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ON THE DIMENSIONAL CAPACITY OF COMPACT SEMILATTICES

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The question of topological dimension is settled for compact abelian groups with the following classical result [7]:

PROPOSITION 0.1 *If G is a compact abelian group with character group* $\bar{G}$ *and n a natural number, the following statements are equivalent:*

(1) *The (Lebesgue covering or cohomological) dimension of G is n.*

(2) *The (torsion free) rank of* $\bar{G}$ *is n.*

(3) *There is a quotient morphism* $G \to \mathbf{T}^n$ *(with zero dimensional kernel), where* $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

(4) *There is an injective morphism* $\mathbf{Z}^n \to \bar{G}$ *(with a torsion cokernel).*

While in general it is not feasible to assign to a compact space a transfinite cardinal as topological dimension, the preceding theorem allows us to do precisely that for compact abelian groups, because statements (2), (3) and (4) remain meaningful for arbitrary cardinals n. Thus, let us make the following definition, denoting with $|X|$ the cardinal of a set X:

DEFINITION 0.2. For a compact abelian group G we set $\#G = \sup\ \{|X| :$ there is a continuous surmorphism $G \to \mathbf{T}^X\}$ and we call this cardinal the (*generalized*) *dimension* of G.

We then have the following conclusive result:

PROPOSITION 0.3. *For each compact abelian group G we have* $\#G = \text{rank}\ \bar{G} = \dim_{\mathbf{Q}} \mathbf{Q} \otimes \bar{G}$, *and there is a continuous surmorphism* $G \to \mathbf{T}^{\#G}$ *with zero dimensional kernel.*

If we now turn to the much larger class of compact abelian monoids, topological dimension becomes much more elusive. This study is a contribution to the question of

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approaching this concept through algebraic tools in the spirit which we have illustrated for the classical case of groups above. Previous work [2] indicates that the problem has to be attacked for compact semilattices as a first order of business, and so we will proceed. We recall first that in our terminology [1-5] a *semilattice* is a commutative idempotent monoid.

DEFINITION 0.4. For a compact semilattice S we set $\#S = \sup \{ |X| :$ there is a continuous surmorphism $S \to I^X\}$, where $I = [0,1]$ with the minimum multiplication; and we call #S the *dimensional capacity* of S.

The Cantor semilattice C (i.e. the ordinary Cantor set $C \subseteq I$ under the induced minimum multiplication) has dimensional capacity 1 and is itself topologically zero dimensional, thus illustrating right in the beginning that the deviations from group theory will be considerable. As a further example let us observe that, at variance with the group situation, dimensional capactiy may not be "attained", i.e. in the definition we may not replace "sup" by "max" :

EXAMPLE 0.5 Consider $T = \prod_{n=1}^{\infty} I^n$ and define $L_n \subseteq T$ by $(x_m)_{m=1,\dots} \in L_n$ if $x_m = 0$ for $m \neq n$. We let $D \subseteq T$ be the diagonal consisting of all $(x_m)_{m=1,\dots}$ for which there is a $t \in I$ with $x_m = (t,\dots,t)$ (m times) for all $m = 1,\dots$. Set $S = D \cup [\bigcup_{n=1}^{\infty} L_n]$. Then we have projections $pr_n \colon S \to I^n$, $n = 1,\dots$; but if we had a surmorphism $f : S \to I^{\mathbf{N}}$, by the Baire category theorem, at least one of the $f(L_n)$ would have interior points, entailing thereby the existence of some surmorphism $L_n \to I^{\mathbf{N}}$, $L_n \cong I^n$, which is not possible, for instance for reasons of breadth. Thus $\#S = \aleph_0$, but no cube quotient of S is infinite dimensional.

Lawson has observed that an n-dimensional compact Lawson similattice S has I^n as a quotient ([6], Corollary 2.3), and it then follows that #S is the sup of the dimensions of those Lawson semilattices which are quotients of S.

The following definition leads to an analog to Proposition 0.3.

DEFINITION 0.6. Let Q be the semilattice of all rationals in [0,1] under min and denote with ${}^XQ$ the coproduct of X copies of Q. If L is any semilattice, we let

$$\text{rank } L = \sup \{ |X| : \text{there is an injective morphism } {}^XQ \to L\}$$

and call this cardinal the *rank* of L.

