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CONSTRUCTING LOW DEGREE HYPERBOLIC SURFACES IN $\mathbb{P}^3$

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Abstract. We describe a new method of constructing Kobayashi-hyperbolic surfaces in complex projective 3-space based on deforming surfaces with a "hyperbolic non-percolation" property. We use this method to show that general small deformations of certain singular abelian surfaces of degree 8 are hyperbolic. We also show that a union of 15 planes in general position in projective 3-space admits hyperbolic deformations.

1. Introduction

A compact complex manifold is hyperbolic in the sense of Kobayashi if every holomorphic map from the complex line $\mathbb{C}$ to the manifold is constant, as is the case for compact complex curves of genus ≥ 2. In 1970, S. Kobayashi conjectured that generic hypersurfaces in $\mathbb{P}^n$ of sufficiently high degree are hyperbolic. Some progress has been made towards this conjecture. Demailly and El Goul [DE], and independently McQuillan [Mc] (with a slightly bigger degree estimate) proved that a very generic surface of degree at least 21 in $\mathbb{P}^3$ is hyperbolic in the sense of Kobayashi. Previously, Clemens [Cl] showed that very generic hypersurfaces of degree $d \geq 2n-1$ contain no rational curves, which is a necessary condition for hyperbolicity. Actually, for $n \geq 3$, this also holds for $d = 2n-2$, and very generic hypersurfaces of degree $d \geq 2n-1$ contain neither rational nor elliptic curves (Voisin [Vo]; see also [CR, CLR, Ei, Pa, Xu1, Xu2]). Thus it is natural to suppose that $2n-1$ is the minimal degree for Kobayashi's conjecture.

Many examples have been given of low degree hyperbolic projective hypersurfaces (e.g., [SZ1, SZ2] and the references therein). The examples of hyperbolic

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surfaces in $\mathbb{P}^3$ of lowest degree found to date are of degree 8 and were discovered independently by Duval [Du] and Fujimoto [Fu]. (A similar example of degree 10 was previously found by Shirosaki [Sh].)

In this paper, we provide a new approach to constructing hyperbolic surfaces in $\mathbb{P}^3$, giving another example of degree 8. Although this example is considerably more complicated than the Duval-Fujimoto example, we hope that our technique can be applied in the future to construct examples of lower degree. We also show that certain small deformations of 15 planes in general position in $\mathbb{P}^3$ are hyperbolic surfaces.

Our technique involves showing that small deformations X_t of certain singular surfaces $X_0 \subset \mathbb{P}^3$ are hyperbolic. The surfaces X_0 that we deform, while not hyperbolic, satisfy a "hyperbolic non-percolation" property. In particular, we consider surfaces X_0 with "double curve" $\bar{S}$; i.e., the singular locus of X_0 is a 1-dimensional subvariety $\bar{S}$, and X_0 has two branches at general points of $\bar{S}$. If nearby surfaces in a linear pencil $\{X_t\}$ were not hyperbolic, then we can find a sequence $t_n \to 0$ and a sequence of Brody curves $f_{t_n} : \mathbb{C} \to X_{t_n}$ converging to a Brody curve $f : \mathbb{C} \to X_0$. Recall that a *Brody curve* in a Hermitian complex manifold M is a non-constant entire holomorphic curve $g : \mathbb{C} \to M$ such that $\|g'(\zeta)\|$ is bounded above by $\|g'(0)\|$. Brody [Br] proved that a compact complex manifold is hyperbolic iff it does not contain any Brody curves.

Our approach is to show using Hurwitz's theorem that either

- $f(\mathbb{C}) \subset \bar{S} \setminus \{p_j\}$, where the p_j are the multiple points of $\bar{S}$, or
- $f(\mathbb{C}) \subset (X_0 \setminus \bar{S}) \cup D$, where D is a finite subset of $\bar{S}$.

We say that $X_0 \setminus \bar{S}$ *has the property of hyperbolic non-percolation through* D if there are no Brody curves $g : \mathbb{C} \to (X_0 \setminus \bar{S}) \cup D$. Hence, if in addition,

- $\bar{S} \setminus \{p_j\}$ is hyperbolic, and
- $X_0 \setminus \bar{S}$ has the property of hyperbolic non-percolation through D,

then small deformations X_t are hyperbolic. We illustrate our construction with two examples.

