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LOCALIZATION AND DUALITY IN ADDITIVE CATEGORIES

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By a *duality* between two categories we shall mean an equivalence between one and the opposite of the other. We propose to establish a common categorical setting for a number of well-known duality theorems in mathematics. While we do not claim that the proof of any one of these theorems will thus be simplified, the categorical approach should help the lazy mathematician who is interested in more than one of these classical results. As a byproduct of our machinery we also obtain some new duality theorems.

The mathematical literature abounds with examples of duality. It is clearly impossible to discuss them all here, but we shall consider some cases of duality for non-additive categories in a sequel to this paper.

1. **Categorical setting for duality.** Keimel and Hofmann [2] have pointed out the importance of adjoint functors in connection with duality theorems. It is fairly obvious that, if $\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}$ is a pair of adjoint functors with adjunctions η: id $\to$ UF and ϵ: FU $\to$ id, they induce an equivalence between the full subcategories

$$\mathrm{Fix}(\mathrm{UF},\eta) = \{ A \in \underline{A} \mid \eta(A) \text{ is iso} \}$$

of $\underline{A}$ and

$$\mathrm{Fix}(\mathrm{FU},\epsilon) = \{ B \in \underline{B} \mid \epsilon(B) \text{ is iso} \}$$

of $\underline{B}$. Surprisingly, the condition which ensures that Fix(UF,η) is a reflective subcategory of $\underline{A}$ will also ensure that Fix(FU,ϵ) is a coreflective subcategory of $\underline{B}$.

THEOREM 1.1. *Let $\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}$ be a pair of adjoint functors with adjunctions η: id $\to$ UF and ϵ: FU $\to$ id. Then these induce an equivalence between Fix(UF,η) and Fix(FU,ϵ). Moreover, the following statements are equivalent:*

(1) *the triple (UF,η,UϵF) on $\underline{A}$ is idempotent,*

(2) *ηU is a natural isomorphism,*

(3) *the cotriple (FU,ϵ,FηU) on $\underline{B}$ is idempotent,*

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(4) *ϵF is a natural isomorphism.*

If these conditions hold, Fix(UF,η) is a reflective subcategory of $\underline{A}$ and Fix(FU,ϵ) is a coreflective subcategory of $\underline{B}$.

PROOF. Suppose $A \in \mathrm{Fix}(UF,\eta)$, that is, $\eta(A)$ is an isomorphism. Since $(\epsilon F(A))(F\eta(A)) = 1$, $\epsilon F(A)$ is an isomorphism, that is $F(A) \in \mathrm{Fix}(FU,\epsilon)$. Thus F induces a functor

$$F' \colon \mathrm{Fix}(UF,\eta) \to \mathrm{Fix}(FU,\epsilon),$$

and similarly U induces a functor U' which will be right adjoint to F'. Clearly, $U'F' \cong \mathrm{id}$ and $F'U' \cong \mathrm{id}$.

We shall now show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

We note that if $\eta(A)\colon A \to UF(A)$ has a left inverse g, then the following is a split equalizer diagram:

$$\begin{tikzcd} A \arrow[r, "\eta(A)"] & UF(A) \arrow[l, bend left, "g"] \arrow[r, shift left, "\eta UF(A)"] \arrow[r, shift right, "UF\eta(A)"'] & (UF)^2(A) \arrow[l, bend left=50, "UF(g)"] \end{tikzcd}$$

If $(UF,\eta,U\epsilon F)$ is an idempotent triple, it is known [15] that $\eta UF = UF\eta$, and it follows that $\eta(A)$ is an isomorphism.

Now $\eta U(B)$ always has a left inverse $U\epsilon(B)$, hence (1) ⇒ (2).

Now assume (2). Then, for each object B of $\underline{B}$, $\epsilon FU(B)$ is an isomorphism, as was shown at the beginning of this proof. Thus (2) ⇒ (3).

We observe that $F^{op}\colon \underline{A}^{op} \to \underline{B}^{op}$ (indistinguishable from F) is the right adjoint of $U^{op}\colon \underline{B}^{op} \to \underline{A}^{op}$ (indistinguishable from U). Therefore also (3) ⇒ (4) ⇒ (1).

