi = \operatorname{span}\{e_1^- + e_1^+\}$. Since $\ker\tilde\phi$ is totally isotropic, $\ker\tilde\phi$ contains no spacelike vectors and no timelike vectors so $R_{\tilde\phi}$ has constant spacelike and timelike rank 2. We define mixed 2 planes

$$\pi_1 := \operatorname{span}\{e_1^-, e_1^+\} \text{ and } \pi_2 := \operatorname{span}\{e_2^-, e_2^+\}.$$

Since $\ker\tilde\phi \cap \pi_1 \neq \{0\}$ and $\ker\tilde\phi \cap \pi_2 = \{0\}$, $R_{\tilde\phi}$ does not have constant mixed rank. Assertion (3) now follows. □

Let R be an algebraic curvature tensor of constant spacelike rank r on a vector space of signature (p, q). In Theorem 1.1, we noted that $r = 2$ in many cases if $p = 0$, $p = 1$, or $p = 2$. There are, however, examples where $r = 4$ if $p \geq q$.

Lemma 2.3. *Let V be a vector space of signature (p, q) where $p \geq q \geq 2$.*

(1) *If $p = q$, then there exists an algebraic curvature tensor R on V which has constant spacelike and timelike rank 4, and which does not have constant mixed rank.*

(2) *If $p > q$, then there exists an algebraic curvature tensor R on V which has constant spacelike rank 4, and which does not have constant timelike or mixed rank 4.*

Proof. Let $\mathcal{B}$ be a normalized basis for V. We suppose that $p \geq q$ and define a linear self-adjoint map ϕ of V by setting:

$$\begin{aligned}\phi(e_i^-) &= -e_i^+ \text{ for } 1 \leq i \leq q,\\ \phi(e_i^-) &= \quad 0 \text{ for } q < i \leq p, \text{ and}\\ \phi(e_j^+) &= \quad e_j^- \text{ for } 1 \leq j \leq q.\end{aligned}$$

Expand any vector $v \in V$ in the form $v = \sum_i c_i e_i^- + \sum_j d_j e_j^+$. Then

$$(v, v) = -\textstyle\sum_i c_i^2 + \sum_j d_j^2 \text{ and } (\phi v, \phi v) = \sum_{i\leq q} c_i^2 - \sum_j d_j^2.$$

Thus if v is spacelike, ϕv is timelike. Furthermore, if $p = q$, then ϕ is a para-isometry. Let $R := R_{\mathrm{id}} + R_\phi$. Then

$$R(x, y)z = (y, z)x - (x, z)y + (\phi y, z)\phi x - (\phi x, z)\phi y.$$

If $\{x, y\}$ spans a spacelike 2 plane π, then $\{\phi x, \phi y\}$ spans a timelike 2 plane $\phi\pi$. Thus $\{x, y, \phi x, \phi y\}$ is a linearly independent set and we may use Lemma 2.1 to see that R has spacelike rank 4. If $p = q$, then the roles of $+$ and $-$ are symmetric so R also has timelike rank 4. If $p > q$, then $\ker\phi$ contains a timelike vector and the same argument used to prove Lemma 2.2 shows R does not have constant timelike rank. Let $\pi_1 := \operatorname{span}\{e_1^-, e_1^+\}$ and $\pi_2 := \operatorname{span}\{e_1^-, e_2^+\}$ be mixed 2 planes. We show that R does not have mixed rank 4 by noting that:

$$\operatorname{rank} R_\phi(\pi_1) \leq 2 \text{ and } \operatorname{rank} R_\phi(\pi_2) = 4.$$

□

We omit the proof of the following result as the proof is straightforward.

Lemma 2.4. *Let V be a vector space of signature (p, q).*

(1) *If T is a linear map of V with $T^2 = 0$, 0 is the only eigenvalue of T.*
(2) *If T_1 and T_2 are linear maps of V with $T_1^2 = T_2^2 = 0$, then T_1 is Jordan equivalent to T_2 if and only if* $\operatorname{rank}(T_1) = \operatorname{rank}(T_2)$.

We use Lemma 2.4 to show that the concepts IP, spacelike Jordan IP, and timelike Jordan IP are inequivalent.

Lemma 2.5. *Let V have signature (p, q) where $p \geq 3$ and $q \geq 3$.*

(1) *There exists an algebraic curvature tensor R on V which is IP, not spacelike Jordan IP, not timelike Jordan IP, and not mixed Jordan IP.*
(2) *If $p = q$, then there exists an algebraic curvature tensor R on V which is spacelike Jordan IP, timelike Jordan IP, and not mixed Jordan IP.*
(3) *If $p > q$, then there exists an algebraic curvature tensor R on V which is spacelike Jordan IP, not timelike Jordan IP, and not mixed Jordan IP.*

Proof. Let $\mathcal{B}$ be a normalized basis for V. For $k \leq q$, we define:

$$\begin{aligned} \phi_k(e_i^-) &= \quad e_i^- + e_i^+ \text{ for } i \leq k, \ \phi_k(e_i^-) = 0 \text{ for } i > k, \\ \phi_k(e_i^+) &= -e_i^- - e_i^+, \text{ for } i \leq k, \ \phi_k(e_i^+) = 0 \text{ for } i > k. \end{aligned}$$

Then ϕ_k is self-adjoint and $\operatorname{range}\phi_k$ is totally isotropic, i.e. the metric is trivial on $\operatorname{range}\phi_k$. Since $(\phi_k u, \phi_k v) = 0$ for all $u, v \in V$, we have $R_{\phi_k}(x, y)^2 = 0$ for any $x, y \in V$. Since $R_{\phi_k}(\pi)$ is nilpotent, we apply Lemma 2.4 to see that 0 is the only eigenvalue of R_{ϕ_k} and hence R is IP. We set $k = 2$ to prove assertion (1). Let

$$\begin{aligned} \pi_1 &:= \operatorname{span}\{e_1^-, e_2^-\}, \ \pi_2 := \operatorname{span}\{e_1^+, e_2^+\}, \ \pi_3 := \operatorname{span}\{e_1^-, e_2^+\}, \\ \tilde{\pi}_1 &:= \operatorname{span}\{e_1^-, e_3^-\}, \ \tilde{\pi}_2 := \operatorname{span}\{e_1^+, e_3^+\}, \ \tilde{\pi}_3 := \operatorname{span}\{e_1^-, e_3^+\}. \end{aligned}$$

