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HOMOGENEOUS PLANE CONTINUA

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ABSTRACT. A space is *homogeneous* if for each pair p, q of its points there exists a homeomorphism of the space onto itself that takes p to q. R H Bing [3] proved that every homogeneous plane continuum that contains an arc is a simple closed curve. F. Burton Jones [9] showed that every homogeneous decomposable plane continuum is either a simple closed curve or a circle of homogeneous nonseparating plane continua. Using these results, we prove that every decomposable subcontinuum of a homogeneous indecomposable plane continuum contains a homogeneous indecomposable continuum. It follows that Bing's theorem remains true if the word "arc" is replaced by "hereditarily decomposable continuum."

1. **Introduction.** The simple closed curve, the pseudo-arc [2] [11], and the circle of pseudo-arcs [4] are the only plane continua known to be homogeneous. Does there exist a fourth homogeneous plane continuum? A history of this unsolved problem is given in [4] and [13]. According to Bing's theorem [3], if a fourth homogeneous plane continuum exists, it does not contain an arc. Our generalization (Theorem 2, Section 5) of Bing's result asserts that such a continuum cannot have a hereditarily decomposable subcontinuum. An example of a hereditarily decomposable plane continuum that does not contain an arc is given in [1].

2. **Definitions and preliminaries.** In this paper, a *continuum* is a nondegenerate compact connected metric space. A continuum is *decomposable* if it is the union of two proper subcontinua; otherwise, it is *indecomposable.* A continuum is *hereditarily decomposable* if all of its subcontinua are decomposable.

A continuum T is called a *triod* if it contains a subcontinuum Z such that T - Z is the union of three nonempty disjoint open sets. When a continuum does not contain a triod, it is said to be *atriodic.*

A continuum is *unicoherent* provided that if it is the union of two subcontinua E and F, then $E \cap F$ is connected. A continuum is *hereditarily unicoherent* if all of its subcontinua are unicoherent.

LEMMA 1. *If M is a homogeneous indecomposable plane continuum, then M is atriodic and hereditarily unicoherent.*

PROOF. Since M is indecomposable, it has uncountably many mutually exclusive composants [12, Theorems 138 and 139, p. 59]. Suppose M contains a triod. Since M is homogeneous, it follows that each of its composants contains a triod. This violates the fact that the plane does not contain uncountably many mutually exclusive triods [12, Theorem 84, p. 222]. Thus M is atriodic.

Suppose M contains a continuum F that is not unicoherent. Then each composant of M contains a continuum that is homeomorphic to F. Note that F separates the plane [12, Theorem 22, p. 175]. Since the plane is separable and M has uncountably many composants, there exists a proper subcontinuum of M having two complementary domains that intersect M. This contradicts the fact that each composant is dense in M [12, Theorem 135, p. 58]. Hence M is hereditarily unicoherent.

3. Decomposition properties. Let $\underline{D}$ be a collection of compact subsets of a continuum M such that each point of M is contained in one and only one element of $\underline{D}$. The collection $\underline{D}$ is said to be a *decomposition* of M. If in addition, for each open set Q of M, the set $\cup \{D \in \underline{D} : D \subset Q\}$ is open, then $\underline{D}$ is said to be an *upper semi-continuous decomposition* of M. Let k be the function that assigns to each point x of M the element of $\underline{D}$ that contains x. The function k is called the *quotient map* associated with $\underline{D}$. The collection $\underline{D}$ can be topologized by calling a subset V of $\underline{D}$ open if and only if $k^{-1}[V]$ is open. The space thus obtained is called the *quotient space* associated with $\underline{D}$.

Following E. S. Thomas [14], we define a continuum to be of *type A* provided it is irreducible between two of its points and admits an upper semi-continuous decomposition whose elements are connected sets and whose quotient space is the unit interval [0,1]. Such a decomposition is called *admissible.* If a continuum M is of type A and has an admissible decomposition, each of whose elements has void interior (relative to M), then M is said to be of *type A′*.