Finally, we observe that, when $(UF,\eta,U\epsilon F)$ is idempotent, $A \mapsto UF(A)$ is a reflector from $\underline{A}$ to Fix(UF,η) with reflection $\eta(A)\colon A \to UF(A)$. Similarly $B \mapsto FU(B)$ is a coreflector from $\underline{A}$ to Fix(FU,ϵ) with coreflection $\epsilon(B)\colon FU(B) \to B$.

REMARK 1.2. In applications we often put $\underline{B} = \underline{C}^{op}$, so that $F\colon \underline{A} \to \underline{C}^{op}$ being left adjoint to $U\colon \underline{C}^{op} \to \underline{A}$ implies that $U^{op}\colon \underline{C} \to \underline{A}^{op}$ is left adjoint to $F^{op}\colon \underline{A}^{op} \to \underline{C}$, a more symmetrical situation. The equivalent statements of Theorem 1.1 then involve two idempotent triples and the conclusion asserts the duality between their fixed subcategories.

Isbell [3] has called a pair of adjoint functors between $\underline{A}$ and $\underline{C}^{op}$ a "Galois connection" between $\underline{A}$ and $\underline{C}$ if it satisfies conditions (2) and (4) of Theorem 1.1. He mentions the duality between the reflective subcategories when both conditions hold but does not point out the equivalence of the two conditions.

Many, if not all, equivalence and duality theorems in mathematics can be put into the above setting. The real work will then consist in proving that one of the triples is idempotent and in identifying the two fixed subcategories.

This job can be made a little easier, if we begin by searching for idempotent triples. We recall the following result from [10, 11], which shows that many triples give rise to idempotent triples by a simple process due to Fakir [1].

PROPOSITION 1.3. *Let* $\underline{A} \underset{U}{\overset{F}{\rightleftarrows}} \underline{B}$ *be a pair of adjoint functors with adjunction* $\eta\colon id \to UF$. *Assuming* $\underline{A}$ *has equalizers, let* $\chi\colon Q \to UF$ *be the equalizer of the two natural transformations* $UF \underset{UF\eta}{\overset{\eta UF}{\rightrightarrows}} (UF)^2$, *and let* $\lambda\colon id \to Q$ *be the unique morphism such that* $\chi\lambda = \eta$. *Then the following statements are equivalent:*

(1) *(Q,λ) is an idempotent triple;*

(2) *for all A in $\underline{A}$ and B in $\underline{B}$ and each $f\colon Q(A) \to U(B)$ there exists $g\colon UF(A) \to U(B)$ such that $g\chi(A) = f$;*

(3) *$Fix(Q,\lambda)$ is the limit closure of the class of all $U(B)$ with B in $\underline{B}$.*

In explanation of (1) we remark that, while a triple requires three data, an idempotent triple requires only two. The *limit closure* of a class of objects is the smallest full subcategory of $\underline{A}$ which contains this class and which is closed under limits.

When the equivalent conditions of Proposition 1.3 hold we sometimes call (Q,λ) the *idempotent co-approximation* of $(UF,\eta,U\epsilon F)$.

REMARK 1.4. Of special interest is the case where $\underline{B}$ is the category of sets, I is an object of $\underline{A}$ all powers of which exist, $F = Hom(-,I)$ and $U = I^{(-)}$. Then (2) and (3) may be replaced by the following:

(2′) for all A in $\underline{A}$ and each $f\colon Q(A) \to I$ there exists $g\colon UF(A) \to I$ such that $g\chi(A) = f$;

(3′) $Fix(Q,\lambda)$ is the limit closure of I.

Because of (2′), we call the object I *χ-injective*. We call Q the *localization* functor associated with I.

It was shown in [10], that $\chi(A)\colon Q(A) \to UF(A)$ may also be characterized as the joint equalizer of all pairs of morphisms φ,ψ: $UF(A) \Rightarrow I$ such that $\varphi\eta(A) = \psi\eta(A)$.

We also recall from [10, Proposition 2], the following notion. An object P of $\underline{A}$ is called a *regular generator* if for every object A of $\underline{A}$ there is a regular epimorphism (that is, a morphism which happens to be a coequalizer) from some sum of copies of P to A. It was shown that, when P is a χ-projective regular generator and Q is the colocalization functor associated with P, then the canonical morphism $Q(A) \to A$ is an isomorphism.