2. Deformation of 15 planes

In 1989, the second author [Za] showed that the complements of certain smooth, irreducible small deformations of 5 lines in $\mathbb{P}^2$ are complete hyperbolic and hyperbolically embedded. We begin by using our technique to give, as a parallel example, hyperbolic deformations of 15 planes in general position in $\mathbb{P}^3$.

Let $L_j,\ j=1,\dots,15$, be linear functions on $\mathbb{C}^4$ defining hyperplanes

$$H_j := \{z \in \mathbb{P}^3 : L_j(z) = 0\}$$

in general position; i.e., any 4 of the L_j are linearly independent, or equivalently, every point of $\mathbb{P}^3$ is contained in at most 3 of the H_j. Let $D = \{z \in \mathbb{P}^3 : Q(z) = 0\}$ be a general quintic, and consider the linear pencil of surfaces:

$$G_t = \left\{\prod_{j=1}^{15} L_j + tQ^3 = 0\right\} \subset \mathbb{P}^3 \ .$$

Theorem 2.1. *The surface G_t is hyperbolic for sufficiently small $t \neq 0$.*

Proof. Suppose on the contrary that there exist $t_n \to 0$ such that G_{t_n} is not hyperbolic. Then we can find a sequence of Brody curves $f_n : \mathbb{C} \to G_{t_n}$ with $\sup \|f_n'\| = \|f_n'(0)\| = 1$, where the norm is computed with respect to the Fubini-Study metric on $\mathbb{P}^3$. Then we can choose a subsequence, which we also denote by $\{f_n\}$, converging to a Brody curve $f : \mathbb{C} \to \bigcup_{j=1}^{15} H_j$. Assume without loss of generality that $f(\mathbb{C}) \subset H_{15}$.

We first show that

$$f(\mathbb{C}) \subset \left(H_{15} \setminus \bigcup_{j=1}^{14} H_j\right) \cup D \ . \tag{1}$$

To verify (1), suppose that D does not pass through any of the points $H_i \cap H_j \cap H_k$ $(i,j,k$ distinct), and that on the contrary $f(\zeta_0) \in H_k \setminus D$, where $k \neq 15$. Let Δ be a small disk about ζ_0 so that $f(\bar{\Delta})$ does not intersect D; i.e., $Q \circ f(\zeta) \neq 0$ for $\zeta \in \bar{\Delta}$. Hence

$$\prod_j L_j \circ f_n(\zeta) = -t_n Q^3 \circ f_n(\zeta) \neq 0, \qquad \zeta \in \Delta,\ n \gg 0 \ .$$

Since $L_k \circ f(\zeta_0) = 0$, it follows from Hurwitz's Theorem that $L_k \circ f \equiv 0$; i.e., $f(\mathbb{C}) \subset H_k \cap H_{15}$. If $j \notin \{k, 15\}$ and $f(\mathbb{C})$ passes through the point $p_{jk} := H_j \cap H_k \cap H_{15} \notin D$, then by the above argument (replacing k with j), we again conclude from Hurwitz's Theorem that $f(\mathbb{C}) \subset \{p_{jk}\}$, contradicting the fact that f is non-constant. Hence

$$f(\mathbb{C}) \subset H_k \cap H_{15} \setminus \{p_{jk} : 1 \leq j \leq 14, j \neq k\} \approx \mathbb{P}^1 \setminus \{13 \text{ points}\} \ ,$$

which implies that f is constant, a contradiction. Therefore (1) holds.

We assume that the curves $D \cap H_j$ are smooth (or have at most 4 double points), so that the degree 5 curve $D \cap H_{15}$ is hyperbolic and hence $f(\mathbb{C}) \not\subset D$.