LEMMA 2. *Let M be a continuum that is irreducible between a pair of points. A necessary and sufficient condition that M be of type A′ is that every subcontinuum of M with nonvoid interior be decomposable [10, Theorem 3, p. 216] [14, Theorem 10,* *p. 15].*

LEMMA 3. *If M is a continuum of type A, then M has a unique minimal admissible decomposition (relative to the partial ordering by refinement) [14, Theorem 3, p. 8 and Theorem 6, p. 10].*

4. Homeomorphisms near the identity. *A topological transfromation group* (G,M) is a topological group G together with a topological space M and a continuous mapping $(g,x) \rightarrow gx$ of $G \times M$ into M such that $ex = x$ for all $x \in M$ (e denotes the identity of G) and $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in M$.

For each $x \in M$, let G_x be the isotropy subgroup of x in G (that is, the set of all $g \in G$ such that $gx = x$). Letting G/G_x be the left coset space with the usual topology, the mapping of G/G_x onto Gx that sends gG_x to gx is one-to-one and continuous. The set Gx is called the *orbit* of x.

Hereafter, M is a continuum with metric ρ and G is the topological group of homeomorphisms of M onto itself with the topology of uniform convergence [10, p. 88]. E. G. Effros [5, Theorem 2.1] proved that each orbit is a set of the type G_δ in M if and only if for each $x \in M$, the mapping $gG_x \rightarrow g(x)$ of G/G_x onto Gx is a homeomorphism.

LEMMA 4. *Suppose M is a homogeneous continuum, ϵ is a given positive number, and x is a point of M. Then x belongs to an open subset W of M having the following property. For each pair y, z of points of W there exists a homeomorphism h of M onto M such that $h(y) = z$ and $\rho(v, h(v)) < \epsilon$ for all v belonging to M.*

PROOF. Since M is homogeneous, the orbit of each point of M is M, a G_δ-set. Hence the mapping T_x: $g \rightarrow g(x)$, being the composition of the natural open mapping of G onto G/G_x and a homeomorphism of G/G_x onto M, is an open mapping of G onto M.

Let U be the open set consisting of all $g \in G$ such that $\rho(v, g(v)) < \epsilon/2$ for each $v \in M$. Define W to be the open set $T_x[U]$. Since the identity e belongs to U and $T_x(e) = x$, the set W contains x.

Assume y and z are points of W. Let f and g be elements of U such that $T_x(f) = y$ and $T_x(g) = z$. Since $f(x) = y$ and $g(x) = z$, the homeomorphism $h = gf^{-1}$ of M onto M has the property that $h(y) = z$ and $\rho(v, h(v)) < \epsilon$ for all $v \in M$. This completes the proof.

In [15], G. S. Ungar used the mapping T_X to prove that every 2-homogeneous continuum is locally connected. For other applications of T_X see [6] and [7].

5. Principal results.

THEOREM 1. *Suppose M is a homogeneous indecomposable plane continuum and A is a decomposable subcontinuum of M. Then A contains a homogeneous indecomposable continuum.*

PROOF. Since A is decomposable, there exist proper subcontinua B and C of A such that $A = B \cup C$. Let b and c be points of $B - C$ and $C - B$ respectively. Let E be a continuum in A that is irreducible between b and c.

The continuum E does not have an indecomposable subcontinuum with nonvoid interior (relative to E). To see this assume the contrary. Let I be an indecomposable continuum in E that contains a nonempty open subset Q of E. Since M is hereditarily unicoherent and I is indecomposable, I is contained in $E \cap B$ or $E \cap C$. Assume without loss of generality that I is a subcontinuum of $E \cap B$. Let F be the c-component of $E - Q$. Since F is a continuum that does not contain I and M is hereditarily unicoherent, F meets only one composant of I. Let x be a boundary point of Q that belongs to F. By Lemma 4, there exist homeomorphisms f and g of M onto M (near the identity) such that (1) x, f(x), and g(x) belong to distinct composants of I, (2) $\{ c, f(c), g(c) \}$ is a subset of $M - B$, and (3) Q is not contained in $f[F] \cup g[F]$. Note that F, f[F], and g[F] are mutually disjoint. It follows that $B \cup F \cup f[F] \cup g[F]$ is a triod, which contradicts Lemma 1. Hence every subcontinuum of E with nonvoid interior is decomposable.