2. **Duality and equivalence in additive categories.** A category $\underline{A}$ is called *additive* if Hom(A,B) is an abelian group, for each pair of objects A,B, and if composition of morphisms is bilinear. An object I of an additive category $\underline{A}$ is called *co-small* if the functor Hom(-,I) sends products of $\underline{A}$ to coproducts in the category of abelian groups. This means that, for every family $\{A_x | x \in X\}$ of objects in $\underline{A}$, each morphism $\prod_{x \in X} A_x \to I$ factors through some finite sub-product.

THEOREM 2.1. *Let $\underline{A}$ be a complete additive category, I a χ-injective co-small object, E its ring of endomorphisms. Then the functor $F = Hom(-,I)$: $\underline{A} \to (E\,Mod)^{op}$ has a right adjoint U, (UF,η) is an idempotent triple on $\underline{A}$ (in fact, UF is the localization functor associated with I), (FU,ϵ) is an idempotent cotriple on $(E\,Mod)^{op}$, and F induces an equivalence between $Fix(UF,\eta)$ and $Fix(FU,\epsilon)$. Moreover, $Fix(UF,\eta)$ is the limit closure of I in $\underline{A}$.*

PROOF. The existence of the right adjoint U is well-known (see for instance [6, Proposition 1.2] for the dual situation). In view of Theorems 1.1 and 1.3, we need only show that the triple $(UF,\eta,U\epsilon F)$ coincides with the idempotent triple (Q,λ), where Q is the localization functor associated with I. The fact that $Fix(UF,\eta)$ is the limit closure of I follows from Remark 1.4.

A proof that $UF = Q$ was sketched in [10, Example 5]. We shall give another proof here, which uses less of the theory of triples. We pass to the dual statement, which is more useful in some applications.

THEOREM 2.1*. *Let $\underline{A}$ be a cocomplete additive category, P a χ-projective small object, E its ring of endomorphisms. Then the functor $U = Hom(P,-)$: $\underline{A} \to Mod\,E$ has a left adjoint F, (FU,ϵ) is an idempotent cotriple on $\underline{A}$ (in fact, FU is the colocalization functor Q associated with P), (UF,ϵ) is an idempotent triple on Mod E, and U induces an equivalence between $Fix(FU,\epsilon)$ and $Fix(UF,\eta)$. Moreover, $Fix(FU,\epsilon)$ is the colimit* *closure of P in $\underline{A}$.*

PROOF. We need only show that $FU = Q$. Let U_0: Mod E $\to$ Set be the usual forgetful functor with left adjoint F_0, given by $F_0(X) = \sum_{x \in X} E$, and adjunction ϵ_0: $F_0U_0 \to \mathrm{id}$. Then $U' = U_0U$ has left adjoint F' with $F'(X) = \sum_{x \in X} P$. We can choose F so that F' and FF_0 are equal, not merely isomorphic. We must choose $F(E) = P$, $F(\sum_{x \in X} E) = \sum_{x \in X} P$. Then the adjunction for F' is given by $\epsilon'(A) = (\epsilon(A))(F\epsilon_0 U(A))$: $\sum_{f:P\to A} P \to A$.

We note that $\epsilon_0 U(A)$: $\sum_{f:P\to A} E \to U(A)$ is the joint cokernel of all v: $E \to \sum_{f:P\to A} E$ such that $(\epsilon_0 U(A))v = 0$. This is clear, e.g. in view of Remark 1.4, because E is a χ-projective regular generator of Mod E.

Since F preserves coequalizers, $F\,\epsilon_0 U(A)$: $\sum_{f:P\to A} P \to FU(A)$ is the joint cokernel of all F(v): $P \to \sum_{f:P\to A} P$ such that $(\epsilon_0 U(A))v = 0$.

Given any morphism g: $P \to A$, by adjointness there is a unique homomorphism g': $E \to U(A)$ such that $\epsilon(A)F(g') = g$. Taking $g = \epsilon'(A)F(v)$, we see that $g' = (\epsilon_0 U(A))v$. Consequently

$$(\epsilon_0 U(a))v = 0 \quad \Leftrightarrow \quad \epsilon'(A)F(v) = 0.$$

Now, by Remark 1.4, $\chi(A)$: $\sum_{f:P\to A} P \to Q(A)$ is the joint cokernel of all morphisms $f:P \to \sum_{f:P\to A} P$ such that $\epsilon'(A)f = 0$. Thus, in any case, we obtain a unique morphism $\sigma(A)$: $FU(A) \to Q(A)$ such that

$$\sigma(A)(F\epsilon_0 U(A)) = \chi(A).$$

Moreover, $\sigma(A)$ will be an isomorphism if every morphism g: $P \to \sum_{f:P\to A} P$ has the form F(v) for some v: $P \to \sum_{f:P\to A} E$. Since P is small, g can be factored thus:

$$\begin{tikzcd} P \arrow[rr, "g"] \arrow[dr, "h"'] & & \sum_{f:P\to A} P \\ & \sum_{f\in G} P \arrow[ur, "i_G"'] & \end{tikzcd}$$

where G is a finite subset of Hom(P,A) and i_G is the canonical injection of a subsum.