We have $\{\pi_1, \tilde{\pi}_1\}$ are timelike, $\{\pi_2, \tilde{\pi}_2\}$ are spacelike, and $\{\pi_3, \tilde{\pi}_3\}$ are mixed 2 planes. For $1 \leq i \leq 3$, we apply Lemma 2.1 to prove assertion (1) by observing:

$$\begin{aligned} \pi_i \cap \ker\phi_2 = \{0\} &\text{ so } \operatorname{rank} R_{\phi_2}(\pi_i) = 2 \\ \tilde{\pi}_i \cap \ker\phi_2 \neq \{0\} &\text{ so } \operatorname{rank} R_{\phi_2}(\tilde{\pi}_i) < 2. \end{aligned}$$

Suppose $p \geq q$. We set $k = q$ to prove assertions (2) and (3). Since $\ker\phi_q$ contains no spacelike vectors, $\operatorname{rank} R_{\phi_q}(\pi) = 2$ for any spacelike 2 plane so by Lemma 2.4, R_{ϕ_q} is spacelike Jordan IP. Since $\ker\phi_q$ contains no timelike vectors if and only if $p = q$, R_{ϕ_q} is timelike Jordan IP if and only if $p = q$. We study $\pi_1 := \operatorname{span}\{e_1^-, e_2^+\}$ and $\pi_2 := \operatorname{span}\{e_1^-, e_1^+\}$ to see R_{ϕ_q} is never mixed Jordan IP. $\square$

3. IP Manifolds

Let V be a vector space of signature (r,s), let M be a simply connected smooth manifold of dimension $m=r+s-1$, and let $F:M\to\mathbb{R}^{(r,s)}$ be an immersion of M. We assume the induced metric g on M is non-degenerate; (M,g) is said to be a *non-degenerate hypersurface*. Let ν be a unit normal along M. If $(\nu,\nu)=+1$, then (M,g) has signature $(r,s-1)$. If $(\nu,\nu)=-1$, then (M,g) has signature $(r-1,s)$. We define the *second fundamental form* L and *shape operator* S by:

$$L(x,y):=(xyF,\nu)\text{ and }(Sx,y):=L(x,y).$$

The following is well known so we omit the proof in the interests of brevity.

Lemma 3.1. *Let V be a vector space of signature (r,s), let (M,g) be a non-degenerate hypersurface in V, let ν be a unit normal along M, and let S be the associated shape operator. Then ${}^gR=(\nu,\nu)R_S$.*

We say that a pseudo-Riemannian manifold (M,g) has *constant sectional curvature* κ if ${}^gR=\kappa R_{\mathrm{id}}$. Let V be a vector space of signature (r,s). For $\varrho>0$, let $S^\pm(r,s;\varrho)$ be the pseudo-spheres of spacelike and timelike vectors of length $\pm\varrho^{-1}$:

$$S^\pm(r,s;\varrho):=\{v\in V:(v,v)=\pm\varrho^{-2}\}.$$

The following result is well known so we omit the proof in the interests of brevity.

Lemma 3.2. *Let $\varrho>0$. Then $S^\pm(r,s;\varrho)$ has constant sectional curvature $\pm\varrho$ and is a rank 2 spacelike, timelike, and mixed Jordan IP pseudo-Riemannian manifold. Any pseudo-Riemannian manifold of constant sectional curvature is either flat or is locally isometric to one of these manifolds.*

It is also possible to construct examples of rank 2 spacelike Jordan IP pseudo-Riemannian manifolds by taking twisted products. We introduce the following notational conventions. Let $\varrho>0$, let $\varepsilon=\pm1$, and let $\delta=\pm1$ be given. Let

$$\begin{aligned} &M:=I\times S^\delta(r,s;\varrho), && f(t):=\varepsilon\kappa t^2+At+B,\\ &ds_M^2:=\varepsilon dt^2+f(t)d^2_{S^\delta(r,s;\varrho)}, && C(t):=f^{-2}\{f\kappa-\tfrac14\varepsilon f_t^2\},\\ &\phi:=-\mathrm{id}\ \text{ on } TS^\delta(r,s;\varrho), && \phi(\partial_t):=\partial_t. \end{aligned}$$

Choose $\{\kappa,A,B\}$ so $f\kappa-\frac14\varepsilon f_t^2\neq0$ or equivalently so that $A^2-4\varepsilon\kappa B\neq0$. Choose the interval I so $f(t)\neq0$ on I.

Lemma 3.3. *We have that (M,g_M) is a rank 2 spacelike, timelike, and mixed Jordan IP pseudo-Riemannian manifold with ${}^gR=CR_\phi$.*

PROOF. Let $m = r + s$. Fix $P \in S := S^\delta(r, s; \varrho)$. Choose local coordinates $x = (x_1, ..., x_{m-1})$ for S which are centered at P. We let $x_0 = t$ to define local coordinates $(x_0, ..., x_{m-1})$ on M. We let indices a, b, c, and d range from 1 to $m-1$ and index the local coordinate frame for S. We let indices i, j, k, and ℓ range from 0 to $m-1$ and index the full local coordinate frame for M. Let g_{ij} and $\tilde g_{ab}$ denote the components of the metric tensors on M and S relative to this local coordinate frame. We normalize the coordinates on S so that $\partial_a \tilde g_{bc}(P) = 0$. We have $g_{ab} = f\tilde g_{ab}$, $g_{0a} = 0$, and $g_{00} = \varepsilon$. Let $f_t := \partial_t f$ and $f_{tt} = \partial_t^2 f$. Let Γ and $\tilde\Gamma$ be the Christoffel symbols:

