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ON A CLASS OF COMPACTA

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ABSTRACT. Using the notion of nearly extendable maps which has been introduced in [5], in connection with the theory of fixed points, a class of compacta (called NE-sets) is defined and investigated. This class is more general than the class of all approximative absolute neighborhood retracts (called AANR-sets), but it is less general than the class of all movable compacta.

1. Introduction. It is well known that if the shape of a compactum X is trivial and if X has a sufficiently regular topological structure (in particular, if $X \in ANR$), then each map $f:X \to X$ has a fixed point. But an analogous statement fails if one omits the hypothesis that the structure of X is regular. However, the hypothesis on the regular structure of X can be omitted if we restrict, in an appropriate way, the class of considered maps. In [5], a class of nearly extendable maps (called NE-maps) is introduced and it is shown there that

(1.1) For every compactum X with trivial shape, every NE-map $f:X \to X$ has a fixed point.

The class of NE-maps is quite large. In particular, it is known (see [5]) that

(1.2) If at least one of the compacta X,Y is an ANR-set, then every map $f:X \to Y$ is an NE-map.

In the present note, we define and study a class of compacta (called NE-sets) given by the following

(1.3) DEFINITION. A compactum X is said to be an *NE-set* if every map of X into any compactum Y is an NE-map (see Section 2 for the definition of NE-maps).

It follows from (1.2) that every (compact) ANR-space is an NE-set. But the class of NE-sets is much more general than the class of all ANR's. Observe that (1.1) implies that every NE-set with a trivial shape has the fixed point property.

By a map we understand here a continuous function, and a space always means a metrizable space. By AR-sets and ANR-sets we understand compact absolute retracts and compact absolute neighborhood retracts, respectively.

2. **Some properties of NE-maps.** A map f of a compactum X into another compactum Y is said to be an *NE-map* if there exist AR-spaces M and N containing X and Y, respectively, and there is a map $\tilde{f}:M \to N$ satisfying the condition

(2.1) $\tilde{f}(x) = f(x)$ for every $x \in X$,

and such that

(2.2) For every $\epsilon > 0$, there is a neighborhood U of X in M such that, for every neighborhood V of Y in N, there is a map $g:U \to V$ with $\rho(\tilde{f}(x),g(x)) < \epsilon$ for every $x \in U$.

One shows (see [5]) that the choice of spaces $M,N \in AR$ containing X and Y, respectively, and also the choice of a map $\tilde{f}:M \to N$ satisfying (2.1) are immaterial. Moreover, it is shown in [5] that:

(2.3) If at least one of the maps $f:X \to Y$ and $g:Y \to Z$ is an NE-map, then $gf:X \to Z$ is an NE-map.

(2.4) For every compacta X,Y, the set of all NE-maps $f:X \to Y$ is closed in the functional space Y^X.

(2.5) If X,Y are compacta and if all values of a map $f:X \to Y$ belong to an ANR-set $A \subset Y$, then f is an NE-map.

3. **Some properties of NE-sets.** The following propositions are direct consequences of (2.3):

(3.1) *X is an NE-set if and only if the identity map* $i_X:X \to X$ *is an NE-map.*

(3.2) *Y is an NE-set if and only if for every compactum X all maps* $f:X \to Y$ *are NE-maps.*

Now let us formulate a condition which characterizes NE-sets among all compacta.

(3.3) CONDITION. There exists an AR-space M containing X and such that, for every $\epsilon > 0$, there is a neighborhood U of X in M and a map $g:U \to X$ such that $\rho(x,g(x)) < \epsilon$ for every $x \in U$.