Now, clearly, $i_G = F(i'_G)$, where i'_G is the corresponding injection $\sum_{f\in G} E \to \sum_{f:P\to A} E$ in Mod E. Moreover, if k_f: $P \to \sum_{f\in G} P$ and p_f: $\sum_{f\in G} P \to P$ are the usual canonical injections and projections, we have

$$h = \sum_{f\in G} k_f p_f h = \sum_{f\in G} k_f h_f,$$

where $h_f = p_f h \in E$. Hence

$$h = \Sigma\, F(k'_f)\, F(h'_f),$$

where k'_f is the corresponding injection $E \to \sum_{f \in G} E$ in Mod E and h'_f: $E \to E$ is defined by left multiplication with h_f. This is easily seen from the explicit construction of F.

COROLLARY 2.2. *Under the assumptions of Theorem 2.1*, we have:*

(a) *Fix(FU,ϵ) = $\underline{A}$ if and only if P is a regular generator.*

(b) *Fix(UF,η) = Mod E if and only if the restriction of U to Fix(FU,ϵ) preserves colimits.*

(c) *U is an equivalence between $\underline{A}$ and Mod E if and only if P is a regular generator and U preserves colimits.*

PROOF. (a) the notion of "regular generator" was discussed in Remark 1.4.

(b) Suppose the restriction of U to Fix(FU,ϵ) preserves colimits, then it takes colimits in Fix(FU,ϵ) (which are also colimits in $\underline{A}$) to colimits in Mod E. Now F takes any colimit of Mod E to a colimit of $\underline{A}$ which lies in Fix(FU,ϵ). Hence UF preserves colimits, and so Fix(UF,η) is closed under colimits in Mod E. Since this subcategory contains E, it is equal to Mod E.

The converse is clear.

(c) Put together (a) and (b).

We note that U preserves colimits if and only if it preserves coproducts, that is, P is small, and it preserves coequalizers. When $\underline{A}$ is abelian, the second condition means that U preserves epimorphisms, that is, P is projective. Moreover, in that case every generator is a regular generator. Thus we have the following theorem of Mitchell and Gabriel as a special case (this is not the easiest proof):

COROLLARY 2.3. *An additive category is equivalent to a module category if and only if it is cocomplete Abelian and has a small projective generator.*

This last result has several important applications. Taking $\underline{A}$ = Mod R, one obtains Morita equivalence. Taking $\underline{A}$ to be the opposite of the category of compact Abelian groups, with P = $\mathbf{R}/\mathbf{Z}$, one obtains Pontrjagin duality. In the last case, the crux of the proof consists in showing that P is a small projective.

We note that the conclusions of Theorem 2.1* remain valid if we replace the assumption that P is χ-projective and small by the assumptions that P is a generator and that $\underline{A}$ is Abelian with exact direct limits. In fact, in that case UF is the identity functor. This result is one half of the Gabriel-Popescu Theorem [6, Corollary to

Proposition 4.4], the other half asserting that F is exact. We shall resist the temptation of inserting another proof of this result here.

3. **Duality for modules.** Given an associative ring R with unity element, we shall consider the category Cont R defined as follows: its objects are right R-modules which are at the same time topological abelian groups on which the elements of R act continuously; its morphisms are continuous R-homomorphisms. We shall take I to be a quasi-injective module in Mod R equipped with the discrete topology.

I is called *quasi-injective* if, for every submodule B of I, every homomorphism $B \to I$ can be extended to $I \to I$. Harada had proved that, for every finite n and every submodule B of I^n, every homomorphism $B \to I$ can then be extended to $I^n \to I$ [9, Lemma 4.1]. It was shown in [9, Proposition 5.2] that, for every set X and every regular submodule B of I^X in Cont R, every continuous homomorphism $B \to I$ can be extended to $I^X \to I$. Consequently, I is χ-injective in Cont R.