$$\begin{array}{ll}
\Gamma_{ijk} = \frac12\{\partial_j g_{ik} + \partial_i g_{jk} - \partial_k g_{ij}\}, & \tilde\Gamma_{abc} = \frac12\{\partial_b \tilde g_{ac} + \partial_a \tilde g_{bc} - \partial_c \tilde g_{ab}\},\\
\Gamma_{0ab} = \Gamma_{a0b} = \frac12 f_t \tilde g_{ab}, & \Gamma_{ab0} = -\frac12 f_t \tilde g_{ab}, \quad \Gamma_{abc} = f\tilde\Gamma_{abc},\\
\Gamma_{0a}{}^b = \Gamma_{a0}{}^b \equiv \frac12 f^{-1} f_t \tilde g_a{}^b, & \Gamma_{ab}{}^0 = -\frac12 \varepsilon f_t \tilde g_{ab},\ \Gamma_{ab}{}^c = \tilde\Gamma_{ab}{}^c;
\end{array}$$

the Christoffel symbols where 0 appears twice or three times vanish. Thus:

$$\begin{array}{rcl}
R_{ijk}{}^l &=& \partial_i \Gamma_{jk}{}^l - \partial_j \Gamma_{ik}{}^l + \sum_n \Gamma_{in}{}^l \Gamma_{jk}{}^n - \sum_n \Gamma_{jn}{}^l \Gamma_{ik}{}^n,\\
R_{abc}{}^d &\equiv& \{\partial_a \tilde\Gamma_{bc}{}^d - \partial_b \tilde\Gamma_{ac}{}^d\} + \Gamma_{a0}{}^d \Gamma_{bc}{}^0 - \Gamma_{b0}{}^d \Gamma_{ac}{}^0\\
&\equiv& \tilde R_{abc}{}^d - \frac14 \varepsilon f_t^2 f^{-1}\{\tilde g_a{}^d \tilde g_{bc} - \tilde g_{ac} \tilde g_b{}^d\} \bmod O(|x|),\\
R_{abcd} &\equiv& (f\kappa - \frac14 \varepsilon f_t^2)\{\tilde g_{ad}\tilde g_{bc} - \tilde g_{ac}\tilde g_{bd}\}\\
&\equiv& f^{-2}(f\kappa - \frac14 \varepsilon f_t^2)\ \{g_{ad}g_{bc} - g_{ac}g_{bd}\} \bmod O(|x|).
\end{array}$$

Since S is a symmetric space, there is an isometry of S fixing P which acts as -1 on $T_P S$. This extends to an isometry of S. Thus $R_{0abc} = 0$ for any a, b, and c. Let $a \neq b$. The isotropy group of isometries of S fixing P acts on $T_P M$ as $O(p, q)$. Thus we can find an isometry of S which sends $e_a \to e_a$ and $e_b \to -e_b$. This implies $R_{0ab0} = 0$. We compute the remaining curvature:

$$\begin{array}{rcl}
R_{0aa}{}^0 &\equiv& \partial_0 \Gamma_{aa}{}^0 - \Gamma_{ab}{}^0 \Gamma_{0a}{}^b \equiv -\frac12 \varepsilon f_{tt} \tilde g_{aa} + \frac14 \varepsilon f^{-1} f_t^2 \tilde g_{ab} \tilde g_a{}^b\\
R_{0aa0} &\equiv& f^{-2}\varepsilon\{-\frac12 f f_{tt} + \frac14 f_t^2\}\{g_{aa}g_{00} - g_{0a}g_{a0}\}.
\end{array}$$

We wish to find $C(t)$ and ϕ so that ${}^gR = CR_\phi$. If

$$(f\kappa - \tfrac14 \varepsilon f_t^2) = \varepsilon\{-\tfrac12 f f_{tt} + \tfrac14 f_t^2\},$$

then we may take $\phi = \mathrm{id}$; the resulting metric then has constant sectional curvature. Since this does not give rise to a new family of metrics, we define instead:

$$\phi(\partial_t) = \partial_t, \text{ and } \phi(\partial_a) = -\partial_a.$$

To ensure that $R = CR_\phi$, we solve the equation:

$$(f\kappa - \tfrac14 \varepsilon f_t^2) = -\varepsilon\{-\tfrac12 f f_{tt} + \tfrac14 f_t^2\} \text{ i.e. } \kappa = \tfrac12 \varepsilon f_{tt}.$$

This implies that the warping function is quadratic so $f(t)=\varepsilon\kappa t^2+At+B$. $\square$

4. The proof of Theorem 1.5

Throughout this section, we will let (M,g) be a connected pseudo-Riemannian manifold of signature (p,q) with $q\geq 5$ and dimension $m:=p+q$ which is rank 2 spacelike Jordan IP. We use Theorem 1.3 to express ${}^gR_P=C(P)R_{\phi(P)}$ where ϕ is self-adjoint and $C\neq 0$. We suppose for the moment that ${}^gR$ is never nilpotent so that we can normalize ϕ so $\phi^2=\pm\mathrm{id}$. The maps $P\to C(P)$ and $P\to\phi(P)$ can then be chosen to be smooth, at least locally. To have a unified notation, we complexify the tangent bundle and extend ϕ, $(\cdot,\cdot)$, and the curvature tensor ${}^gR$ to be complex multi-linear. Let:

$$\tilde\phi:=\left\{\begin{array}{rl}\phi & \text{if } \phi^2=\ \ \mathrm{id}\,,\\ \sqrt{-1}\phi & \text{if } \phi^2=-\mathrm{id}\,.\end{array}\right.$$

Since $\tilde\phi^2=\mathrm{id}$, the eigenvalues of $\tilde\phi$ are ±1 and $\tilde\phi$ is diagonalizable. Let

$$\mathcal{F}^\pm:=\{X\in TM\otimes\mathbb{C}:\tilde\phi X=\pm X\}\text{ and }\nu^\pm:=\dim\mathcal{F}^\pm.$$