(3.4) REMARK. Notice that the choice of an AR-space M containing X is inessential in (3.3). In fact, if M′ is another AR-space containing X, then there exists a map

$$\alpha:M' \to M$$

such that $\alpha(x) = x$ for every $x \in X$. If ϵ, U and $g:U \to X$ are as in condition (3.3), then there exists a neighborhood U' of X in M' such that $\alpha(U') \subset U$ and $\rho(x',\alpha(x')) < \epsilon$ for every $x' \in U'$. Setting

$$g'(x') = g\alpha(x') \text{ for every } x' \in U',$$

one gets a map $g':U' \to X$ such that

$$\rho(x',g'(x)) = \rho(x',g\alpha(x')) \leqslant \rho(x',\alpha(x')) + \rho(\alpha(x'),g\alpha(x')) < 2\epsilon,$$

because $\rho(x',\alpha(x')) < \epsilon$ and $\rho(\alpha(x'),g\alpha(x')) < \epsilon$.

Now let us prove the following

(3.5) THEOREM. *A compactum X is an NE-set if and only if X satisfies condition (3.3).*

PROOF. It is clear that (3.3) implies that the identity map $i_X:X \to X$ is an NE-map, and we infer by (3.1) that $X \in NE$.

On the other hand, if X is an NE-set, then the identity map $i_X:X \to X$ is an NE-map. Assume that $X \subset M \in AR$. Then, for each $\epsilon > 0$ and $n = 1,2,...$, there exists a neighborhood U_n of X in M and a map $f_n:U_n \to U_{n+1}$ such that

$$\rho(x,f_n(x)) < \epsilon \cdot 2^{-n} \text{ for every } x \in U_n.$$

Moreover, we can assume that $U_{n+1} \subset U_n$ for $n = 1,2,...$ and that

$$X = \bigcap_{n=1}^{\infty} U_n.$$

Setting

$$g_n = f_n f_{n-1} \cdots f_1(x) \text{ for every } x \in U_1,$$

one gets a sequence of maps $g_n:U_1 \to U_{n+1}$ uniformly converging to a map $g:U_n \to X$ which staisfies the inequality

$$\rho(x,g(x)) < \sum_{n=1}^{\infty} \epsilon \cdot 2^{-n} = \epsilon \text{ for every } x \in U_1.$$

Thus condition (3.3) is satisfied and the proof of theorem (3.5) is completed.

4. NE-sets and a class of compacta introduced by M. H. Clapp. One sees easily that, for compacta X, condition (3.3) is equivalent to the following one:

(4.1) CONDITION. For every homeomorphism h which maps X onto a subset h(X) of a metric space M, there exists, for each $\epsilon > 0$, a neighborhood U of h(X) in M and a map $g:U \to h(X)$ such that $\rho(y,g(y)) < \epsilon$ for every $y \in h(X)$.

Compacta satisfying condition (4.1) were introduced and studied by M. H. Clapp [7]. By theorem (3.5), these compacta are the same as NE-sets. Thus several results due to M. H. Clapp imply some properties of NE-sets. In particular, the approximative absolute neighborhood retracts (i.e., AANR-sets in the sense of A. Gmurczyk, see [8] and also [11]) are a special kind of the NE-sets (see [7], p. 118). Moreover, if 2^Q_c denotes the space of all non-empty compacta lying in the Hilbert cube Q which is metrized by the metric of continuity ρ_c (see [1], p. 169), we get (see [7], p. 122) the following proposition:

(4.2) *The NE-sets coincide with the homeomorphic images of compacta belonging to the closure of the subset of 2^Q_c consisting of all polyhedra.*

5. NE-sets which are not AANR's. It is known that all the Betti numbers of each AANR-set are finite (see [8], p. 14). However, there exist NE-sets for which some Betti numbers are infinite. This follows from examples due to M. H. Clapp (see [7], p. 199). Moreover, we have the following

(5.1) THEOREM. *Every locally connected plane continuum is an NE-set.*

PROOF. Let X be a locally connected continuum contained in the plane E^2. Since an empty set is an NE-set, we may assume that $X \neq \emptyset$. For any given $\epsilon > 0$, there exists a positive number $\eta > 0$ such that

(5.2) Every subset of X with diameter $< \eta$ is contained in a locally connected subcontinuum of X with diameter $< \frac{1}{2}\epsilon$.