According to Lemma 2, E is of type A′. By Lemma 3, E has a unique minimal admissible decomposition $\underline{D}$, each of whose elements has void interior. Let $k: E \to [0,1]$ be the quotient map associated with $\underline{D}$.

There exists a number $s\,(0 < s < 1)$ such that $k^{-1}(s)$ is not degenerate; for otherwise, E would contain an arc [14, Theorem 21, p. 29] and M would be a simple closed curve [3], which contradicts the assumption that M is indecomposable. Let Y denote the continuum $k^{-1}(s)$.

Let p and q be distinct points of Y. We shall prove that Y is a homogeneous subcontinuum of A by establishing the existence of a homeomorphism of Y onto itself that takes p to q.

Let r and t be numbers such that $0 < r < s < t < 1$. Define ϵ to be $\rho(k^{-1}[[r,t]], k^{-1}(0) \cup k^{-1}(1))$.

Let $\underline{W}$ be an open cover of Y such that for each $W \in \underline{W}$, if y, $z \in W$, then there exists a homeomorphism h of M onto M such that $h(y) = z$ and $\rho(v, h(v)) < \epsilon$ for all $v \in M$ (Lemma 4). Since Y is a continuum, there exists a finite sequence $\{W_i\}_{i=1}^{n}$ of elements of $\underline{W}$ such that $q \in W_1$, $p \in W_n$, and $W_i \cap W_{i+1} \neq \emptyset$ for $1 \leqslant i < n$.

Choose $\{P_i\}_{i=0}^{n}$ such that $p_0 = q$, $p_n = p$, and $p_i \in W_i \cap W_{i+1}$ for $0 < i < n$. For each i $(1 \leqslant i \leqslant n)$, let h_i be a homeomorphism of M onto M such that $h_i(p_i) = p_{i-1}$ and $\rho(v, h_i(v)) < \epsilon$ for all $v \in M$.

Each h_i maps Y into itself. To see this, assume $h_i[Y]$ is not contained in Y. Note that $h_i k^{-1}[[r,t]]$ does not meet $k^{-1}(0) \cup k^{-1}(1)$. Since $Y \cap h_i[Y] \neq \emptyset$ and M is atriodic and hereditarily unicoherent, $h_i[Y] \subset h_i k^{-1}[[r,t]] \subset E$. Let d and e be numbers such that $kh_i[Y] = [d,e]$. Since E is irreducible between b and c, each element of $\underline{H} = \{k^{-1}(u) : d < u < e\}$ is in $h_i[Y]$.

Because $h_i[Y] \cap h_i[k^{-1}(r) \cup k^{-1}(t)] = \emptyset$, the set $h_i[k^{-1}(r) \cup k^{-1}(t)]$ is in $k^{-1}[[0,d] \cup [e,1]]$. Let R and T be the components of $h_i[E] - h_i[Y]$ that contain $h_i k^{-1}(r)$ and $h_i k^{-1}(t)$ respectively. Note that $R \cup T$ is in $k^{-1}[[0,d] \cup [e,1]] \cup h_i k^{-1}[[0,r] \cup [t,1]]$. Since $h_i k^{-1}(r)$ separates $h_i k^{-1}[[0,r)]$ from $h_i[Y]$ in $h_i[E]$, and since $h_i k^{-1}(t)$ separates $h_i k^{-1}[(t,1]]$ from $h_i[Y]$ in $h_i[E]$, the closure of $R \cup T$ does not meet an element of $\underline{H}$. Hence $h_i[Y]$ contains a nonempty open subset of $h_i[E]$, which contradicts the fact that $h_i[Y]$ is an element of $\{h_i k^{-1}(u) : 0 \leqslant u \leqslant 1\}$, the minimal admissible decomposition of $h_i[E]$. It follows that $h_i[Y]$ is a subset of Y.