If $\nu^-=0$ or $\nu^+=0$, then $\tilde\phi=\phi=\pm\mathrm{id}$ and (M,g) has constant sectional curvature and assertion (2) of Theorem 1.5 holds. Thus, we suppose $\nu^-\geq1$ and $\nu^+\geq1$. Let $u^\pm\in\mathcal{F}^\pm$. As $(u^+,u^-)=(\tilde\phi u^+,u^-)=(u^+,\tilde\phi u^-)=-(u^+,u^-)$, $\mathcal{F}^-$ and $\mathcal{F}^+$ are non-degenerate orthogonal complex distributions. As we are working over $\mathbb{C}$, we can find a local orthonormal frame $\mathcal{B}:=\{e_1,...,e_m\}$ for $TM\otimes\mathbb{C}$ so

$$\mathcal{F}^-=\operatorname{span}\{e_1,...,e_{\nu^-}\},\ \mathcal{F}^+=\operatorname{span}\{e_{\nu^-+1},...,e_m\},\text{ and }(e_i,e_j)=\delta_{ij}.$$

Let Roman indices a, b, etc. range from 1 to ν^- and index the frame for $\mathcal{F}^-$, let Greek indices α, β, etc. range from ν^-+1 to m and index the frame for $\mathcal{F}^+$, and let Roman indices i, j, etc. range from 1 to m and index the frame for $TM\otimes\mathbb{C}$. Let $\tilde\phi_{ij;k}$ be the components of $\nabla\tilde\phi$ relative to such a normalized basis.

Lemma 4.1. *Let (M,g) be a connected pseudo-Riemannian manifold of signature (p,q) with $q\geq5$ so that ${}^gR_P=C(P)R_{\phi(P)}$ where $\phi(P)^2=\pm\mathrm{id}$ and $C(P)\neq0$. Assume that $\nu^+\geq1$ and $\nu^-\geq1$.*

1. *We have $\tilde\phi_{ij;k}=\tilde\phi_{ji;k}$, $\tilde\phi_{ab;k}=0$, $\tilde\phi_{\alpha\beta;k}=0$, and $\tilde\phi_{a\alpha;i}=-2\Gamma_{ia\alpha}$.*
2. *If i, j, and k are distinct indices, then $\tilde\phi_{ij;k}=\tilde\phi_{ik;j}$.*
3. *The only non-zero components of $\nabla\tilde\phi$ are $\tilde\phi_{a\alpha;a}=\tilde\phi_{\alpha a;a}=-2\Gamma_{aa\alpha}$ and $\tilde\phi_{a\alpha;\alpha}=\tilde\phi_{\alpha a;\alpha}=-2\Gamma_{\alpha a\alpha}$.*

PROOF. We adapt arguments from [5] to prove this result. As the frame is orthonormal, $\Gamma_{kij} = -\Gamma_{kji}$. Let $\varepsilon_i := (\phi e_i, e_i) = \pm 1$; $\varepsilon_a = -1$ and $\varepsilon_\alpha = +1$. We prove assertion (1) by computing:

$$\begin{aligned}\tilde{\phi}_{ij;k} &= (\nabla_{e_k}\tilde{\phi}e_i - \tilde{\phi}\nabla_{e_k}e_i, e_j) = (\nabla_{e_k}\tilde{\phi}e_i, e_j) - (\nabla_{e_k}e_i, \tilde{\phi}e_j)\\ &= (\varepsilon_i - \varepsilon_j)(\nabla_{e_k}e_i, e_j) = (\varepsilon_i - \varepsilon_j)\Gamma_{kij}.\end{aligned}$$

Let ${}^gR_{ijkl;n}$ be the components of $\nabla^g R$. With our normalizations, we can express:

$$\begin{aligned}&{}^gR_{ijkl} = \varepsilon C\{\tilde{\phi}_{il}\tilde{\phi}_{jk} - \tilde{\phi}_{ik}\tilde{\phi}_{jl}\} \text{ where } \phi^2 = \varepsilon\mathrm{id} \text{ and,}\\ &{}^gR_{ijkl;n} = \varepsilon C(\tilde{\phi}_{il;n}\tilde{\phi}_{jk} + \tilde{\phi}_{il}\tilde{\phi}_{jk;n} - \tilde{\phi}_{ik;n}\tilde{\phi}_{jl} - \tilde{\phi}_{ik}\tilde{\phi}_{jl;n})\\ &\qquad\qquad + C_{;n}(\tilde{\phi}_{il}\tilde{\phi}_{jk} - \tilde{\phi}_{ik}\tilde{\phi}_{jl}).\end{aligned}$$

We use the *second Bianchi identity*:

$$0 = {}^gR_{ijkl;n} + {}^gR_{ijln;k} + {}^gR_{ijnk;l}.$$

Fix distinct indices i, j, and k. As $q \geq 5$, we may choose an index l which is distinct from i, j, and k. We have $\tilde{\phi}_{ij} = 0$ for $i \neq j$. Since $C \neq 0$, we may prove assertion (2) by computing:

$$0 = {}^gR_{illj;k} + {}^gR_{iljk;l} + {}^gR_{ilkl;j} = \varepsilon C\varepsilon_\ell\tilde{\phi}_{ij;k} + 0 - \varepsilon C\varepsilon_\ell\tilde{\phi}_{ik;j}.$$

If i, j, and k are distinct, then $\tilde{\phi}_{ij;k} = \tilde{\phi}_{ik;j} = \tilde{\phi}_{jk;i}$. Since at least two of the indices must index elements of $\mathcal{F}^-$ or $\mathcal{F}^+$, $\tilde{\phi}_{ij;k} = 0$ by assertion (1). Thus (i, j, k) is a permutation of (a, a, α) or (a, α, α). Assertion (3) follows as

$$\tilde{\phi}_{aa;\alpha} = \tilde{\phi}_{\alpha\alpha;a} = 0 \text{ and } \tilde{\phi}_{a\alpha;i} = \tilde{\phi}_{\alpha a;i}$$

. □

We continue our study of $\nabla\tilde{\phi}$.