Consider, in E^2, a square M containing X in its interior and let $M_1, M_2, \ldots, M_n$ be a system of squares with diameters $< \frac{1}{2}\eta$ such that $M = M_1 \cup M_2 \cup \cdots \cup M_n$ and such that, for $i \neq j$, the interiors of M_i and of M_j are disjoint. We can order these squares so that there are natural number $k \leqslant m \leqslant n$ such that:

$M_i \subset X$ for $i < k$,

$M_i \neq M_i \cap X \neq \emptyset$ for $k \leqslant i \leqslant m$,

$M_i \cap X = \emptyset$ for $m < i \leqslant n$.

Then, the set

$$A = \bigcup_{i \leqslant m} M_i$$

is a polyhedron containing X in its interior. If $k \leqslant i \leqslant m$, then the interior of M_i has a point $a_i \in M \backslash X$. It follows that the set

$$U = A \backslash \bigcap_{i=k}^{m} (a_i)$$

is a neighborhood of X in M.

Let α_i denote the projection of $M_i \backslash (a_i)$ onto the boundary B_i of the square M_i (for $k \leqslant i \leqslant m$). Then

$$\rho(\alpha_i(x),x) < \tfrac{1}{2}\eta < \tfrac{1}{2}\epsilon \text{ for every } x \in M_i\backslash(a_i).$$

Using (5.2), one sees easily (see [10], p. 347) that there exists a map $\beta_i: B_i \to X$ such that $\beta_i(x) = x$ if $x \in B_i \cap X$, and $\rho(\beta_i(x),x) < \tfrac{1}{2}\epsilon$ for every point $x \in B_i$. Setting

$$\begin{cases} g(x) = x \text{ for every point } x \in M_i, \text{ where } i < k, \\ g(x) = \beta_i\alpha_i(x) \text{ for every } x \in M_i\backslash(a_i), \text{ where } k \leqslant i \leqslant m, \end{cases}$$

one gets a map $g: U \to X$ satisfying the condition

$$\rho(g(x),x) < \epsilon \text{ for every } x \in U.$$

Hence condition (3.3) is satisfied and we infer, by theorem (3.5), that X is an NE-set.

Now let us give a simple example of a plane continuum that is not an NE-set.

(5.3) EXAMPLE. Consider the subset X of the plane consisting of all points $(0,x_2)$ with $-1 \leqslant x_2 \leqslant 2$ and of all points $(x_1, \sin \frac{\pi}{x_1})$ with $0 < x_1 \leqslant 1$. Suppose that X is an NE-set; then, by (3.5), condition (3.3) must be satisfied. Consequently, there exists, in any space $M \in AR$ containing X, a neighborhood U of X and a map $g: U \to X$ such that

$$\rho(x,g(x)) < \tfrac{1}{2} \text{ for every } x \in U.$$

Then, in the component of U containing X, there is an arc L with endpoints $a = (0,2)$ and $b = (1,0)$. It follows that g(L) is a locally connected continuum in X which contains both points g(a) and g(b). Since $\rho(a,g(a)) < \tfrac{1}{2}$ and $\rho(b,g(b)) < \tfrac{1}{2}$, we see at once that such a locally connected continuum does not exist. Hence the supposition $X \in NE$ fails.

Let us add that, by virtue of (1.1), any compactum with trivial shape and without the fixed point property is not an NE-set.

6. NE-sets with trivial shape. We know already that each AANR-set is an NE-set and that the converse is not true. The situation,however, is different among compacta with trivial shape.

(6.1) THEOREM. *Every NE-set with trivial shape is an AANR-set.*

PROOF. Let X be an NE-set with trivial shape. Then there exists (see [9], p. 92 and also [6], p. 182) a sequence of AR-sets $M = M_1 \supset M_2 \supset \cdots$ such that