(In [2] this number was denoted Br L.) Then we have

PROPOSITION 0.7. [2] *For each compact zero dimensional semilattice S we* *have* $\#S = \text{rank}\,\overline{S}$.

Here $\overline{S}$ is the character semilattice Hom(S,2), $2 = \{0,1\}$. A more sophisticated characterization is called for if one wishes to consider compact semilattices which are not zero dimensional (see Proposition 2.1.i below).

In the case of groups one knows that the torsion free rank, like vector space dimension, is an additive invariant; i.e. it satisfies rank $\Pi_J A_J = \Sigma_J$ rank A_j for any family $\{A_j : j \in J\}$ of abelian groups. But then duality and Proposition 0.3 immediately yield the fact that generalized dimension is logarithmic.

PROPOSITION 0.8. *For any family of compact abelian groups* $\{G_j : j \in J\}$ *one has*

$$\# \Pi_J G_j = \Sigma_J \# G_j.$$

We do not know of a direct proof that the rank of semilattices (Definition 0.6) is additive, but even if such a proof does exist, duality and Proposition 0.7 would establish the logarithmic property of dimensional capacity only for compact zero-dimensional semilattices. However, it is the purpose of this paper to establish the following result:

MAIN THEOREM 0.9 *For any family* $\{S_j : j \in J\}$ *of compact semilattices one has*

$$\# \Pi_J S_j = \Sigma_J \# S_j.$$

REMARK 0.10. If, in the Main Theorem, one of $|J|$ or $\sup_J \# S_j$ at least is infinite, then

$$\# \Pi_J S_j = \max \{ |J|, \sup_J \# S_j \}.$$

As a first application, it is now easy to derive that the rank of semilattices is additive from the main theorem:

COROLLARY 0.11. *For any family* $\{L_j : j \in J\}$ *of semilattices one has*

$$\text{rank}\ \Pi_J L_j = \Sigma_J\ \text{rank}\ L_j.$$

PROOF. Using duality [3] and Proposition 0.7 above we calculate rank $\Pi_J L_j = \# \Pi_J \overline{L_j} = \Sigma_J \# \overline{L_j} = \Sigma_J$ rank L_j, invoking the Main Theorem.

As a second application, we prove one of the major results of [2]. We recall that a compact monoid S is dimensionally stable if no quotient raises topological dimension. Thus, a compact zero dimensional semilattice S is dimensionally stable iff $\#S = 0$. As an immediate consequence of the Main Theorem we obtain

COROLLARY 0.12. [2] *The product of compact zero-dimensional, dimensionally stable semilattices is dimensionally stable.*

Our methods will provide a proof of the additivity of the breadth of semilattices in the same spirit as we have established the additivity of the rank. We recall [2]:

DEFINITION 0.13. If L is a semilattice, we let

$$\text{br } L = \sup \{ |X| : \text{there is an injective morphism } {}^X2 \to L \}$$

and call this cardinal the *breadth* of L.

One notes that ${}^X2$ is the free semilattice in X generators.

Duality and Definition 0.4 both motivate the introduction of a dual cardinal invariant:

DEFINITION 0.14. If S is a compact semilattice, we let

$$\text{cobr } S = \sup \{ |X| : \text{there is a continuous surmorphism } S \to 2^X \}$$

and call this cardinal the *co-breadth* of S.

PROPOSITION 0.15. *For any compact semilattice S we have*

$$\text{cobr } S = \text{br } \overline{S}.$$

PROOF. Let $S \mapsto \widetilde{S}$ denote the left reflection of the category $\underline{C}$ of all compact semilattices into the subcategory $\underline{Z}$ of all compact zero-dimensional semilattices, i.e. $\widetilde{S}$ is the quotient of S modulo the connectivity congruence. Since $2^X \in \underline{Z}$, then every surmorphism $S \to 2^X$ factors uniquely through the quotient map $S \to \widetilde{S}$ with a surmorphism $\widetilde{S} \to 2^X$. Thus cobr S = cobr $\widetilde{S}$. Further, since $2 \in \underline{Z}$, the front adjunction $S \to \widetilde{S}$ induces an isomorphism $(\widetilde{S})^- = \text{Hom}(\widetilde{S},2) \to \text{Hom}(S,2) = \overline{S}$, whence br $\overline{S} = \text{br}(\widetilde{S})^-$. But Pontryagin duality between $\underline{Z}$ and the category $\underline{S}$ of semilattices yields cobr $\widetilde{S} = \text{br}(\widetilde{S})^-$. The assertion follows.