Lemma 4.2. *Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) with $q \geq 5$ so that ${}^gR_P = C(P)R_{\phi(P)}$ where $\phi(P)^2 = \pm\mathrm{id}$ and $C(P) \neq 0$. Assume that $\nu^+ \geq 1$ and $\nu^- \geq 1$.*

1. *If $\nu^- \geq 2$, then $C_{;a} = C\tilde{\phi}_{\alpha a;\alpha}$ and $\Gamma_{\alpha a\alpha} = -\frac{1}{2}\frac{C_{;a}}{C}$.*
2. *If $\nu^- \geq 3$, then $C_{;a} = 0$, $C_{;\alpha} = -2C\tilde{\phi}_{a\alpha;a}$, and $\Gamma_{aa\alpha} = \frac{C_{;\alpha}}{C}$.*
3. *If $\nu^+ \geq 2$, then $C_{;\alpha} = -C\tilde{\phi}_{a\alpha;a}$ and $\Gamma_{aa\alpha} = \frac{1}{2}\frac{C_{;\alpha}}{C}$.*
4. *If $\nu^+ \geq 3$, then $C_{;\alpha} = 0$, $C_{;a} = +2C\tilde{\phi}_{a\alpha;\alpha}$, and $\Gamma_{\alpha a\alpha} = -\frac{C_{;a}}{C}$.*

Proof. If $\nu^- \geq 2$, then we may choose $a \neq b$ and use Lemma 4.1 and the second Bianchi identity to prove assertion (1) by computing:

$$0 = {}^gR_{a\alpha\alpha a;b} + {}^gR_{a\alpha ab;\alpha} + {}^gR_{a\alpha b\alpha;a} = -\varepsilon C_{;b} + \varepsilon C\tilde{\phi}_{\alpha b;\alpha} + 0.$$

If $\nu^- \geq 3$, then we may choose distinct indices a, b, and c and use Lemma 4.1 and the second Bianchi identity to compute

$$\begin{aligned} 0 &= {}^gR_{cbbc;a} + {}^gR_{cbca;b} + {}^gR_{cbab;c} = \varepsilon C_{;a} + 0 + 0 \text{ and} \\ 0 &= {}^gR_{cbbc;\alpha} + {}^gR_{cbc\alpha;b} + {}^gR_{cb\alpha b;c} = \varepsilon C_{;\alpha} + \varepsilon C\tilde{\phi}_{b\alpha;b} + \varepsilon C\tilde{\phi}_{c\alpha;c}. \end{aligned}$$

Thus $C_{;a} = 0$ and $C_{;\alpha} = -C(\phi_{b\alpha;b} + \phi_{c\alpha;c})$. Similarly $C_{;\alpha} = -C(\phi_{a\alpha;a} + \phi_{c\alpha;c})$. Thus $\phi_{b\alpha;b} = \phi_{a\alpha;a} = \phi_{c\alpha;c}$ and assertion (2) follows.

We use the same argument to prove assertions (3) and (4) with appropriate changes of sign. If $\nu^+ \geq 2$, then we may choose $\alpha \neq \beta$ to compute:

$$0 = {}^gR_{\alpha aa\alpha;\beta} + {}^gR_{\alpha a\alpha\beta;a} + {}^gR_{\alpha a\beta a;\alpha} = -\varepsilon C_{;\beta} - \varepsilon C\tilde{\phi}_{a\beta;a} + 0.$$

If $\nu^+ \geq 3$, then we may choose distinct indices α, β, and γ to compute

$$\begin{aligned} 0 &= {}^gR_{\gamma\beta\beta\beta\gamma;\alpha} + {}^gR_{\gamma\beta\gamma\alpha;\beta} + {}^gR_{\gamma\beta\alpha\beta;\gamma} = \varepsilon C_{;\alpha} + 0 + 0 \text{ and} \\ 0 &= {}^gR_{\gamma\beta\beta\gamma;a} + {}^gR_{\gamma\beta\gamma a;\beta} + {}^gR_{\gamma\beta a\beta;\gamma} = \varepsilon C_{;a} - \varepsilon C\tilde{\phi}_{\beta a;\beta} - \varepsilon C\tilde{\phi}_{\gamma a;\gamma c}. \end{aligned}$$

□

We can now show that $\phi^2 = \text{id}$ if $C \neq 0$.

Lemma 4.3. *Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) with $q \geq 5$ so that ${}^gR_P = C(P)R_{\phi(P)}$ where $\phi(P)^2 = \pm\text{id}$ and $C(P) \neq 0$. Assume that $\nu^+ \geq 1$ and $\nu^- \geq 1$.*

1. *We do not have $\nabla\tilde{\phi} = 0$. Furthermore either $\nu^- \leq 2$ or $\nu^+ \leq 2$.*
2. *We have $\phi^2 = \text{id}$. Furthermore either $\nu^- = 1$ or $\nu^+ = 1$.*

Proof. Assume that $\nabla\tilde{\phi} = 0$. We use Lemma 4.1 to see that $\Gamma_{ia\alpha} = -\frac{1}{2}\tilde{\phi}_{a\alpha;i} = 0$. Thus $\nabla e_a \in \mathcal{F}^-$ so ${}^gR(e_a, e_\alpha)e_a \in \mathcal{F}^-$. This shows:

$$0 =^g R(e_a, e_\alpha, e_a, e_\alpha) = \varepsilon C.$$

This shows that $C = 0$ which is false. Thus $\nabla\tilde{\phi} \neq 0$. Next suppose that $\nu^- \geq 3$ and $\nu^+ \geq 3$. We may then apply Lemma 4.2 to see that $dC = 0$ and $\nabla\tilde{\phi} = 0$ which is false. This establishes assertion (1).

Suppose that $\phi^2 = -\text{id}$. Then $(\phi v, \phi v) = (\phi^2 v, v) = -(v, v)$ so ϕ interchanges the roles of spacelike and timelike vectors. Thus $p = q \geq 5$. Since $\tilde{\phi} = \sqrt{-1}\phi$, $\tilde{\phi}$ is purely imaginary. Thus conjugation interchanges the distributions $\mathcal{F}^+$ and $\mathcal{F}^-$ so $\nu^+ = \nu^- = q \geq 5$. This contradicts assertion (1). Consequently $\phi^2 = \text{id}$.