Then by the Cartan Second Main Theorem [Ca] (see also [Ko, §3.B]) applied to the map $f : \mathbb{C} \to H_{15} \approx \mathbb{P}^2$ and the 14 lines $H_j \cap H_{15}$, we have

$$(14-3)T_f(r) \leq \sum_{j=1}^{14} N_2(H_j, r) + O(\log T_f(r)) \ . \tag{2}$$

(Note that since Brody curves are of finite order ≤ 2, the inequality holds without exceptional intervals.) As we have assumed that D does not pass through any of the points $H_i \cap H_j \cap H_k$, (1) implies that

$$\sum_{j=1}^{14} N_2(H_j, r) \leq 2 \sum_{j=1}^{14} N_1(H_j, r) \leq 2N(D, r) \ . \tag{3}$$

Furthermore, by the Carlson-Griffiths First Main Theorem [CG] (see also [NO, §5.2], [Ko, §8.4]) applied to the divisor D, we have

$$N(D, r) \leq 5T_f(r) + O(1) \ . \tag{4}$$

Combining the inequalities (2)–(4), we arrive at a contradiction.

□

Remark. In the second part of the proof of Theorem 2.1, we showed that the complement of 14 general lines in $\mathbb{P}^2$ has the property of hyperbolic non-percolation through (the intersection of these lines with) a general quintic. This should also hold for fewer lines; e.g., an open problem is whether the complement of 5 general lines in $\mathbb{P}^2$ has the property of hyperbolic non-percolation through a general sextic curve. This would imply that a general small deformation of 6 planes in general position in $\mathbb{P}^3$ is a hyperbolic sextic surface.

3. Deformation of a singular abelian surface

We now use our hyperbolic non-percolation technique to construct a new example of a hyperbolic surface of degree 8. This time, instead of deforming a reducible surface, we deform an irreducible surface X_0 with self-intersections.

The surface X_0 is described in [LB] and is defined as follows. Let A be a simple abelian surface with an ample line bundle $L \to A$ of type $(1, 4)$. Recall that an abelian variety is said to be simple if it does not contain any proper abelian subvariety. See [LB, p. 47] for the definition of line bundles of type (d_1, d_2) on an abelian surface.

It follows from the Riemann-Roch Theorem and the Kodaira vanishing theorem that $h^0(A, L) = \chi(L) = \frac{1}{2} L \cdot L = 4$ and hence $\deg L := L \cdot L = 8$ (see [LB, p. 289]).

We let $X_0 = \varphi_L(A)$, where $\varphi_L : A \to \mathbb{P}^3$ is the rational map defined by the linear system $|L|$. We shall establish the following result:

Theorem 3.1. *General small deformations of the surface $X_0 \subset \mathbb{P}^3$ are hyperbolic surfaces of degree* 8.

We begin with the description of X_0. By [LB, pp. 308–312], the surface X_0 is birational to A and is given by

$$X_0 = \{z \in \mathbb{P}^3 : Q = 0\}\ ,$$

where

$$\begin{aligned} Q \ = \ & \lambda_1{}^2\left(z_0{}^4z_1{}^4 + z_2{}^4z_3{}^4\right) + \lambda_2{}^2\left(z_0{}^4z_2{}^4 + z_1{}^4z_3{}^4\right) + \lambda_3{}^2\left(z_0{}^4z_3{}^4 + z_1{}^4z_2{}^4\right) \\ & +2\,\lambda_1\lambda_2\left(z_0{}^2z_1{}^2 + z_2{}^2z_3{}^2\right)\left(-z_0{}^2z_2{}^2 + z_1{}^2z_3{}^2\right) \\ & +2\,\lambda_1\lambda_3\left(z_0{}^2z_1{}^2 - z_2{}^2z_3{}^2\right)\left(z_0{}^2z_3{}^2 - z_1{}^2z_2{}^2\right) \\ & +2\,\lambda_2\lambda_3\left(z_0{}^2z_2{}^2 + z_1{}^2z_3{}^2\right)\left(z_0{}^2z_3{}^2 + z_1{}^2z_2{}^2\right) + \lambda_0{}^2z_0{}^2z_1{}^2z_2{}^2z_3{}^2\ . \end{aligned}$$

In fact, general choices of $(\lambda_0, \ldots, \lambda_3) \in \mathbb{C}^4$ give singular abelian surfaces (see Remark 3.3 in [LB, p. 301]).