We shall prove

THEOREM 0.16. *For any family* $\{S_j : j \in J\}$ *of compact semilattices one has*

$$\text{cobr } \Pi_J S_j = \Sigma_J \text{ cobr } S_j .$$

COROLLARY 0.17. *For any family* $\{L_j : j \in J\}$ *of semilattices one has*

$$\text{br } \Pi_J L_j = \Sigma_J \text{ br } L_j.$$

PROOF. Apply 0.16 to $\{L_j : j \in J\}$, recalling $(\Pi_J L_j)^- \cong \Pi_J \overline{L_j}$ and 0.15.

Finally a word on our methods: In the case of groups, the results 0.1, 0.3 are proved by Pontryagin duality. For compact semilattices in general a duality of such a simple nature is not available. We therefore resort to techniques which have become recently available through the use of Galois connections for compact semilattices [5]. These methods have already proved successful in [1, 2, 4, 5] and turn out to be very useful in the present context. We complement the principal results which we outlined above by a discussion of some related cardinality invariants which in many cases permit concrete calculations of dimensional capacity and co-breadth.

1. Background information on Galois connections. Much of what follows is based on results proved in [5]. The first is fairly elementary:

LEMMA 1.1. *Let S and T be partially ordered sets. Then for any two functions* $d : S \to T$ *and* $g : T \to S$ *the following statements are equivalent:*

(1) $g(t) \geqslant s$ *iff* $t \geqslant d(s)$ *for all* $t \in T$, $s \in S$.

(2) $d(s) = \inf g^{-1}(\uparrow s)$ *for all* $s \in S$, *where* $\uparrow s = \{ x \in S : s \leqslant x \}$.

(3) $g(t) = \sup d^{-1}(\downarrow t)$ *for all* $t \in T$, *where* $\downarrow t = \{ x \in T : x \leqslant t \}$.

If these conditions are satisfied, then one component in the pair (g,d) determines the other uniquely according to (2,3), and d preserves all existing sups and g all existing infs. Also, d is injective [surjective] iff g is surjective [injective].

The second, however, is less elementary:

THEOREM 1.2 *Let S and T be compact semilattices and T a Lawson semilattice. Then the pairing* $g \leftrightarrow d$ *described in Lemma 1.1 establishes a bijection between the set of all continuous semilattice morphisms* $g : T \to S$ *and the set of all functions* $d : S \to T$ *satisfying these conditions:*

(I) *d preserves arbitrary sups.*

(II) $d(\operatorname{int} \uparrow s) \subseteq \operatorname{int} \uparrow d(s)$ *for all* $s \in S$.

NOTATION 1.3. A pair of functions (g,d) as described in 1.1 is called a *Galois connection* between S and T. The function g is called the *left adjoint* (gauche) of d, and d is called the *right adjoint* (droit) of g. Any map $d : S \to T$ between compact semilattices satisfying (I, II) of 1.2 will be called a *co-morphism*.

LEMMA 1.4. *Let* $d : S \to T$ *be a co-morphism between compact semilattices and suppose that T is a Lawson semilattice. If* $e \in S$, *then* $d(\uparrow e) \subseteq \uparrow d(e)$, *and the restriction and corestriction* $d_1 : \uparrow e \to \uparrow d(e)$ *is a co-morphism.*

PROOF. Let $g : T \to S$ be the left adjoint of d defined by 1.1.(3). Since d is monotone by 1.1.1, $e \leqslant s$ implies $d(e) \leqslant d(s)$, whence $d(\uparrow e) \subseteq \uparrow d(e)$. Conversely let $t \in \uparrow d(e)$, i.e. $t \geqslant d(e)$. Then $g(t) \geqslant e$ by 1.1.1, i.e. $g(t) \in \uparrow e$. Thus $g(\uparrow d(e)) \subseteq \uparrow e$. If

$g_1 : \uparrow d(e) \to \uparrow e$ is the restriction and corestriction of g, then (g_1, d_1) evidently is a Galois connection, since it satisfies 1.3.1. Thus g_1 is a left adjoint, and since $\uparrow d(e)$ is a Lawson semilattice, 1.2 shows that d_1 is a co-morphism.