By replacing ϕ by $-\phi$, we may assume $\nu^- \leq \nu^+$. To establish the final assertion, we must rule out the case $\nu^- = 2$. We may then use assertions (1), (3), and (4) of Lemma 4.2 to see $dC = \nabla\tilde{\phi} = 0$ which contradicts assertion (1). □

Assertion (1) of Theorem 1.5 follows from Lemma 4.3 and from:

Lemma 4.4. *Let (M, g) be a connected pseudo-Riemannian manifold of signature (p, q) for $q \geq 5$. Assume that (M, g) is rank 2 spacelike Jordan IP. Then either ${}^gR$ is nilpotent for all points of M or ${}^gR$ is nilpotent at no point of M.*

Proof. Let S_1 be the set of all points of M where ${}^gR$ is nilpotent and let S_2 be the complementary subset of all points of M where ${}^gR$ is not nilpotent. We assume that both S_1 and S_2 are non-empty and argue for a contradiction. Since S_1 is closed, since S_2 is open, and since M is connected, we may conclude that S_2 is not closed. Thus we may choose points $P_i \in S_2$ so that $P_i \to P_\infty \in S_1$. Let R_n be the curvature tensor at P_n and let V_n be the tangent space at P_n.

Let $\mathcal{B} := \{f_1^-, ..., f_p^-, f_1^+, ..., f_q^+\}$ be a normalized local orthonormal frame for the tangent bundle near P_∞ and let $V^\pm$ be the associated maximal spacelike and timelike distributions. We set $V_n^\pm := V^\pm(P_n)$. Since $P_n \in S_2$, we can express $R_n = C_n R_{\phi_n}$. Since $\mathrm{Tr}\{R_n(\pi)^2\} = -C_n^2$ for any spacelike 2 plane π, we have $C_n \to 0$ since R_∞ is nilpotent.

We apply Lemma 4.3 to see $\phi_n^2 = \mathrm{id}$ and thus we do not need to complexify to define the sub-bundles $\mathcal{F}_n^\pm$ of V_n. By replacing ϕ_n by $-\phi_n$ if necessary, we may apply Lemma 4.3 to see $\dim \mathcal{F}_n^- \leq 1$. Consequently $\dim \mathcal{F}^+ \geq p + q - 1$ so $\dim \mathcal{F}^+ \cap V_n^+ \geq q - 1 \geq 4$. Choose elements $v_n, w_n \in V_n^+ \cap \mathcal{F}_n^+$ so that $\{v_n, w_n\}$ forms an orthonormal spacelike set. We choose a compact neighborhood of P_∞ over which $S(V^+)$ is compact. By passing to a subsequence, we can suppose that $v_n \to v_\infty$ and $w_n \to w_\infty$. Let $\pi_n := \mathrm{span}\{v_n, w_n)$. Then we have that $\pi_n \to \pi_\infty := \mathrm{span}\{v_\infty, w_\infty\}$. The planes π_n and π_∞ are spacelike. Let $z_\infty \in V_\infty$. Choose elements $z_n \in V_n$ so $z_n \to z_\infty$. As $\phi_n v_n = v_n$ and $\phi_n w_n = w_n$, we have:

$$\begin{aligned} R(\pi_\infty) z_\infty &= \lim_{n\to\infty} R_n(v_n, w_n) z_n \\ &= \lim_{n\to\infty} C_n\{(\phi w_n, z_n)\phi v_n - (\phi v_n, z_n)\phi w_n\} \\ &= \lim_{n\to\infty} C_n\{(w_n, z_n) v_n - (v_n, z_n) w_n\} \\ &= \{\lim_{n\to\infty} C_n\} \cdot \{(w_\infty, z_\infty) v_\infty - (v_\infty, z_\infty) w_\infty\} = 0. \end{aligned}$$

This contradicts the assumption that R has spacelike rank 2. □

Since $\phi^2 = \mathrm{id}$ so the distributions $\mathcal{F}^\pm$ are real. By replacing ϕ by $-\phi$ if necessary, we may suppose that $\nu^- \leq \nu^+$ and thus $\nu^- \leq 1$. If $\nu^- = 0$, then

(M,g) has constant sectional curvature and assertion (2) of Lemma 1.5 holds. We therefore suppose $\nu^- = 1$. The distributions $\mathcal{F}^\pm$ are non-degenerate. Let $\{e_1\}$ be a local orthonormal section to $\mathcal{F}^-$ and let $\{e_2, ..., e_m\}$ be a local orthonormal frame for $\mathcal{F}^+$; since we are working over $\mathbb{R}$, we no longer impose the normalization that $(e_i, e_i) = +1$. By replacing g by $-g$, we may assume without loss of generality that e_1 is spacelike. If $\alpha \neq \beta$, then we may apply Lemma 4.1 to compute:

$$\begin{aligned}([e_\alpha, e_\beta], e_1) &= \Gamma_{\alpha\beta 1} - \Gamma_{\beta\alpha 1} = \Gamma_{\beta 1\alpha} - \Gamma_{\alpha 1\beta}\\ &= -\tfrac{1}{2}(\tilde{\phi}_{1\alpha;\beta} - \tilde{\phi}_{1\beta;\alpha}) = 0.\end{aligned}$$

Thus the foliation $\mathcal{F}^+$ is integrable. Let $y = (y^1, ..., y^{m-1})$ be local coordinates on a leaf of this foliation. We define geodesic tubular coordinates on M by setting:

$$T(t, y) := \exp{}_y(te_1(y)).$$

Assertion (3) of Theorem 1.5 will follow from:

Lemma 4.5. *Let (M,g) be a connected pseudo-Riemannian manifold of signature (p,q) for $q \geq 5$. Assume that ${}^gR = CR_\phi$ where $\phi^2 = \mathrm{id}$ and $C \neq 0$. Assume that $\nu^- = 1$ and that $\mathcal{F}^-$ is spacelike.*

1. *For fixed y_0, the curves $t \to T(t, y_0)$ are unit speed geodesics in (M,g) which are perpendicular to the leaves of the foliation $\mathcal{F}^+$.*
2. *For fixed t_0, the surfaces $T(t_0, y)$ are leaves of the foliation $\mathcal{F}^+$ and inherit metrics of constant sectional curvature.*
3. *Locally $ds^2 = dt^2 + f ds^2_\kappa$ where $f(t)$ is non-zero smooth function and ds^2_κ is a metric of constant sectional curvature κ.*
4. *The warping function $f(t) = \kappa t^2 + At + B$ where $A^2 - 4\kappa B \neq 0$.*