PROPOSITION 3.1. *Let I be a quasi-injective right R-module equipped with the discrete topology. Then I is co-small in Cont R.*

PROOF. Let $A = \prod_{x \in X} A_x$ with projections $p_x: A \to A_x$ and consider any $f \in \mathrm{Cont}_R(A,I)$. Since I is discrete, ker f is an open neighborhood of zero; so, in view of the way the product topology on A is defined,

$$\begin{aligned} \ker f &\supseteq \ker p_{x_1} \cap \cdots \cap \ker p_{x_n} \\ &= \ker p, \end{aligned}$$

where $p = \langle p_{x_1},\ldots,p_{x_n} \rangle : A \to \prod_{i=1}^{n} A_{x_i}$. Therefore, there exists an R-homomorphism $g: \prod_{i=1}^{n} A_{x_i} \to I$ such that $gp = f$.

We are now ready to apply Theorem 2.1 to the situation in hand, but a little more preparation is necessary if we want to identify the fixed subcategory of $(E\ \mathrm{Mod})^{op}$.

The I-*adic topology* on a module $A_0 \in \mathrm{Mod}\ R$ has a fundamental system of open neighborhoods of zero consisting of all kernels of homomorphisms $A_0 \to I^n$. If only some of these kernels are contained in the system of neighborhoods we shall speak of a *sub*-I-*adic* topology. Thus, both the I-adic topology and the indiscrete topology are sub-I-adic topologies.

PROPOSITION 3.2. *Let A_0 be an R-module, I a quasi-injective R-module. Then there is a lattice anti-isomorphism between the lattice of sub-I-adic topologies on A_0*

and the lattice of E-submodules of $Hom_R(A_0,I)$:

(1) *with each E-submodule B of* $Hom_R(A_0,I)$ *associate the topology* T_B *on* A_0 *which has a fundamental system of open neighborhoods of zero of the form* $\ker b_1 \cap \cdots \cap \ker b_n$ *with* $b_1,\dots,b_n \in B$;

(2) *with each sub-I-adic topology T on* A_0 *associate* $B_T = Cont_R((A_0, T),I)$, *where I is endowed with the discrete topology.*

PROOF. Starting with $B \subseteq \mathrm{Hom}_R(A_0,I)$, we note that, in fact, $B \subseteq \mathrm{Cont}_R((A_0,T_B),I)$. Suppose $c \in \mathrm{Cont}_R((A_0,T_B),I)$, then

$$\ker c \supseteq \ker b_1 \cap \cdots \cap \ker b_n = \ker b,$$

where $b = \langle b_1,\dots,b_n\rangle\colon A_0 \to I^n$. Consequently, there exists $g \in \mathrm{Hom}_R(\mathrm{im}\, b, I)$ such that $gb(a) = c(a)$, for all $a \in A$. By Harada's Lemma, we may extend g to $h\colon I^n \to I$. Let $\chi_i\colon I \to I^n$ and $\pi_i\colon I^n \to I$ be the canonical injections and projections, for $i = 1,\dots,n$. Then

$$c = hb = h \sum_{i=1}^{n} \chi_i\pi_i b = \sum_{i=1}^{n} (h\chi_i)(\pi_i b),$$

and this belongs to B, since $h\chi_i \in E$. Therefore $B = \mathrm{Cont}_R((A_0,T_B),I)$.

On the other hand, let T be any sub-I-adic topology on A_0, then T_{B_T} has a fundamental system of open neighborhoods of zero of the form $\ker b_1 \cap \cdots \cap \ker b_n$, where $b_1,\dots,b_n \in B_T = \mathrm{Cont}_R((A_0,T),I)$. Since the b_i are continuous, $\ker b_i \in T$. Thus $T_{B_T} \subseteq T$. Conversely, any fundamental open neighborhood of zero in T has the form $\ker c$, for some $c \in \mathrm{Hom}_R(A_0,I^n)$. Then $c \in \mathrm{Cont}_R((A_0, T,I^n)$, hence $\pi_1 c,\dots,\pi_n c \in B_T$ and so $\ker c \in T_{B_T}$. Therefore $T_{B_T} = T$.