We will now see that various functions occuring naturally in the context of products are components of Galois connections.

PROPOSITION 1.5. *Let* $\{S_j : j \in J\}$ *be a family of compact semilattices, let* $P = \Pi_J S_j$ *and denote with* $p_k : P \to S_k$ *the projection and with* $g_k : S_k \to P$ *the map given by* $p_j g_k(s) = s$ *for* $j = k$ *and* $= 1$ *otherwise. Set* $d_k : S_k \to P$, $p_j d_k(s) = s$ *for* $j = k$ *and* $= 0$ *otherwise. Then*

(i) g_k *is the left adjoint of* p_k,

(ii) d_k *is the right adjoint of* p_k.

In particular, any product projection is always a morphism and a co-morphism at the same time.

PROOF. (i) $g_k(s) \geqslant (s_j)_{j \in J}$ iff $s \geqslant s_k$ (by the definition of g_k) iff $s \geqslant p_k((s_j)_{j \in J})$.

(ii) $p_k((s_j)_{j \in J}) \geqslant s$ iff $s_k \geqslant s$ iff $(s_j)_{j \in J} \geqslant d_k(s)$ (by the definition of d_k).

In the setting of 1.5 let $K \subseteq J$ and denote with $p_K : P \to S_K = \Pi_K S_j$ the projection and with $g_K : S_K \to P$ the map given by $p_j g_K((s_i)_{i \in K}) = s_j$ for $j \in K$ and $= 1$ for $j \notin K$. Set $d_K : S_K \to P$, $p_j d_K((s_i)_{i \in K}) = s_j$ for $j \in K$ and $= 0$ for $j \notin K$. The following is then an immediate consequence of 1.5:

COROLLARY 1.6. *Under the hypotheses of 1.5 we have*

(i) g_K *is the left adjoint of* p_K,

(ii) d_K *is the right adjoint of* p_K.

We now provide some technical information involving products. Let $\{S_j : j \in J\}$ be a family of compact semilattices. For $K \subseteq J$ we abbreviate $S_K = \Pi_K S_j$. Let $T_k \subseteq S_k$ be a sup-closed subset containing 0_k and 1_k, define T_K accordingly and let $\psi_k : T_k \to S_k$ and $\psi_K = \Pi_K \psi_k : T_K \to S_K$ be the inclusion morphisms. Let $g_k^K : S_k \to S_K$ be the left adjoint of the projection $S_K \to S_k$ and let $\bar{g}_k^K : T_k \to T_K$ be obtained by restricting and corestricting g_k^K.

Now assume that $d : S_J \to S$ is a sup-preserving map between compact semilattices and define $\phi = d d_K \psi_K$. Since ψ_K, d_K and d all preserve sups, ϕ preserves sups. The situation is best visualized by the following commutative diagram:

$$\begin{array}{ccccc} S_k & \xrightarrow{g_k^K} & S_K & \xrightarrow{d_K} & S_J \\ \uparrow{\scriptstyle \psi_k} & & \uparrow{\scriptstyle \psi_K} & & \downarrow{\scriptstyle d} \\ T_k & \xrightarrow[\bar{g}_k^K]{} & T_K & \xrightarrow[\phi]{} & S \end{array},$$

LEMMA 1.7. *The following statements are equivalent:*

(1) ϕ *is injective.*

(2) $\phi\bar{g}_k^K$ *is injective for all* $k \in K$.

Moreover, these conditions are implied by

(3) $dg_k^J\psi_k$ *is injective for all* $k \in K$.

PROOF. (1) $\Rightarrow$ (2) is trivial since $\bar{g}_k^K$ is injective.