Proof. We choose a local orthonormal frame $\{e_i\}$ for TM so e_1 spans $\mathcal{F}^-$ and $\{e_2, ..., e_m\}$ spans $\mathcal{F}^+$. We set $\varepsilon_i := (e_i, e_i) = \pm 1$; by assumption $\varepsilon_1 = +1$. We have $\dim(\mathcal{F}^+) = m - 1 \geq 3$. Taking into account the fact that $(e_\alpha, e_\alpha) = \pm 1$, we may apply Lemmas 4.1 and 4.2 to see that

$$C_{;\alpha} = 0, \ \Gamma_{\alpha 1\beta} = -(e_\alpha, e_\beta)\tfrac{C_{;1}}{C}, \text{ and } \Gamma_{11\alpha} = \tfrac{1}{2}\tfrac{C_{;\alpha}}{C} = 0.$$

Let $\gamma(t, y_0)$ be an integral curve for e_1 starting at a point y_0 on the leaf of the foliation $\mathcal{F}^+$. Since e_1 is a unit vector, $\Gamma_{111} = 0$. As $\Gamma_{11\alpha} = 0$, γ is a geodesic. Thus $\gamma(t, y_0) = T(t, y_0)$ so $\partial_t = e_1$. We compute

$$\partial_t(\partial_t, \partial^y_\alpha) = (\partial_t, \nabla_{\partial_t}\partial^y_\alpha) = (\partial_t, \nabla_{\partial^y_\alpha}\partial_t) = \tfrac{1}{2}\partial^y_\alpha(\partial_t, \partial_t) = 0.$$

This shows $(\partial_t, \partial^y_\alpha) = 0$ so the ∂^y_α span the perpendicular distribution $\mathcal{F}^+$ and the manifolds $T(t_0, y)$ are leaves of the foliation $\mathcal{F}^+$. Since $C_{;\alpha} = 0$, C is constant on the leaves of $\mathcal{F}^+$. Let $\tilde{R}$ be the curvature of the induced metric on the leaves of the foliation $\mathcal{F}^+$. We show that $\tilde{R}$ has constant sectional curvature by computing:

$$\begin{aligned}\tilde{R}(e_\alpha, e_\beta, e_\gamma, e_\sigma) &= R(e_\alpha, e_\beta, e_\gamma, e_\sigma) + \Gamma_{\alpha 1 \sigma}\Gamma_{\beta\gamma}{}^1 - \Gamma_{\beta 1 \sigma}\Gamma_{\alpha\gamma}{}^1 \\ &= \{C(t) + (\tfrac{C_{;1}}{C})^2\}\{(e_\beta, e_\gamma)(e_\alpha, e_\sigma) - (e_\alpha, e_\gamma)(e_\beta, e_\sigma)\}.\end{aligned}$$

Let $\partial_\alpha^y = \Sigma_\gamma a_{\alpha\gamma} e_\gamma$. We show that the metric is a warped product by computing:

$$\begin{aligned}(\nabla_{\partial_t}\partial_\alpha^y, \partial_\beta^y) &= (\nabla_{\partial_\alpha^y}\partial_t, \partial_\beta^y) = \Sigma_{\gamma\sigma} a_{\alpha\gamma} a_{\beta\sigma}(\nabla_{e_\gamma}\partial_t, e_\sigma) \\ &= \Sigma_{\gamma\sigma} a_{\alpha\gamma} a_{\beta\sigma}(e_\gamma, e_\sigma)\tfrac{C_{;1}}{C} = \tfrac{C_{;1}}{C} g_{\alpha\beta} \text{ so} \\ \partial_t g_{\alpha\beta} &= (\nabla_{\partial_t}\partial_\alpha^y, \partial_\beta^y) + (\partial_\alpha^y, \nabla_{\partial_t}\partial_\beta^y) = \tfrac{2C_{;1}}{C} g_{\alpha\beta}.\end{aligned}$$

The argument used to prove Lemma 3.3 now shows the warping function is quadratic. □

5. Nilpotent spacelike Jordan IP pseudo-Riemannian manifolds

The rank 2 spacelike Jordan IP pseudo-Riemannian manifolds whose curvature operators are not nilpotent for at least one point of M are classified in Theorem 1.5. In this section, we study the remaining case and present some preliminary results. We focus our attention on the balanced setting $p = q$.

Lemma 5.1. *Let (M, g) be a connected pseudoRiemannian manifold of signature (p, p) for $p \geq 5$. Assume that (M, g) is spacelike rank 2 nilpotent Jordan IP.*

(1) *We have ${}^gR = \pm R_\phi$ where ϕ is self-adjoint and where* $\ker\phi = \operatorname{range}\phi$.
(2) $\operatorname{range}\phi$ *is an integrable distribution of the tangent bundle of M.*

Proof. We apply Theorem 1.3 to write $R = \pm R_\phi$ where we normalize ϕ by requiring that $C = \pm 1$. The map $P \to \phi$ can then be chosen to vary smoothly with P, at least locally. Since ϕ contains no spacelike vectors, $\dim\{\ker(\phi)\} \leq p$. Since ϕ is self-adjoint and $\phi^2 = 0$, we show $\operatorname{range}(\phi) = \ker(\phi)$ and complete the proof of the first assertion by computing:

$$\begin{aligned}&\operatorname{range}(\phi) \subset \ker(\phi), \\ &\dim\{\operatorname{range}(\phi)\} \leq \dim\{\ker(\phi)\}, \text{ and} \\ &2p = \dim\{\operatorname{range}(\phi)\} + \dim\{\ker(\phi)\} \leq 2\dim\{\ker(\phi)\} \leq 2p.\end{aligned}$$