Let H_j denotes the coordinate plane $\{z_j = 0\}$, and let

$$p_j := (\delta_0^j : \delta_1^j : \delta_2^j : \delta_3^j) \in X_0 \qquad \text{(so that } \ p_0 = (1:0:0:0),\ \text{etc.)}$$

denote the vertices of the coordinate tetrahedron $\{H_0, H_1, H_2, H_3\}$ $(0 \le j \le 3)$. The singular locus of X_0 consists of 4 double curves $\bar{S}_j := X_0 \cap H_j$, $j = 0, 1, 2, 3$. The equation for, say, $\bar{S}_3 \subseteq H_3$ is

$$-\lambda_1 z_0{}^2z_1{}^2 + \lambda_2 z_0{}^2z_2{}^2 + \lambda_3 z_1{}^2z_2{}^2 = 0\,.$$

It is known [LB, p. 312] that $\bar{S}_3$ is an irreducible rational curve with 3 ordinary double points at $\{p_0, p_1, p_2\}$. Generic points of $\bar{S}_3$ are ordinary double points of the surface X_0; i.e., X_0 is the union of two transversal smooth surface germs at generic points of $\bar{S}_3$. The set of points of $\bar{S}_3$ which are not ordinary double points of X_0 consists of the 3 double points $\{p_0, p_1, p_2\}$ of $\bar{S}_3$ together with 12 smooth points of $\bar{S}_3$, which are pinch points of X_0 (see [LB, p. 312]). The same description applies to the other double curves $\bar{S}_0, \bar{S}_1, \bar{S}_2$.

We need to know the structure of X_0 at the 4 vertices $\{p_j\}$. Let $\bar{S} = \bigcup_{j=0}^3 \bar{S}_j$. The singular set of $\bar{S}$ consists of the 4 points $\{p_j\}$. We shall show that each p_j is an ordinary 6-fold singularity of $\bar{S}$; i.e., the germ of $\bar{S}$ at p_j consists of 6 smooth local curves $\{B_j^i\}_{1 \le i \le 6}$ with distinct tangents $(j = 0, 1, 2, 3)$. For example, $\bar{S}_1, \bar{S}_2, \bar{S}_3$

pass through p_0, each contributing 2 local components of the germ of $\bar{S}$ at p_0. To describe the tangents to the B_0^i, we write

$$\mu_1 = \sqrt{\lambda_1}\,, \qquad \mu_2 = \sqrt{\lambda_2}\,, \qquad \mu_3 = \sqrt{-\lambda_3}\,,$$

and we use the affine coordinates

$$x = z_1/z_0,\ y = z_2/z_0,\ z = z_3/z_0$$

about $p_0 = (0,0,0)$. Then the 6 tangent lines are

$$\ell_1^{\pm} = \{x = 0\,,\ \mu_2 y = \pm\mu_3 z\},\ \ \ell_2^{\pm} = \{y = 0\,,\ \mu_1 x = \pm\mu_3 z\},\ \ \ell_3^{\pm} = \{z = 0\,,\ \mu_1 x = \pm\mu_2 y\}\,,$$

where the 2 lines ℓ_j^+, ℓ_j^- are tangent to $\bar{S}_j$ at p_0 ($j = 1,2,3$).

Lemma 3.2. *Let $0 \le j \le 3$. The germ of X_0 at p_j consists of 4 smooth surface germs $Y_j^1, \dots, Y_j^4$. Each of these surfaces contains exactly 3 of the 6 components $B_j^1, \dots B_j^6$ of the germ of $\bar{S}$ at p_j, and each of these components B_j^k is contained in exactly 2 of the Y_j^i. The intersection of any 3 of the Y_j^i is the germ of the point p_j.*

