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A REMARK ON THE CHERN-MOSER TENSOR

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Abstract. We compute the fourth order Chern-Moser tensor for real hypersurfaces of revolution in complex Euclidean space.

One of the most important biholomorphic differential invariants of a Levi nondegenerate real hypersurface M^{2n+1} in $\mathbf{C}^{n+1}$, $n \geq 2$, is the fourth order Chern-Moser tensor S [1]. (For $n = 1$, it is replaced by the Cartan invariant.) Unfortunately, it is usually rather difficult to compute. In particular, it is hard to locate the points on M where S vanishes, the so-called umbilic points.

In this short note we compute S for real hypersurfaces of revolution, i. e. those admitting unitary $U(n)$ symmetry. Relative to variables $(z, w) \in \mathbf{C}^n \times \mathbf{C}$, we may write the hypersurface and domain it bounds as

$$M : r = 0, \quad D : r < 0, \quad r = p(z, \overline{z}) + q(w, \overline{w}), \tag{1}$$

$$p(z, \overline{z}) = h_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta, \quad q = \overline{q}. \tag{2}$$

We use the convention that repeated greek indices are summed from 1 to n, and generally follow the notations of [1]. The positive definite hermitian matrix $h_{\alpha\overline{\beta}}$ may be taken to be the identity matrix.

We have the auxiliary curve and domain in $\mathbf{C}$,

$$M_0 : q = 0, \quad D_0 : q < 0, \tag{3}$$

where we assume $dq \neq 0$ on $q = 0$. It is easy to compute that M is strictly pseudoconvex if and only if, on D_0,

$$h = -(\log q)_{w\overline{w}} = (q_w q_{\overline{w}} - q q_{w\overline{w}})/q^2 > 0. \tag{4}$$

Assuming this, we have the (positive definite) hermitian metric on D_0,

$$ds^2 = h dw d\overline{w}. \tag{5}$$

Our goal is to express the CR invariants of M in terms of the Riemannian invariants of ds^2.

It turns out that the metric ds^2 is complete, and its Gaussian curvature K approaches -2 as we approach the boundary M_0. By the $U(n)$ symmetry of M, the w-axis meets M in a chain, along which $S = 0$. Also, if $q = w\overline{w} - 1$, then $K \equiv -2$ and $S \equiv 0$. This motivates the following result.

Theorem 1. *Let $w \in D_0$ and $(z, w) \in M$. Then, at points where $dq \neq 0$, $S(z, w) = 0$ if and only if $K(w) = -2$.*

Although the proof is a matter of computation, it has some interesting aspects, and gives more than is stated.

Since the variables z and w are separated in r, we may use a computational procedure developed in [2]. Following the notation of [2], we introduce the one-forms and quantities

$$\theta = i\partial r, \quad \theta^\alpha = dz^\alpha + i\eta^\alpha\theta, \tag{6}$$

$$\eta^\alpha = g^{\alpha\overline{\beta}}\eta_{\overline{\beta}}, \quad \eta_\alpha = -Qp_\alpha, \quad Q = q_{w\overline{w}}/(q_w q_{\overline{w}}). \tag{7}$$

We have

$$d\theta = ig_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}}, \quad g_{\alpha\overline{\beta}} = -h_{\alpha\overline{\beta}} - Qp_\alpha p_{\overline{\beta}}; \tag{8}$$

which says that the coframe (6) is admissible for the pseudo-hermitian structure (M, θ).

With the dual vector fields $X_\alpha = \partial_\alpha - (p_\alpha/q_w)\partial_w$, as in [2] we may write the pseudo-hermitian curvature tensor as

$$\begin{aligned} R_{\beta\overline{\alpha}\rho\overline{\sigma}} &= -X_{\overline{\sigma}}X_\rho g_{\beta\overline{\alpha}} + g^{\gamma\overline{\mu}}X_\rho g_{\beta\overline{\mu}}X_{\overline{\sigma}}g_{\gamma\overline{\alpha}} \\ &+ g_{\rho\overline{\sigma}}\eta^\gamma(X_\beta g_{\gamma\overline{\alpha}} - X_\gamma g_{\beta\overline{\alpha}}) - g_{\rho\overline{\alpha}}X_{\overline{\sigma}}\eta_\beta - g_{\beta\overline{\sigma}}X_\rho\eta_{\overline{\alpha}} \\ &- g_{\rho\overline{\sigma}}X_\beta\eta_{\overline{\alpha}} - \eta_\beta\eta_{\overline{\alpha}}g_{\rho\overline{\sigma}} - \eta_\gamma\eta^\gamma g_{\beta\overline{\sigma}}g_{\rho\overline{\alpha}}. \end{aligned} \tag{9}$$