(6.2) $X = \bigcap_{n=1}^{\infty} M_n.$

Since X, as an NE-set, satisfies condition (3.3), we infer, by remark (3.4), that, for every $\epsilon > 0$, there exists a neighborhood U of X in M and a map $g: U \to X$ such that

$$\rho(x,g(x)) < \epsilon \text{ for every } x \in U.$$

By (6.2), there is an index n_ϵ such that $M_{n_\epsilon} \subset U$. Since $M_{n_\epsilon} \in AR$, there is a retraction $r: M \to M_{n_\epsilon}$. Setting

$$f(x) = gr(x) \text{ for every } x \in M,$$

one gets a map $f: M \to X$ satisfying the condition $\rho(x,f(x)) < \epsilon$ for every $x \in X$, since if $x \in X$, then $r(x) = x$ and consequently

$$\rho(x,f(x)) = \rho(x,g(x)) < \epsilon.$$

Using remark (4.2), we conclude that $X \in AANR$ and (6.1) is proved.

It is well known (see [4], p. 274) that compacta with trivial shape are the same as fundamental absolute retracts, i.e., FAR-sets. More general is the class of fundamental absolute neighborhood retracts, i.e., FANR-sets.

(6.3) PROBLEM. Does there exist among FANR-sets an NE-set which is not an AANR-set?

7. Neighborhood retracts of NE-sets. Recall that a set $Y \subset X$ is said to be a *neighborhood retract* of X (see [2], p. 14) if there exists a neighborhood W of Y in X and a retraction $r: W \to Y$.

(7.1) THEOREM. *Every neighborhood retract of an NE-set is an NE-set.*

PROOF. Let M be an AR-space containing $X \in NE$ and let $Y \subset X$ be a retract of a neighborhood W of Y in X. Consider a retraction $r: W \to Y$, and let ϵ be a positive number. Then there is a neighborhood $W_0 \subset W$ of Y in X such that

(7.2) $\rho(y,r(y)) < \frac{1}{2}\epsilon$ for every $y \in W_0$.

Moreover, there is a positive number $\eta < \frac{1}{2}\epsilon$ such that

(7.3) If $x \in X$, $y \in Y$ and $\rho(x,y) < \eta$, then $x \in W_0$.

Since X, as an NE-set, satisfies condition (3.3), there exists a neighborhood U of X in M and a map $f: U \to X$ such that

(7.4) $\rho(x,f(x)) < \eta$ for every $x \in U$.

It follows, by (7.3) and (7.4), that there exists a neighborhood $V \subset U$ of Y in M such that $f(V) \subset W_0$. Setting $g(y) = rf(y)$ for every $y \in V$, one gets a map $g: V \to Y$ such that

$$\rho(y,g(y)) \leqslant \rho(y,f(y)) + \rho(f(y),rf(y)) < \eta + \tfrac{1}{2}\epsilon < \epsilon.$$

Thus we have shown that condition (3.3) (in which X is replaced by Y and U by V) is satisfied. By theorem (3.5), Y is an NE-set.

(7.5) COROLLARY. *Every retract of an NE-set is an NE-set.*

8. Movability of NE-sets. The property of the movability (see [3], p. 142) is a shape invariant which eliminates the most complicated global singularities of compacta. Let us prove the following

(8.1) THEOREM. *Every NE-set is movable.*

PROOF. Assume that X is an NE-set lying in the Hilbert cube Q. If U is a neighborhood of X in Q, then there is an $\epsilon > 0$ and a neighborhood W of X in Q such that if $x \in W$, $y \in Q$ and $\rho(x,y) < \epsilon$, then the segment $[x,y]$ with endpoints x,y lies in U. Since X is an NE-set, there exists a neighborhood $U_o \subset W$ of X in Q such that, for every neighborhood V of X in Q, there is a map $g: U_o \to V$ such that $\rho(x,g(x)) < \epsilon$ for every $x \in U_o$. Then $[x,g(x)] \subset U$, and we infer that g is homotopic, in U, to the inclusion $i: U_o \to U$. Hence X is movable.

9. Cartesian product of NE-sets. Let us establish the following

(9.1) THEOREM. *The Cartesian product $X = X_1 \times X_2 \times \cdots \neq \emptyset$ is an NE-set if and only if $X_n \in NE$ for every $n = 1,2,\ldots$.*

PROOF. Since X_n is homeomorphic with a retract of X, we see, by (7.5), that $X \in NE$ implies $X_n \in NE$ for every $n = 1,2,\ldots$.