(2) $\Rightarrow$ (1): We assume (2) and suppose $\phi((s_i')_{i\in K}) = \phi((s_i)_{i\in K})$ with $s_i \leqslant s_i'$ in T_i for $i \in K$; we must show $s_k = s_k'$ for all $k \in K$. Let $k \in K$ be arbitrary and define an element $(e_i)_{i\in K} \in S_K$ (depending on k) by

$$e_j = 0 \text{ for } j = k \text{ and } = 1 \text{ for } j \neq k.$$

We then use the definition of $\bar{g}_k^K$ and the sup-preservation of ϕ to calculate $\phi\bar{g}_k^K(s_k) = \phi((s_i)_{i\in K} \vee (e_i)_{i\in K}) = \phi((s_i)_{i\in K}) \vee \phi((e_i)_{i\in K}) = \phi((s_i')_{i\in K}) \vee \phi((e_i)_{i\in K}) = \phi((s_i')_{i\in K} \vee (e_i)_{i\in K}) = \phi\bar{g}_k^K(s_k')$. Invoking (2) we conclude $s_k = s_k'$.

(3) $\Rightarrow$ (2): We assume (3) and suppose $s \leqslant s'$ in T_k with $\phi\bar{g}_k^K(s) = \phi\bar{g}_k^K(s')$ in S; we must show $s = s'$. We now introduce elements $(s_j)_{j\in J}, (s_j')_{j\in J}, (f_j)_{j\in J} \in S_J$ as follows:

$$s_j = \begin{cases} s \\ 1 \\ 0 \end{cases}, \quad s_j' = \begin{cases} s' \\ 1 \\ 0 \end{cases}, \quad f_j = \begin{cases} 0 \\ 0 \\ 1 \end{cases} \text{ for } \begin{array}{l} j = k \\ j \neq k,\ j \in K. \\ j \notin K \end{array}$$

Then

(i) $(s_j)_{j\in J} = d_K g_k^K \psi_k(s) = d_K \psi_K \bar{g}_k^K(s)$, and $(s_j')_{j\in J} = d_K g_k^K \psi_k(s') = d_K \psi_K \bar{g}_k^K(s')$, by the commutativity of diagram (D). Recalling $\phi = dd_K\psi_K$ and the hypothesis on s and s′ we obtain

(ii) $d((s_j)_{j\in J}) = \phi\bar{g}_k^K(s) = \phi\bar{g}_k^K(s') = d((s_j')_{j\in J})$.

Our definitions also yield

(iii) $g_k^J\psi_k(s) = (s_j)_{j\in J} \vee (f_j)_{j\in J}$, and $g_k^J\psi_k(s') = (s_j')_{j\in J} \vee (f_j)_{j\in J}$.

We apply d to both identities in (iii), recall that d preserves sups and obtain from (ii) the relation

(iv) $dg^J_k\psi_k(s) = dg^J_k\psi_k(s')$.

But then condition (3) implies $s = s'$ as desired.

LEMMA 1.8. *Let $d\colon S \to \Pi_J S_j$ be a co-morphism between compact semilattices and suppose that $s \in S$ is such that int $\uparrow s \neq \emptyset$. Then there is a finite set $F \subseteq J$ and a factorization*

$$\begin{array}{ccc} \downarrow s & \longrightarrow & \Pi_F S_j \\ {\scriptstyle \text{incl}}\downarrow & & \downarrow{\scriptstyle d_F} \\ S & \underset{d}{\longrightarrow} & \Pi_J S_j \end{array}$$

PROOF. By 1.1.2 we must have $d(\text{int}\uparrow s) \subseteq \text{int}\uparrow d(s)$. In order that $\uparrow d(s)$ have a non-empty interior in $\Pi_J S_j$ it is necessary and sufficient that there be a finite set $F \subseteq J$ such that $p_j d(s) = 0$ for $j \notin F$ and that $\uparrow p_j d(t)$ have non-empty interior in S_j for $j \in F$. Thus $d(\downarrow s) \subseteq d_F(\Pi_F S_j)$. Since d is monotone and $d_F(\Pi_F S_j)$ is an ideal, the assertion follows.