Let $\mathcal{K} := \operatorname{range}\phi$ and let $\mathcal{S}$ be a maximal local spacelike distribution. Since $\mathcal{K}$ is totally isotropic, $\mathcal{K} \cap \mathcal{S} = \{0\}$. Thus $\mathcal{K} = \operatorname{range}\phi = \phi\mathcal{S}$ and $TM = \mathcal{S} \oplus \phi\mathcal{S}$. Let $L(\cdot, \cdot) := (\phi\cdot, \cdot)$ be the associated bilinear form. We have:

$$(\phi s, \phi\tilde{s}) = (s, \phi^2\tilde{s}) = 0.$$

Thus if $0 \neq s \in \mathcal{S}$, then there must exist $\tilde{s} \in \mathcal{S}$ so that $(\phi s, \tilde{s}) \neq 0$. Thus L is a non-degenerate bilinear form. We complexify and choose a frame $\{s_a\}$ for $\mathcal{S}$ so:

$$(\phi e_a, e_b) = \delta_{ab}.$$

Let Roman indices a, b, etc. range from 1 to p and index this frame for $\mathcal{S}$. Let Greek indices α, β, etc. range from $p+1$ to $2p$ and index the frame $e_\alpha := \phi(e_{\alpha-p})$ for $\mathcal{K}$. Let Roman indices i, j, etc. range from 1 to $2p$ and index the frame $\{e_1, ..., e_p, \phi e_1, ..., \phi e_p\}$. We have $\phi_{ab} = \delta_{ab}$, $\phi_{a\beta} = 0$, and $\phi_{\alpha\beta} = 0$. Let α, β, and i be indices which need not be distinct. Choose $a \neq i$ and use the second Bianchi identity to compute:

$$\begin{aligned} R_{ijkl;n} &= \phi_{il;n}\phi_{jk} + \phi_{il}\phi_{jk;n} - \phi_{ik;n}\phi_{jl} - \phi_{ik}\phi_{jl;n}, \\ 0 &= R_{a\alpha\beta a;i} + R_{a\alpha i\beta;a} + R_{a\alpha ai;\beta} = \phi_{\alpha\beta;i} + 0 - \phi_{\alpha i;\beta}, \text{ and} \\ \phi_{\alpha i;\beta} &= \phi_{\alpha\beta;i} = (\nabla_{e_i}\phi(\phi e_a), \phi e_b) - (\phi\nabla_{e_i}\phi e_a, \phi e_b) = 0. \end{aligned}$$

We clear the previous notation. Let $\alpha = a+p$, $\beta = b+p$, and $\gamma = c+p$ be indices which are not necessarily distinct. We show that $\mathcal{K}$ is an integrable distribution by showing that:

$$\begin{aligned} &([e_\alpha, e_\beta], e_\gamma) = (\nabla_{e_\alpha}\phi e_b, \phi e_c) - (\nabla_{e_\beta}\phi e_a, \phi e_c) \\ = \;& ((\nabla_{e_\alpha}\phi - \phi\nabla_{e_\alpha})e_b, \phi e_c) - ((\nabla_{e_\beta}\phi - \phi\nabla_{e_\alpha})e_a, \phi e_c) \\ = \;& \phi_{b\gamma;\alpha} - \phi_{a\gamma;\beta} = 0 \text{ so } [e_\alpha, e_\beta] \in C^\infty(\mathcal{K}^\perp) = C^\infty(\mathcal{K}). \end{aligned}$$

□

We now construct a nilpotent rank 2 Jordan IP pseudo-Riemannian manifold.

Lemma 5.2. *Let $\{e_1, ..., e_p, \tilde{e}_1, ..., \tilde{e}_p\}$ be a basis for a vector space V and let $\{x^1, ..., x^p, \tilde{x}^1, ..., \tilde{x}^p\}$ be the corresponding dual basis for V^*. Define an inner product of signature (p,p) on V by $(e_i, \tilde{e}_j) = \delta_{ij}$ and $(e_i, e_j) = (\tilde{e}_i, \tilde{e}_j) = 0$ for all i, j. Give $W := V \oplus \mathbb{R}$ the direct sum inner product which has signature $(p, p+1)$. Let $f(\tilde{x}_1, ..., \tilde{x}_p)$ be a real valued function so that $df(0) = 0$ and so that $\det(\partial_i^{\tilde{x}}\partial_j^{\tilde{x}} f)(0) \neq 0$. The embedding $F := \text{id} \oplus f$ of V into W defines a hypersurface (M, g) of W so that (M, g) is a spacelike and timelike rank 2 Jordan IP nilpotent pseudo-Riemannian manifold of signature (p,p) close to the origin.*

Proof. Since $df(0) = 0$, the natural identification of T_0M with V is an isometry. Thus the metric is non-degenerate and has signature (p,p) sufficiently close to the origin. Let L_P and S_P be the associated second fundamental form and shape operators at P. Then Lemma 3.1 shows that ${}^gR = R_{S_P}$. Since $\partial_i^x F = 0$, $L_P(\partial_i^x, *) = 0$ for any point of the manifold. Thus $S_P(\partial_i^x) = 0$ so

$$\operatorname{span}\{\partial_i^x\} \subset \ker(S_P).$$

Since the Hessian of f is non-singular at 0, S_0 is invertible on $\operatorname{span}\{\partial_i^{\tilde{x}}\}$. Thus $\dim\ker(S_0) = p$ and hence by shrinking the neighborhood $\mathcal{O}$ if necessary we may suppose that $\dim\ker(S_P) \le p$ for $P \in \mathcal{O}$. It now follows that

$$\ker(S_P) = \operatorname{span}\{\partial_i^x\} \text{ for all } P \in \mathcal{O}.$$

Since $F_*(\partial_i^x) = e_i$, we have $g(\partial_i^x, \partial_j^x) = 0$ for all i, j and hence $\ker(S_P)$ is totally isotropic and in particular $\ker(S_P)$ contains no spacelike or timelike vectors. Since $\dim\{\ker(S_P)\} = p$, $\ker(S_P) = \ker(S_P)^\perp = \operatorname{range}(S_P)$ so $S_P^2 = 0$. Thus ${}^gR$ is nilpotent. $\square$

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