Assume now that $X_n \in NE$ for $n = 1,2,\ldots$. Let M_n be an AR-space containing X_n. Then $M = M_1 \times M_2 \times \cdots$ is an AR-space containing X. We may assume that the diameter of M_n is $< n^{-2}$. Then the distance in the space M may be given by the formula

(9.2) $\rho((y_1,y_2,\ldots),(y'_1,y'_2,\ldots)) = \sum_{n=1}^{\infty} \rho(y_n,y'_n)$.

Since $X \neq \emptyset$, we can select a point a_n in each X_n. Then, for every $\epsilon > 0$, there exists an index $k = k(\epsilon)$ such that

$$\sum_{n=k+1}^{\infty} n^{-2} < \tfrac{1}{2}\epsilon.$$

Setting

$$\varphi(y) = (y_1,\ldots,y_k,a_{k+1},a_{k+2},\ldots) \text{ for every } y = (y_1,y_2,\ldots) \in M,$$

one gets a map $\varphi: M \to M$ satisfying the condition

(9.3) $\rho(y,\varphi(y)) < \tfrac{1}{2}\epsilon$ for every $y \in M$.

Since $X_n \in NE$, we infer by (3.3) that there exists a neighborhood U_n of X_n in M and a map $g_n: U_n \to X_n$ such that

(9.4) $\rho(y_n,g_n(y_n)) < \frac{1}{2k}\epsilon$ for every $y_n \in U_n$.

Then the set $U = U_1 \times \cdots \times U_k \times M_{k+1} \times M_{k+2} \times \cdots$ is a neighborhood of X in M and the formula

$$g(y) = \varphi(g_1(y_1),g_2(y_2),\ldots) \text{ for every } y = (y_1,y_2,\ldots) \in U$$

defines a map $g: U \to X$, because

$$g(y) = (g_1(y_1),\ldots,g_k(y_k),a_{k+1},a_{k+2},\ldots) \in X.$$

Moreover, we conclude from (9.2), (9.3) and (9.4) that

$$\rho(\varphi(y),g(y))$$
$$= \rho((y_1,\ldots,y_k,a_{k+1},a_{k+2},\ldots),(g(y_1),\ldots,g_k(y_k),a_{k+1},a_{k+2},\ldots))$$
$$= \sum_{n=1}^{k} \rho(y_n,g_n(y_n)) < k\cdot\frac{1}{2k}\epsilon = \frac{1}{2}\epsilon.$$

It follows, by (9.3), that

$$\rho(y,g(y)) \leqslant \rho(y,\varphi(y)) + \rho(\varphi(y),g(y)) < \epsilon \text{ for every point } y \in U.$$

Thus the condition (3.3) is satisfied, whence $X \in NE$ and the proof of theorem (9.1) is complete.

10. Suspension of NE-sets. Let us show that the class of NE-sets is closed with respect to the operation of the suspension.

(10.1) THEOREM. *The suspension of every NE-set is an NE-set. (The referee points out that this theorem can be reversed. Indeed, it follows from (7.1) that if the suspension of a compactum X is an NE-set, then X is also an NE-set. (Editor)).*

PROOF. Let Q_o denote the subset of the Hilbert space consisting of all points $y = (y_1,y_2,\ldots)$ with $|y_n| \leqslant \frac{1}{n}$ for $n = 1,2,\ldots$, and let M denote the set of all points $y \in Q_o$ with $y_1 = 0$. Then M is an AR-space homeomorphic to Q_o. Let δ denote the diameter of M. We may assume that $X \subset M$. Setting

$$a = (-1,0,0,\ldots),\ b = (1,0,0,\ldots),$$

let us assign, to every set $Z \subset M$, the suspension $\widetilde{Z}$ of it, which we define as the union of all the segments [az] and [bz] with $z \in Z$. Then the suspension $\widetilde{X}$ of X is contained in $\widetilde{M} \in AR$.