LEMMA 1.9. *Let $\{T_x\colon x \in X\}$ and $\{S_j\colon j \in J\}$ be families of compact semilattices and let $d\colon \Pi_X T_x \to \Pi_J S_j$ be a co-morphism. Let $y \in X$ and suppose that $t \in T_y$ is such that int $\uparrow t \neq \emptyset$. Then there exists a finite subset $F \subseteq J$ and a factorization*

$$\begin{array}{ccc} \downarrow t & \longrightarrow & \Pi_F S_j \\ {\scriptstyle \text{incl}}\downarrow & & \\ T_y & & \downarrow{\scriptstyle d_F} \\ {\scriptstyle g_y}\downarrow & & \\ \Pi_X T_x & \longrightarrow & \Pi_J S_j \end{array}$$

PROOF. Let e be the zero of the filter $g_y(T_y)$ in $\Pi_X T_x$, and set $d(e) = (s_j)_{j \in J}$. Then $\uparrow d(e) = \Pi_J \uparrow s_j$, and the restriction and corestriction $d_1\colon \uparrow e \to \uparrow d(e)$ is a co-morphism by 1.4. We have a factorization

$$\text{(i)} \qquad \begin{array}{ccc} T_y & \overset{d_1}{\longrightarrow} & \Pi_J \uparrow s_j \\ {\scriptstyle g_y}\downarrow & & \downarrow{\scriptstyle \text{incl}} \\ \Pi_X T_x & \underset{d}{\longrightarrow} & \Pi_J S_j \end{array}$$

We apply 1.8 to d_1 and obtain with a suitable finite subset F of J a factorization

$$\text{(ii)}\qquad \begin{array}{ccc} \downarrow t & \longrightarrow & \Pi_F \uparrow s_j \\ \text{incl}\ \big\downarrow & & \big\downarrow \\ T_y & \xrightarrow[d_1]{} & \Pi_J \uparrow s_j \end{array}$$

Since the composition $\Pi_F \uparrow s_j \xrightarrow{d_1} \Pi_J \uparrow s_j \xrightarrow{\text{incl}} \Pi_J S_j$ factors through d_F: $\Pi_F S_j \to \Pi_J S_j$, the assertion follows.

2. The logarithmic property of dimensional capacity. With the preliminaries provided in Section 1 we are ready for the proof of the main results.

PROPOSITION 2.1. *Let S be a compact semilattice.*

(i) *The dimensional capacity is given by* $\#S = \sup \{|X|$: *there is an injective co-morphism* $I^X \to S\}$.

(ii) *The co-breadth is given by* $\text{cobr } S = \sup \{|X|$: *there is an injective co-morphism* $2^X \to S\}$.

PROOF. This is immediate from definitions 0.4 and 0.14 and from Theorem 1.2.

DEFINITION 2.2. Let $\{S_j: j \in J\}$ be a family of compact semilattices and let T be either I or 2. Suppose that $d: T^X \to \Pi_J S_j$ is an injective co-morphism. Let g_x: $T \to T^X$ be the left adjoint of the projection p_x: $T^X \to T$. For each $k \in J$ we define X(k) to be the set of all $x \in X$ such that $p_k d g_x$: $T \to S_k$ is injective on some non-degenerate subinterval of T. We then say that $p_k d g_k$ is *somewhere injective*. (Note that in case T = 2 this property is injectivity itself.)

LEMMA 2.3. (i) *If* $T = I$, *then* $\#S_k \geqslant |X(k)|$ *for each* $k \in J$.

(ii) *If* $T = 2$, *then* $\text{cobr } S_k \geqslant |X(k)|$ *for each* $k \in J$.

PROOF. (i) Let T = I. Since $p_k d g_x$ is somewhere injective, there are elements $0 \leqslant u_x < v_x \leqslant 1$ such that, with $T_x = \{0\} \cup [u_x, v_x] \cup \{1\}$, the function $p_k d g_x \psi_x$: $T_x \to S_k$ is injective, where ψ_x: $T_x \to I$ is the inclusion map. This inclusion map preserves sups. Let $\psi_{X(k)} = \Pi_{X(k)} \psi_x$: $\Pi_{X(k)} T_x \to I^{X(k)}$ and let $d_{X(k)}$: $I^{X(k)} \to I^X$ be the right adjoint of the projection $p_{X(k)}$: $I^X \to I^{X(k)}$ (see 1.6). Define ϕ : $\Pi_{X(k)} T_x \to S_k$ by $\phi = p_k d d_{X(k)} \psi_{X(k)}$. Now $p_k d$ is a co-morphism by 1.5; in particular it preserves sups, thus Lemma 1.7 applies and shows that ϕ is injective in view of the definition of X(k), according to which all $p_k d g_x \psi_x$ are injective for $x \in X(k)$. Each ψ_x: $T_x \to I$ is a co-morphism, so $\psi_{X(k)}$ is a co-morphism as a product of co-morphisms. By 1.6, $d_{X(k)}$ is a co-morphism, by hypothesis d is a co-morphism, and by 1.5 p_k is a co-morphism. Hence the composition ϕ is a co-morphism. Then $\phi|\Pi_{X(k)}(\{0\} \cup [u_x, v_x])$ is a co-morphism into S_k whose domain is lattice isomorphic to $I^{X(k)}$. By 2.1.i we then have $|X(k)| \leqslant \#S_k$.