THEOREM 3.3. *Let I be a quasi-injective right R-module endowed with the discrete topology, E its ring of endomorphisms, then* $F = Cont_R(-,I)\colon Cont\ R \to (E\ Mod)^{op}$ *induces a duality of categories between the limit closure of I in Cont R and the full subcategory of E Mod cogenerated by* ${}_E I$, *that is, consisting of all E-modules isomorphic to submodules of powers of* ${}_E I$.

PROOF. Clearly, F has right adjoint $U = \mathrm{Hom}_E(-,I)$, where $U(B) = \mathrm{Hom}_E(B,I) \subseteq I^B$ has the topology induced by the product topology of I^B. Moreover, $(UF,\eta,U\epsilon F)$ is an idempotent triple on Cont R, as was shown in [9, Proposition 5.3]. This also follows from our Theorem 2.1, in view of the observation that I is χ-injective [9, Proposition 5.2] and co-small, which was shown in Proposition 3.1. Therefore, by

Theorem 2.1, we have a duality between the limit closure of I in Cont R and Fix(FU,ϵ). It remains to identify the latter subcategory. This is done in the following Lemma.

First, let us describe the adjunction morphism $\epsilon(B)$: FU(B) $\rightarrow$ B in $(\text{E Mod})^{\text{op}}$. Passing to E Mod, we have $\epsilon(B)$: B $\rightarrow$ $\text{Cont}_R(\text{Hom}_E(B,I),I)$ given by

$$\epsilon(B)(b)(f) = f(b)$$

for all $b \in B$ and $f \in \text{Hom}_E(B,I)$.

LEMMA 3.4. *Under the assumptions of Theorem 3.3, the following statements concerning a left E-module B are equivalent:*

(1) *$\epsilon(B)$: $B \rightarrow FU(B)$ is an isomorphism of E Mod,*

(2) *B is cogenerated by $_EI$,*

(3) *$\epsilon(B)$ is a monomorphism,*

(4) *B is isomorphic to a submodule of $Hom_R(A_0,I)$ for some A_0 in Mod R,*

(5) *$B \cong F(A)$ for some A in Cont R.*

PROOF. We shall show that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2), since $FU(B) \subseteq I^{U(B)}$

(2) $\Rightarrow$ (3): Suppose $\epsilon(B)(b) = 0$, then $f(b) = 0$ for all $f \in \text{Hom}_E(B,I)$. Suppose $B \subseteq I^X$ and let f be the restriction of p_x to B for $x \in X$, where p_x: $I^X \rightarrow I$ is the canonical projection. Then we deduce that $p_x(b) = 0$ for all $x \in X$, hence $b = 0$.

(3) $\Rightarrow$ (4). Take A_0 to be the underlying module of U(B).

(4) $\Rightarrow$ (5). By Proposition 3.2, any submodule of $\text{Hom}_R(A_0,I)$ has the form F(A) for $A = (A_0,T)$, T being a suitable topology on A_0.

(5) $\Rightarrow$ (1). Since the triple $(UF,\eta,U\epsilon F)$ on Cont R is idempotent, $\epsilon F(A)$ is an isomorphism by Theorem 1.1.

4. Examples of module duality. The fixed subcategories of Cont R and E Mod can be described more neatly in some special cases, particularly the former. To do this we need the following proposition which comprises a number of known density theorems (see [9]).

PROPOSITION 4.1. *Given A in Cont R, and I in Mod R equipped with the discrete topology. Suppose that, for any finite n and any $f \in Cont_R(A,I^n)$, $I^n/f(A)$ is cogenerated by I. Then the image of $\eta(A)$: $A \rightarrow UF(A)$ is dense in the topology of UF(A).*

PROOF. (as in [9, Application 3.4]) For any $f = \langle f_1,\ldots,f_n\rangle \in \mathrm{Cont}_R(A,I^n)$, let f^* be the homomorphism $UF(A) \to I^n$ defined by

$$f^*(s) = (s(f_1),\ldots,s(f_n))$$

for $s \in UF(A) = \mathrm{Hom}_E(\mathrm{Cont}_R(A,I),I)$. $UF(A)$ is topologized as a subspace of the product space $I^{\mathrm{Cont}_R(A,I)}$, so the kernels of the homomorphisms f^* form a fundamental system of open neighborhoods of zero. Thus, we must prove that, for any $s \in UF(A)$ and any $f \in \mathrm{Cont}_R(A,I^n)$, there is an $a \in A$ such that $s\text{-}(\eta(A))(a) \in \ker f^*$.