Proof. Clearly, the orbit of p_0 under $\mathrm{Aut}(X_0)$ consists of the 4 vertices $\{p_j\}$. Thus it suffices to consider $j = 0$. As before, fix the affine coordinates

$$x = z_1/z_0,\ y = z_2/z_0,\ z = z_3/z_0$$

about p_0. Then

$$\begin{aligned} Q &= {\lambda_1}^2\left(x^4 + y^4z^4\right) + {\lambda_2}^2\left(y^4 + x^4z^4\right) + {\lambda_3}^2\left(z^4 + x^4y^4\right) \\ &\quad + 2\,\lambda_1\lambda_2\left(x^2 + y^2z^2\right)\left(-y^2 + x^2z^2\right) + 2\,\lambda_1\lambda_3\left(x^2 - y^2z^2\right)\left(z^2 - x^2y^2\right) \\ &\quad + 2\,\lambda_2\lambda_3\left(y^2 + x^2z^2\right)\left(z^2 + x^2y^2\right) + {\lambda_0}^2x^2y^2z^2 \\ &= az^4 + bz^2 + c\,, \end{aligned}$$

where

$$\begin{aligned} a &= {\lambda_2}^2x^4 + 2\,\lambda_2\lambda_1x^2y^2 + {\lambda_1}^2y^4 + 2\,\lambda_3\lambda_2x^2 - 2\,\lambda_3\lambda_1y^2 + {\lambda_3}^2\,, \\ b &= 2\,\lambda_3\lambda_2x^4y^2 + 2\,\lambda_3\lambda_1x^2y^4 + 2\,\lambda_2\lambda_1x^4 - 2\,\lambda_2\lambda_1y^4 + {\lambda_0}^2x^2y^2 + 2\,\lambda_3\lambda_2y^2 + 2\lambda_3\lambda_1x^2, \\ c &= {\lambda_3}^2x^4y^4 + 2\,\lambda_3\lambda_2x^2y^4 - 2\,\lambda_3\lambda_1x^4y^2 + {\lambda_2}^2y^4 - 2\,\lambda_2\lambda_1x^2y^2 + {\lambda_1}^2x^4 \\ &= \left(\lambda_3x^2y^2 - \lambda_1x^2 + \lambda_2y^2\right)^2\,. \end{aligned}$$

We compute the discriminant

$$\delta := b^2 - 4ac = x^2y^2\left(16\lambda_1\lambda_2{\lambda_3}^2 + \cdots\right)\,.$$

Hence in the local ring ${}_3\mathcal{O}_0 = \mathbb{C}[[x,y,z]]$, we have the factorization $Q = Q^+Q^-$, where

$$\begin{aligned} Q^{\pm} &= \sqrt{a}\, z^2 + \frac{b \pm \sqrt{\delta}}{2\sqrt{a}} &= \lambda_3 z^2 + \lambda_1 x^2 + \lambda_2 y^2 \pm 2\sqrt{\lambda_1\lambda_2}\, xy + R_3(Q^{\pm}) \\ & &= (\mu_1 x \pm \mu_2 y)^2 - (\mu_3 z)^2 + R_3(Q^{\pm})\,. \end{aligned}$$

Here, for any $f \in {}_3\mathcal{O}_0$, we let $R_n(f) = \sum_{k=n}^{\infty} f_k$, where f_k denotes the homogeneous terms of order k in the Taylor expansion of f.

It follows that the tangent cone of X_0 at p_0 is the union of the 4 planes (in general position):

$$\begin{aligned} P^{++} &= \{\mu_1 x + \mu_2 y + \mu_3 z = 0\}\,, \\ P^{+-} &= \{\mu_1 x + \mu_2 y - \mu_3 z = 0\}\,, \\ P^{-+} &= \{\mu_1 x - \mu_2 y + \mu_3 z = 0\}\,, \\ P^{--} &= \{\mu_1 x - \mu_2 y - \mu_3 z = 0\}\,. \end{aligned}$$

The planes P^{++} and P^{+-} are tangent to $\{Q^+ = 0\}$, while P^{-+} and P^{--} are tangent to $\{Q^- = 0\}$.