(ii) Let T = 2. Since $p_k dg_x$ is injective, the preceding proof applies with $T_x = T = T$, $\psi_k = 1_T$, $\psi_{X(k)} = 1_{2(X(k))}$.

LEMMA 2.4. *Let d_j: $I \to S$, $j \in J$ be a finite family of monotone maps between partially ordered sets, and suppose that it separates points. Then there is a $k \in J$ and a non-degenerate interval $T_k \subseteq I$ such that $d_k|T_k$ is injective.*

PROOF. We proceed by induction with respect to $|J|$. For $|J| = 1$ the Lemma is trivial. Suppose $|J| = n > 1$ and assume that the assertion is true whenever $|J| < n$. Pick an $i \in J$ and consider d_i: $I \to S$. If d_i is injective, we are done. If not, then the kernel relation R of d_i has at least one non-degenerate coset R(t). Since d_i is monotone, R(t) is an interval in I, hence contains a non-degenerate closed interval I'. Since $\{d_j|I' : j \in J\}$ separates the points and $d_i|I'$ is constant, then $\{d_j|I' : j \in J \setminus \{i\}\}$ separates the points. The induction hypothesis applies and shows that for some $k \in J \setminus \{i\}$ there is a non-degenerate interval T_k in I' such that $d_k|T_k$ is injective.

LEMMA 2.5. *Under the conditions of 2.2 we have $X = \cup \{X(k) : k \in J\}$.*

PROOF. Let $y \in X$. We must show that there is a $k \in J$ such that $p_k dg_y$ is somewhere injective. If T = I we set $t = 1/2 \in T = I$; if T = 2 we take $t = 1 \in T = 2$. We apply Lemma 1.9 and find a finite set $F \subseteq J$ and a factorization

$$\begin{array}{ccc} T' & \xrightarrow{\quad a \quad} & \Pi_F S_j \\ \text{incl} \downarrow & & \\ T & & \downarrow d_F \\ g_y \downarrow & & \\ X & \xrightarrow[\quad d \quad]{} & \Pi_J S_j \end{array}$$

where $T' = [0,1/2]$ if T = I and $T' = 2$ if T = 2. By Lemma 2.4 applied to $\{p_j a : j \in F\}$ we find a $k \in F$ such that $p_k a$: $T' \to S_k$ is somewhere injective. It follows that $p_k d_F a$: $T' \to S_k$ is somewhere injective with the projection p_k: $\Pi_J S_j \to S$. By the commutativity of the diagram this means that $p_k dg_y$ is somewhere injective.

We can now finish the proofs of the main theorems 0.9 and 0.16. Under the conditions of 2.2, in case (i) T = I we have $|X| \leqslant \Sigma_J |X(j)|$ by 2.5, thus $|X| \leqslant \Sigma_J \#S_j$ by 2.3.i. By 2.1.i we conclude

(1) $\#\Pi_J S_j \leqslant \Sigma_J \#S_j$.

In case (ii), T = 2 we obtain in the same fashion

(2) $\mathrm{cobr}\ \Pi_J S_j \leqslant \Sigma_J\ \mathrm{cobr}\ S_j$.

The following Lemma suffices to prove the easier inverse inequalities in order to establish equality in (1) and (2).