Now let us consider a positive number ϵ and let η be a positive number < 1 so small that

(10.2) $\eta < \frac{1}{2}\epsilon$ and $\eta\cdot\delta < \frac{1}{2}\epsilon$.

Setting

$$\varphi(t) = -1 \text{ for } -1 \leqslant t \leqslant -1 + \eta,$$

$\varphi(t) = \frac{1}{1-\eta} t$ for $-1 + \eta < t < 1 - \eta$,

$\varphi(t) = 1$ for $1 - \eta \leqslant t \leqslant 1$,

we get a map $\varphi: \langle -1,1 \rangle \to \langle -1,1 \rangle$ such that:

(10.3) $\varphi(t) \leqslant 0$ for $-1 \leqslant t \leqslant 0$, $\varphi(0) = 0$, and $\varphi(t) \geqslant 0$ for $0 \leqslant t \leqslant 1$,

(10.4) $|t - \varphi(t)| < \frac{1}{2}\epsilon$ for every $t \in \langle -1,1 \rangle$.

Since $X \in NE$, we infer by (3.3) that there is a neighborhood U of X in M and a map $g: U \to X$ such that

(10.5) $\rho(y,g(y)) < \epsilon$ for every point $y \in U$.

Let us denote by A the set consisting of all the points $(y_1,y_2,...) \in \widetilde{M}$ with $y_1 \leqslant -1 + \eta$, and by B the set of all the points $(y_1,y_2,...) \in \widetilde{M}$ with $y_1 \geqslant 1 - \eta$. Notice that (10.2) implies that the diameters of A and of B are less than ϵ.

Setting

$$W = A \cup B \cup \widetilde{U},$$

one gets a neighborhood of X in M.

Now let us define, for every point $y = (y_1,y_2,...) \in W$, a point f(y) given by the formulas:

(10.6) $f(y) = a$ if $y_1 \leqslant -1 + \eta$,

(10.7) $f(y) = b$ if $y_1 \geqslant 1 - \eta$,

(10.8) $f(y) = -\varphi(y_1)a + (1 + \varphi(y_1)) \cdot g(0,y_2,y_3,...)$ if $-1 + \eta < y_1 \leqslant 0$,

(10.9) $f(y) = \varphi(y_1)b + (1 - \varphi(y_1)) \cdot g(0,y_2,y_3,...)$ if $0 \leqslant y_1 < 1 - \eta$.

If $y \in \widetilde{U} \setminus (A \cup B)$, then $-1 + \eta < y_1 < 1 - \eta$ and $g(0,y_2,y_3,...) = x \in X$. If $y_1 \leqslant 0$, then $\varphi(y_1) \leqslant 0$ and we infer by (10.8) that

$$f(y) = -\varphi(y_1)a + (1 + \varphi(y_1))x \in [ax].$$

If $y_1 \geqslant 0$, then $\varphi(y_1) \geqslant 0$ and we infer by (10.9) that

$$f(y) = \varphi(y_1)b + (1 - \varphi(y_1))x \in [bx].$$

It follows, by virtue of (10.6) and (10.7), that $f: W \to \widetilde{X}$.

Moreover, f is continuous, because if $y = (y_1,y_2,...) \in W$ and $y_1 = -1 + \eta$, then $\varphi(y_1) = -1$ and $-\varphi(y_1)a + (1 + \varphi(y_1)) \cdot g(0,y_2,y_3,...) = a$. And if $y_1 = 1 - \eta$, then $\varphi(y_1) = 1$ and $\varphi(y_1)b + (1 - \varphi(y_1)) \times g(0,y_2,y_3,...) = b$. Finally, if $y_1 = 0$, then $\varphi(y_1) = 0$ and both formulas (10.8) and 10.9) coincide.