We can express this in terms of p_α and $g_{\alpha\overline{\beta}}$ with coefficients in $(w, \overline{w})$. After some simplification, we get

$$\begin{aligned} R_{\beta\overline{\alpha}\rho\overline{\sigma}} &= A(g_{\beta\overline{\alpha}}g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}g_{\beta\overline{\sigma}}) + Bp_\beta p_{\overline{\alpha}}p_\rho p_{\overline{\sigma}}, \\ A &= -Q(1 - Qq)^{-1}, \\ B &= Q_{w\overline{w}}(q_w q_{\overline{w}})^{-1} + 2Q((Q_w/q_w) + (Q_{\overline{w}}/q_{\overline{w}})) \\ &\quad +3Q^3 + |(Q_w/q_w) + Q^2|^2 q(1 - Qq)^{-1}. \end{aligned} \tag{10}$$

From [2] we have the following formula for the Chern-Moser tensor,

$$\begin{aligned} S_{\beta\overline{\alpha}\rho\overline{\sigma}} &= R_{\beta\overline{\alpha}\rho\overline{\sigma}} - (n+2)^{-1}(g_{\beta\overline{\alpha}}R_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}R_{\beta\overline{\sigma}} \\ &+ g_{\beta\overline{\sigma}}R_{\rho\overline{\alpha}} + g_{\rho\overline{\sigma}}R_{\beta\overline{\alpha}}) + R(n+1)^{-1}(n+2)^{-1}(g_{\beta\overline{\alpha}}g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}g_{\beta\overline{\sigma}}), \end{aligned} \tag{11}$$

where the Ricci tensor and scalar curvature are given by

$$R_{\rho\overline{\sigma}} = (n+1)Ag_{\rho\overline{\sigma}} + Bq(1-Qq)^{-1}p_\rho p_{\overline{\sigma}}, \tag{12}$$

$$R = n(n+1)A + Bq^2(1-Qq)^{-2}. \tag{13}$$

Some further simplification gives

$$\begin{aligned} S_{\beta\overline{\alpha}\rho\overline{\sigma}} &= Bq^2(1-Qq)^{-2}(n+1)^{-1}(n+2)^{-1}(g_{\beta\overline{\alpha}}g_{\rho\overline{\sigma}} + g_{\rho\overline{\alpha}}g_{\beta\overline{\sigma}}) \\ &-Bq(1-Qq)^{-1}(n+2)^{-1}(g_{\beta\overline{\alpha}}p_\rho p_{\overline{\sigma}} + g_{\rho\overline{\alpha}}p_\beta p_{\overline{\sigma}} + g_{\beta\overline{\sigma}}p_\rho p_{\overline{\alpha}} + g_{\rho\overline{\sigma}}p_\beta p_{\overline{\alpha}}) \\ &+Bp_\beta p_{\overline{\alpha}}p_\rho p_{\overline{\sigma}}. \end{aligned} \tag{14}$$

Next we compute the Gaussian curvature K of the metric ds^2. By definition, $K = R/h$, where $\overline{\partial}\partial \log h = Rdw \wedge d\overline{w}$. With $k = q_w q_{\overline{w}} - qq_{w\overline{w}}$, we get

$$\begin{aligned} K = -2 + q^3k^{-3}(&kq_{ww\overline{w}\overline{w}} + qq_{ww\overline{w}}q_{w\overline{w}\overline{w}} \\ &-q_{ww\overline{w}}q_{\overline{w}\overline{w}}q_w - q_{w\overline{w}\overline{w}}q_{ww}q_{\overline{w}} + q_{w\overline{w}}q_{ww}q_{\overline{w}\overline{w}}). \end{aligned} \tag{15}$$

A lengthy but straight forward computation shows that

$$B = (K+2)k^2q^{-3}(q_w q_{\overline{w}})^{-2}. \tag{16}$$

Clearly, if $K = -2$ then $B = 0$, and (14) shows that $S = 0$. We set $\alpha = \beta = \rho = \sigma = 1$ and restrict to $z^j = 0$, $2 \le j \le n$. Since $p_1 p_{\overline{1}} = -q$, and $g_{1\overline{1}} = -1 + Qq$, we have

$$S_{1\overline{1}1\overline{1}} = Bq^2(n+1)^{-1}(n+2)^{-1}n(n-1). \tag{17}$$

Since we assume $n > 1$, $K = -2$ if this vanishes. This proves our theorem.

It would be of interest to find a more geometric reason for theorem 1, or at least to simplify the computations. Hypersurfaces and domains of revolution should be of interest relative to other constructions, such as the Bergman or Szegö kernels. The chains on M project to an interesting family of curves in D_0.

References

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