Using the same argument, one can show that if Y is not contained in $h_i[Y]$, then Y has nonvoid interior relative to E, which contradicts the fact that Y is an element of $\underline{D}$. Thus $h_i[Y]$ contains Y. Consequently, each h_i maps Y onto itself.

It follows that $h_1 h_2 \cdots h_n | Y$ is a homeomorphism of Y onto Y that takes p to q. Hence Y is homogeneous.

Since Y is a unicoherent homogeneous plane continuum, it is indecomposable [9, Theorem 2].

COROLLARY. *If M is a homogeneous indecomposable plane continuum, then no subcontinuum of M is hereditarily decomposable.*

THEOREM 2. *The simple closed curve is the only homogeneous plane* *continuum that has a hereditarily decomposable subcontinuum.*

PROOF. Let M be a homogeneous plane continuum that has a hereditarily decomposable subcontinuum H. According to the preceding corollary, M is decomposable.

Assume M is not a simple closed curve. Let G be Jones' decomposition of M (to a circle) having the property that each of its elements is a homogeneous, indecomposable, nonseparating plane continuum [8, Theorem 2] [9, Theorem 2]. It follows from our corollary that H does not lie in one element of G. However, if H meets more than one element of G, then H contains an element of G [9, Theorem 1], which contradicts the assumption that H is hereditarily decomposable. Hence M is a simple closed curve.

ADDED IN PROOF. In a subsequent paper, the author has improved Theorem 1 by proving that no subcontinuum of a homogeneous indecomposable plane continuum is decomposable. It follows that every homogeneous nonseparating plane continuum is hereditarily indecomposable.

REFERENCES

1. R. D. Anderson and G. Choquet, *A plane continuum no two of whose non-degenerate subcontinua are homeomorphic: An application of inverse limits,* Proc. Amer. Math. Soc. 10(1959), 347-353.
2. R H Bing, *A homogeneous indecomposable plane continuum,* Duke Math J. 15(1948), 729-742.
3. ——, *A simple closed curve is the only homogeneous bounded plane continuum that contains an arc,* Canad. J. Math. 12(1960), 209-230.
4. R H Bing and F. B. Jones, *Another homogeneous plane continuum,* Trans. Amer. Math. Soc. 90(1959), 171-192.
5. E. G. Effros, *Transformation groups and C*-algebras,* Ann. of Math. 81(1965), 38-55.
6. C. L. Hagopian, *A fixed-point theorem for homogeneous continua,* Michigan Math. J. 21(1974), 233-234.
7. ——,*The fixed-point property for homeomorphisms of almost chainable homogeneous continua,* (to appear).
8. F. B. Jones, *Certain homogeneous unicoherent indecomposable continua,* Proc. Amer. Math. Soc. 2(1951), 855-859.
9. ——, *On a certain type of homogeneous plane continuum,* Proc. Amer. Math. Soc. 6(1955), 735-740.
10. K. Kuratowski, *Topology Vol. 2, 3rd Ed.,* Monografie Mat., Tom 21, PWN, Warsaw, 1961; English transl., Academic Press, New York; PWN, Warsaw, 1968.
11. E. E. Moise, *A note on the pseudo-arc,* Trans. Amer. Math. Soc. 67(1949), 57-58.
12. R. L. Moore, *Foundations of point set theory,* Rev. Ed., Amer. Math. Soc. Colloq. Publ., Vol. 13, Amer. Math. Soc., Providence, R. I., 1962.
13. J. T. Rogers, Jr., *The pseudo-circle is not homogeneous,* Trans. Amer. Math. Soc. 148(1970),

417-428.

14. E. S. Thomas, Jr., *Monotone decompositions of irreducible continua,* Rozprawy Mat. 50(1966), 1-74.
15. G. S. Ungar, *On all kinds of homogeneous spaces,* Trans. Amer. Math. Soc. (to appear).

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