Furthermore, if $y \in A$, then $f(y) = a \in A$ and we infer that $\rho(y,f(y)) < \epsilon$, because the diameter of A is less that ϵ. Similarly, if $y \in B$, then $\rho(y,f(y)) < \epsilon$. If, however, the point $y = (y_1, y_2, \ldots)$ belongs to $\widetilde{U} \setminus (A \cup B)$, then we see, by (10.8) (if $y_1 \leqslant 0$) or by (10.9) (if $y_1 \geqslant 0$), that

$$\rho(y, f(y)) \leqslant \rho((0, y_2, y_3, \ldots), g(0, y_2, y_3, \ldots)) < \epsilon.$$

Thus we have shown that $f: W \to \widetilde{X}$ is a map satisfying the condition $\rho(y, f(y)) < \epsilon$ for every point $y \in W$. Consequently, $\widetilde{X}$ satisfies condition (3.3) and (10.1) is proved.

11. Components of an NE-set. A relation between the NE-property of a compactum and of its components is given by the following

(11.1) THEOREM. *A compactum whose every component is an NE-set is an NE-set.*

PROOF. Assume that X is a compactum, $X \subset M \in AR$ and let ϵ be a positive number. By our hypothesis, there exists, for every component A of X, a neighborhood W_A of A in M such that, for every neighborhood V of X in M, there is a map $g_A: W_A \to V$ satisfying the condition

$$\rho(x, g_A(x)) < \epsilon \text{ for every } x \in W_A.$$

Since W_A can be replaced by any smalled neighborhood of A, we may assume that W_A is open (in M) and that its boundary

$$B_A = \overline{W}_A \setminus W_A$$

lies in $M \setminus X$. Since X is compact, there exists a finite system $A_1, A_2, \ldots, A_n$ of components of X such that

$$U = \bigcup_{i=1}^{n} W_{A_i}$$

is a neighborhood of X in M. Setting

$$U_k = W_{A_k} \setminus \bigcup_{j=1}^{k-1} \overline{W}_{A_j} \quad \text{for } k = 1, 2, \ldots, n,$$

we get a system of open subsets of M such that

$$U = \bigcup_{k=1}^{n} U_k, \quad U_k \subset W_{A_k} \text{ for } k = 1, 2, \ldots, n$$

and

$$U_k \cap U_{k'} = \emptyset \quad \text{for } k \neq k'.$$

It follows that the formula

$$g(x) = g_{A_k}(x) \text{ for every } x \in U_k,\ k = 1, 2, \ldots, n$$

defines a map $g: U \to V$ satisfying the condition

$$\rho(x, g(x)) < \epsilon \text{ for every } x \in U.$$

Since U is a neighborhood of X, we infer that the identity map $i_X: X \to X$ is an NE-map. By (3.1), we get $X \in NE$.

Theorems (5.1) and (11.1) imply the following

(11.2) COROLLARY. *Every plane compactum with locally connected components is an NE-set.*

Now let us show that

(11.3) There exist NE-sets for which not every component is an NE-set.

In order to see this, consider the continuum X defined as example (5.3) and observe that for every $\epsilon > 0$ there exists a neighborhood U_ϵ of X in the plane E^2 and a map $f_\epsilon : U_\epsilon \to E^2$ such that $f_\epsilon(X)$ is an arc lying in $E^2 \setminus X$ and $\rho(f_\epsilon(x),x) < \epsilon$ for every $x \in U_\epsilon$.

It follows easily that there exists, in $E^2 \setminus X$, a sequence of disjoint arcs $L_1, L_2, \ldots$ such that

$$Y = X \cup \bigcup_{i=1}^{\infty} L_i$$

is a compactum with the property that, for every positive ϵ, there is a natural number n satisfying the following condition: there exists a neighborhood V_ϵ of Y in E^2 and a retraction r_ϵ of V_ϵ to the set $\bigcup_{i=1}^{n} L_i$ with $\rho(x, r_\epsilon(x)) < \epsilon$ for every $y \in V_\epsilon$. By theorem (3.5), Y is an NE-set. However its component X is not an NE-set.