To show that Q^+ and Q^- are reducible in ${}_3\mathcal{O}_0$, we now expand $Q = a'x^4 + b'x^2 + c'$, where

$$\begin{aligned} a' &= {\lambda_3}^2 y^4 + 2\,\lambda_3\lambda_2 z^2 y^2 + {\lambda_2}^2 z^4 - 2\,\lambda_3\lambda_1 y^2 + 2\,\lambda_2\lambda_1 z^2 + {\lambda_1}^2\,, \\ b' &= 2\,\lambda_3\lambda_1 z^2 y^4 + 2\,\lambda_2\lambda_1 z^4 y^2 + 2\,\lambda_3\lambda_2 z^4 + 2\,\lambda_3\lambda_2 y^4 + {\lambda_0}^2 z^2 y^2 + 2\lambda_3\lambda_1 z^2 - 2\lambda_2\lambda_1 y^2, \\ c' &= {\lambda_1}^2 z^4 y^4 - 2\,\lambda_3\lambda_1 z^4 y^2 - 2\,\lambda_2\lambda_1 z^2 y^4 + {\lambda_3}^2 z^4 + 2\,\lambda_3\lambda_2 z^2 y^2 + {\lambda_2}^2 y^4 \\ &= \left(-\lambda_1 z^2 y^2 + \lambda_3 z^2 + \lambda_2 y^2\right)^2\,. \end{aligned}$$

Again we compute the discriminant

$$\delta' = b'^2 - 4a'c' = y^2 z^2 \left(-16{\lambda_1}^2\lambda_2\lambda_3 + \cdots\right)\,.$$

Hence we have the factorization $Q = Q'^+ Q'^-$, where

$$Q'^{\pm} = \sqrt{a'}\, x^2 + \frac{b' \pm \sqrt{\delta'}}{2\sqrt{a'}} = (\mu_1 x)^2 - (\mu_2 y \pm \mu_3 z)^2 + R_3(Q'^{\pm})\,.$$

This time the planes P^{++} and P^{--} are tangent to $\{Q'^+ = 0\}$, while P^{+-} and P^{-+} are tangent to $\{Q'^- = 0\}$.

Since ${}_3\mathcal{O}_0$ is a unique factorization domain, Q^+ and Q'^+ must have a common factor

$$Q^{++} = \mu_1 x + \mu_2 y + \mu_3 z + R_2(Q^{++})$$

with zero set tangent to the plane P^{++}. By considering all such possible pairs, we see that the germ of X_0 at p_0 consists of 4 smooth surfaces, each tangent to one of the planes $P^{++}, P^{+-}, P^{-+}, P^{--}$. One easily checks that each of the lines $\ell_j^{\pm}$ is the intersection of exactly 2 of these planes, and each plane contains exactly 3 of the lines. The conclusion of the lemma immediately follows. $\square$

We recall that a simple complex torus does not contain rational or elliptic curves. Moreover, the following holds.

Proposition 3.3. *Let $f : \mathbb{C} \to T$ be a Brody curve in a simple complex 2-dimensional torus $T = \mathbb{C}^2/\Lambda$. Then for any compact complex curve S in T, the intersection $f(\mathbb{C}) \cap S$ is infinite.*

Proof. Assume without loss of generality that S is irreducible. The lift $\tilde{f} : \mathbb{C} \to \mathbb{C}^2$ of f is also a Brody curve and hence is given by degree 1 polynomials, as observed in [Gr]. By a translation of coordinates, we may suppose that $\tilde{f}$ is linear and hence $f(\mathbb{C})$ is a subgroup of T. We first note that S is not contained in any translate of $f(\mathbb{C})$. For if on the contrary $S' := S + v \subset f(\mathbb{C})$, then $f^{-1}(S')$, being analytic and of positive dimension, must be all of $\mathbb{C}$; i.e., $S' = f(\mathbb{C})$. Hence S', being a compact complex curve and a subgroup of T, must be a complex 1-dimensional subtorus of T, contradicting the assumption that T is simple.

Let Y denote the closure of $f(\mathbb{C})$ in the metric topology on T. As T is a simple Lie group and $Y \subseteq T$ is a closed subgroup, we conclude that either $Y = T$ or Y is a real 3-subtorus of T. Choose new coordinates $\{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2\}$ in $\mathbb{C}^2$ so that $f(\mathbb{C})$ is the image of the axis $\{z_1 = 0\}$ via the projection $\tau : \mathbb{C}^2 \to T$, and in the latter case, Y is the image of $\{y_1 = 0\}$.

Claim: $S \cap Y$ is nonempty.