LEMMA 2.6. *Let* $f_j\colon S_j \to T^{X(j)}$, $j \in J$ *be surmorphisms of compact semilattices with* $T = I$ *or* $T = 2$. *then*

$$\Sigma_J |X(j)| \leqslant \sup\ \{\ |Y|\colon \text{there is a surmorphism } \Pi_J S_j \to T^Y\ \}.$$

PROOF. Without loss of generality, we may assume that the X(j) are disjoint. Then $\psi : \Pi_J T^{X(j)} \to T^X$, $\psi((t_x)_{x \in X[j]})_{j \in J} = (t_x)_{x \in X}$ is an isomorphism and the function $f\colon\ \Pi_J S_j \to\ T^X$, $f((s_j)_{j \in J}) =\ \psi((f_j(s_j))_{j \in J})$ is a surmorphism. From $|X| = \Sigma_J |X(j)|$ the assertion follows.

The proofs of 0.9 and 0.16 are now complete.

3. **Supplementary information on cardinal invariants.** It is sometimes useful to compare dimensional capacity and cobreadth with other cardinal invariants. We consider two such, the first is lattice theoretical, the second topological.

DEFINITION 3.1. Let L be a complete lattice. For $A \subseteq L$ we set $h(A) = \min\{\ |B|\colon B \subseteq A,\ \sup B = \sup A\ \}$ and $H(L) = \sup\ \{h(A) : A \subseteq L\}$. The cardinal H(L) is called the *height* of L.

PROPOSITION 3.2. *If S is a compact semilattice, then* $H(S) \geqslant \#S$, *and* $H(S) \geqslant \mathrm{cobr}\ S$.

PROOF. Let $g\colon\ S \to I^X$ be a surmorphism. If d is its right adjoint, then d is injective and preserves sups. From 3.1 it then follows that $H(I^X) \leqslant H(S)$. From the definition of H in 3.1 we conclude directly $|X| \leqslant H(I^X)$. Thus $|X| \leqslant H(S)$, whence $\#S \leqslant H(S)$.

The second inequality is similar.

DEFINITION 3.3. Let X be a topological space. We set $d(X) = \min\{\ |Y| : Y \subseteq X$ and $\overline{Y} = X\}$ and call d(X) the *density character* of X.

PROPOSITION 3.4. *If S is a compact semilattice, then* $d(S) \geqslant \#S$ *and* $d(S) \geqslant \mathrm{cobr}\ S$.

PROOF. Let $f : S \to I^X$ be a surmorphism. Then $|X| \leqslant d(I^X) \leqslant d(S)$, whence $\#S \leqslant d(S)$.

The second inequality is similar.

PROPOSITION 3.5. *If S is a compact semilattice, then br $S \geqslant \#S$.*

PROOF. Let $f : S \to I^X$ be a surmorphism and $p : {}^X2 \to I^X$ the lattice injection given by ${}^X2 \to 2^X \to I^X$. Since ${}^X2$ is free, it is projective in the category of semilattices so there is a semilattice morphism $q : {}^X2 \to S$ with $fq = p$. Since p is injective, so is q, hence br $S \geqslant X$, whence the assertion.

COROLLARY 3.6. *$\#I^X = |X|$ for all sets X.*

PROOF. By the Main Theorem, $\#I^X = |X|\#I$. Trivially, $1 \leqslant \#I$, but by 3.5 we have $\#I \leqslant$ t r $I = 1$.

PROPOSITION 3.7. *If S is a compact Lawson semilattice, then dim $S \leqslant \#S$, where dim is cohomological dimension.*

PROOF. This follows from a result of Lawson's [6], Corollary 2.3, p.558.

Thus by 3.5 and 3.7, the dimensional capacity of a compact Lawson semilattice is sandwiched between cohomological dimension and breadth. In summing up, we have observed

$$\dim S \leqslant \#S \leqslant \min \{ \text{br } S, H(S), d(S) \} .$$

We record the logarithmic properties of the height and density characters, and leave their proofs as exercises for the reader. They are parallels to 0.9 and 0.16:

PROPOSITION 3.8. *Let $\{ L_j : j \in J \}$ be a family of complete semilattices. Then*

$$H(\Pi_J L_j) = \Sigma_J H(L_j).$$

PROPOSITION 3.9. *Let $\{X_j : j \in J \}$ be a family of topological non-singleton spaces. Then $d(\Pi_J X_j) = \Sigma_J d(X_j)$.*

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