12. Addition of NE-sets. Notice that the plane continuum X considered in example (5.3) is the union of the segment X_1 with endpoints (0,1) and (0,2), and the closure X_2 of the diagram of the function $y = \sin\frac{\pi}{x}$, where $0 < x \leqslant 1$. It is clear that both sets X_1 and X_2 are NE-sets, but their union X is not an NE-set though the set $X_1 \cap X_2$ consists of only one point (0,1). Thus, a theorem similar to well-known theorems on the union of two ANR-sets or two FANR-sets is not true for NE-sets. However, we have the following

(12.1) THEOREM. *Suppose $X = X_1 \cup X_2$, where X_1, X_2 are compacta and $X_1 \cap X_2$ consists of only one point a. The following implications hold:*

If $X \in NE$, then X_1, X_2 are NE-sets.

If $X_1, X_2 \in NE$ and X is locally contractible at the point a, then $X \in NE$.

PROOF. Since X_i (for i = 1,2) is a retract of X, the first part of theorem (12.1) is a direct consequence of corollary (7.5).

Passing to the second part, observe that the hypothesis that X is locally contractible at the point a means that, for every $\epsilon > 0$, there exists a closed neighborhood U of a in X, and a map

$$\varphi: U \times \langle 0,1\rangle \to X$$

such that

$$\varphi(x,0) = a, \quad \varphi(x,1) = x \quad \text{for every } x \in U,$$

and

$$\rho(\varphi(x,t),a) < \frac{1}{2}\epsilon \text{ for every } (x,t) \in U \times \langle 0,1\rangle.$$

It is clear that there exist compacta $A,B \subset X$ such that A is a neighborhood of a in X and $U \cap \overline{X \setminus U} \subset B \subset X \setminus A$. Then there is a map

$$\alpha: U \to \langle 0,1\rangle$$

such that

$$\alpha(x) = 0 \text{ for } x \in A, \text{ and } \alpha(x) = 1 \text{ for } x \in B.$$

Setting

$$\begin{cases} \psi(x) = \varphi(x,\alpha(x)) \text{ for } x \in U, \\ \psi(x) = x \text{ for } x \in X \setminus U, \end{cases}$$

we get a map $\psi: X \to X$ such that

$$\psi(x) = a \text{ for every } x \in A,$$
$$\psi(x) = x \text{ for every } x \in X \setminus U,$$
$$\rho(x,\psi(x)) < \frac{1}{2}\epsilon \text{ for every } x \in X.$$

Moreover, there is a positive number $\eta < \epsilon$ such that

(12.2) If $x_1 \in X_1$, $x_2 \in X_2$ and $\rho(x_1,x_2) < \eta$, then $x_1,x_2 \in A$.

If $X_i \in NE$ for $i = 1,2$, then there exists a neighborhood V_i of X_i in M, and a map $f_i: V_i \to X_i$ such that

$$\rho(f_i(y),y) < \frac{1}{2}\eta \text{ for every } y \in V_i.$$

Observe that if $y \in V_1 \cap V_2$, then

$$\rho(f_1(y),f_2(y)) \leqslant \rho(f_1(y),y) + \rho(y,f_2(y)) < \eta,$$

and since $f_1(y) \in X_1$ and $f_2(y) \in X_2$, we infer, by (12.2), that both points $f_1(y)$, $f_2(y)$ belong to A. It follows that setting

$$f(y) = \psi f_i(y) \text{ for every } y \in V_i,\ i = 1,2,$$

we get a map f of the neighborhood $V = V_1 \cup V_2$ of X in M into X such that, for every point $y \in V_i$ (where $i = 1,2$), one has

$$\rho(y,f(y)) = \rho(y,\psi f_i(y)) \leqslant \rho(y,f_i(y)) + \rho(f_i(y),\psi f_i(y)) < \epsilon.$$

Hence X satisfies condition (3.3) and, consequently, X is an NE-set.

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author's attention that S. A. Bogatyi in the paper, "Approximative and fundamental retracts" (in Russian), Mat. Sbornik 93(135) (1974), pp. 90-102, considers compacta satisfying condition (4.1). As we know, such compacta coincide with the NE-sets. His theorem 2 is equivalent to our theorem (11.1), and his theorem 6 is a little stronger than our theorem (8.1). Thus the priority of those two results belongs to S. A. Bogatyi.

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