Proof of the claim: We need to consider only the case where Y is a real 3-torus. Notice that the universal cover of $T \setminus Y$ can be identified with a strip in $\mathbb{C}^2 = \mathbb{R}^4$ between two parallel hyperplanes $y_1 = 0$ and $y_1 = a > 0$, and that y_1 generates a well-defined bounded harmonic function on $T \setminus Y$. If $S \cap Y = \emptyset$, then $y_1|_S$ =const, whence $z_1|_S$ =const, so S is contained in a translate of $f(\mathbb{C})$, which is impossible as noted above. This completes the proof of the claim.

Now let Δ_ε^2 be a coordinate bidisk centered about a point $s = (s_1, s_2) \in Y \cap S$. We may assume by our choice of coordinates in $\mathbb{C}^2$ that $\Im s_1 = 0$ and

$$\Delta_\varepsilon^2 = \Delta' \times \Delta'' = \{(w_1, w_2) \in \mathbb{C}^2 : |w_j - s_j| < \varepsilon,\ j = 1, 2\} \stackrel{\tau}{\hookrightarrow} T\ .$$

We also have $f(\mathbb{C}) \cap \Delta_\varepsilon^2 = E \times \Delta''$, where E is a dense subset of the disk Δ' if $Y = T$, or is a dense subset of the real interval $I_\varepsilon := \Delta' \cap \mathbb{R}$ if Y is a real 3-torus.

We observe that $S \not\supset \{s_1\} \times \Delta''$. Indeed, if on the contrary S contains the disk $\{s_1\} \times \Delta''$, then a translate S' of S contains a disk $\{s_1'\} \times \Delta'' \subset f(\mathbb{C})$, so the analytic set $f^{-1}(S')$ cannot be 0-dimensional. Hence $f^{-1}(S') = \mathbb{C}$, or equivalently $f(\mathbb{C}) \subset S'$, and therefore $Y = \overline{f(\mathbb{C})} \subset S'$, a contradiction.

Let $\rho_1 : \Delta_\varepsilon^2 \to \Delta'$ denote the projection to the z_1-axis. Since $S \cap (\{s_1\} \times \Delta'')$ must be 0-dimensional, it follows that $\rho_1(S \cap \Delta_\varepsilon^2)$ contains a neighborhood of s_1. Since s_1 is a cluster point of E, the set $E \cap \rho_1(S \cap \Delta_\varepsilon^2)$ must be infinite. Since

$$E \cap \rho_1(S \cap \Delta_\varepsilon^2) = \rho_1((E \times \Delta'') \cap S) = \rho_1(f(\mathbb{C}) \cap S \cap \Delta_\varepsilon^2) ,$$

it follows that $f(\mathbb{C}) \cap S$ is also infinite. □

Remark. (i) Proposition 3.3 can be rephrased as follows: For any compact complex curve S in a simple abelian surface T and for any divisor D on S, the complement $T\backslash S$ has the property of hyperbolic non-percolation through D.

(ii) Note that if S were a rational or elliptic curve in a simple complex torus T, then we would have a Brody curve $f : \mathbb{C} \to S \subset T$, and by the first paragraph of the proof of Proposition 3.3, S must contain a translate of a subgroup of T, contradicting the assumption that T is simple.

Proof of Theorem 3.1. Let $X_\infty = \{z \in \mathbb{P}^3 : F(z) = 0\}$ be a general octic surface and consider the linear pencil of surfaces

$$X_t = \{z \in \mathbb{P}^3 : Q(z) + tF(z) = 0\} .$$

Suppose on the contrary that X_{t_n} is not hyperbolic for some sequence $t_n \to 0$. Then as in the proof of Theorem 2.1, after passing to a subsequence of $\{t_n\}$, we can find a sequence of Brody curves $f_n : \mathbb{C} \to X_{t_n}$ converging to a Brody curve $f : \mathbb{C} \to X_0$.

Claim: $f(\mathbb{C}) \subset (X_0\backslash\bar{S}) \cup (\bar{S} \cap X_\infty) \cup \Gamma$, where Γ is the set of 48 pinch points of X_0.

Proof of the claim: Suppose on the contrary that $f(\zeta_0) = x_0 \in \bar{S} \setminus (X_\infty \cup \Gamma)$. We first consider the case where $x_0 \notin \{p_0, p_1, p_2, p_3\}$, so that x_0 is an ordinary double point of X_0. Choose a small neighborhood $U \subset \mathbb{P}^3 \setminus X_\infty$ of x_0 such that we have a factorization $Q|_U = Q'Q''$, where $Q',\ Q''$ are holomorphic on U and vanish at x_0; hence $X_0 \cap U$ consists of two components $X' = \{Q' = 0\}$, $X'' = \{Q'' = 0\}$. Let Δ be a small disk about ζ_0 such that $f(\bar{\Delta}) \subset U$. Then (by the same argument as in the proof of Theorem 2.1) for n sufficiently large, $f_n(\Delta)$ does not meet X_0 and hence $Q \circ f_n(\zeta) \neq 0$ for $\zeta \in \Delta$. Since $f(\zeta_0) = x_0 \in X' \cap X''$, it follows by Hurwitz's theorem applied to $Q' \circ f_n$ and to $Q'' \circ f_n$ that $f(\Delta) \subset X' \cap X'' = \bar{S} \cap U$, and thus $f(\mathbb{C}) \subset \bar{S}$. We shall complete this case below.

We now turn to the case $x_0 = p_j$ $(j = 0, 1, 2, 3)$. By virtue of Lemma 3.2, this time $X_0 \cap U$ consists of 4 components, and we conclude as above that $f(\mathbb{C})$ is contained in the intersection of these 4 components. But (by Lemma 3.2) this intersection is the point p_j, and hence f is constant, a contradiction.

Returning to the first case, we can now conclude that $f : \mathbb{C} \to \bar{S} \setminus \{p_0, p_1, p_2, p_3\}$. However, the variety $\bar{S} \setminus \{p_0, p_1, p_2, p_3\}$ consists of 4 components $\bar{S}_j \setminus \{p_i : i \neq j\}$, $0 \leq j \leq 3$, each a $\mathbb{P}^1$ with 6 points (corresponding to the 3 double points of S_j) punctured out. Again we conclude that f is constant, and this contradiction completes the proof of the claim.

As the morphism $\varphi_L : A \to X_0$ is birational and proper, it provides a normalization of X_0. Let $\tilde{f} : \mathbb{C} \to A$ be the lift of f to the normalization, and let $S := \varphi_L^{-1}(\bar{S}) \subset A$. The image $\tilde{f}(\mathbb{C})$ meets S inside the finite set $D := S \cap \varphi_L^{-1}(X_\infty \cup \Gamma)$. Although f is a Brody curve, $\tilde{f}$ is not *a priori* Brody, but we can construct a Brody curve from $\tilde{f}$ as follows. Let Δ_n denote the disk of radius n about the origin. By Brody's reparametrization lemma [Br], we can find a sequence of holomorphic maps

$$g_n = \tilde{f} \circ \rho_n \circ \alpha_n : \Delta_n \to \widetilde{f}(\mathbb{C}) \subset (A \setminus S) \cup D\,,$$

with $||g_n'(0)|| = c > 0$ and $||g_n'(\zeta)|| \leq \frac{n^2 c}{n^2 - |\zeta|^2}$, where α_n is a suitably chosen automorphism of Δ_n and $\rho_n(\zeta) = r_n \zeta$, $r_n > 0$ (see [Ko, (3.6.2), (3.6.4)], [NO, (1.6.6)]). After passing to a subsequence, we may suppose that g_n converges as $n \to \infty$ to a Brody curve $g : \mathbb{C} \to A$.

By Proposition 3.3, $g(\mathbb{C}) \cap S$ is infinite. Hence, there is a point $\zeta_0 \in \mathbb{C}$ such that $g(\zeta_0) \in S \setminus D$. We choose a small disk Δ about ζ_0 such that $g_n(\Delta)$ does not meet D and hence $g_n(\Delta) \subset A \setminus S$ for $n \gg 0$. We then conclude as before by Hurwitz's theorem that $g(\mathbb{C}) \subset S$. But since A is a simple abelian variety, S cannot be rational or elliptic. Thus g is constant, a contradiction.

Therefore, X_t is hyperbolic for t sufficiently small. □

Remark. Note that the only condition on X_∞ is that it does not contain any of the p_j.

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