Consider the mapping

$$I^n \xrightarrow{e} I^n/f(A) \xrightarrow{m} I^X \xrightarrow{\pi_x} I$$

where e is the canonical surjection, m the assumed monomorphism and π_x the canonical projection associated with $x \in X$. Then

$$\pi_x\, mef = 0.$$

We have $f_i = p_i f$,

$$f^*(s) = \sum_{i=1}^{n} k_i s(f_i),$$

and

$$\pi_x mef^*(s) = \sum_{i=1}^{n} \pi_x mek_i s(p_i f)$$

$$= s(\sum_{i=1}^{n} \pi_x mek_i p_i f)$$

$$= s(\pi_x mef) = s(0) = 0.$$

This is so, because $\pi_x mek_i \in E$ and s is an E-homomorphism. It follows that $mef^* = 0$, hence $ef^* = 0$, that is, $\mathrm{im}\, f^* \subseteq \mathrm{im}\, f$. This means that, for any $s \in UF(A)$, there exists $a \in A$ such that

$$f^*(s) = f(a) = f^*\eta(A)(a),$$

that is,

$$s - \eta\,(A)(a) \in \ker f^*,$$

as was to be shown.

This result could also have been proved by a variation of the dual of the argument used for Theorem 2.1*.

REMARK 4.2. The assumptions of the above proposition hold in the following known cases:

(1) I is a cogenerator of Mod R,

(2) I is completely reducible,

(3) I is a nice injective and the underlying abstract module of A is I-torsionfree divisible.

PROOF. Case (1) is clear. In case (2) we may take $I^X = I^n$. For case (3) we recall that an abstract module is called I-*torsionfree divisible* if it lies in the limit closure of I in Mod R, and that I is called *nice* if this limit closure is closed under cokernels [7]. We deduce from (3) that f(A) is I-torsionfree divisible, hence that $I^n / f(A)$ is I-torsionfree, that is, cogenerated by I.

PROPOSITION 4.3. *If I is either completely reducible or a quasi-injective cogenerator of Mod R, its limit closure in Cont R consists of all R-modules with a sub-I-adic topology which are Hausdorff and complete in this topology. If I is a nice injective in Mod R, its limit closure in Cont R consists of all I-torsionfree divisible R-modules with a sub-I-adic topology which are complete and Hausdorff in this topology.*

PROOF. Any object in the limit closure of I is of the form U(B), for some B in E Mod. A fundamental open neighborhood of zero in $U(B) = \mathrm{Hom}_E(B,I) \subseteq I^B$ has the form

$$\{ g \in \mathrm{Hom}_E(B,I) | g(b_1) = 0 \& \ldots \& g(b_n) = 0 \} = \ker \beta,$$

where β: $U(B) \to I^n$ is defined by

$$\beta(g) = (g(b_1), \ldots, g(b_n)).$$

Thus the topology of U(B) is sub-I-adic.

Let A be any object in the limit closure of I. Then A is Hausdorff and complete, because the class of all such modules contains I and is closed under limits. Moreover, the underlying abstract module of A is I-torsionfree divisible, because the forgetful functor from Cont R to Mod R preserves limits and because the class of I-torsionfree divisible modules in Mod R contains I and is closed under limits.

On the other hand, if A is Hausdorff in a sub-I-adic topology, then

$$\ker \eta(A) = \cap \{ \ker f | f \in \mathrm{Cont}_R(A,I) \} = 0,$$

hence $A \to \mathrm{im}\, \eta(A)$ is a module isomorphism. It is continuous, as we already know, and open, because

$$\eta(A)(\ker f) = \ker f^* \cap \mathrm{im}\, \eta(A),$$

for any $f \in \mathrm{Cont}_R(A,I^n)$, f^* being defined as in the proof of Proposition 4.1. Thus $A \cong \mathrm{im}\,\eta(A)$ in Cont R.

If A is also complete, then im $\eta(A)$ is a closed submodule of UF(A). Now, under the assumptions of the proposition, in view of Proposition 4.1 and Remark 4.2, im $\eta(A)$ is dense in UF(A). Therefore, im $\eta(A)$ = UF(A), and so $A \in \mathrm{Fix}(UF,\eta)$.

PROPOSITION 4.4. *Let I be an injective Artinian R-module endowed with the discrete topology. Then* $F = \mathrm{Cont}_R(-,I)\colon \mathrm{Cont}\,R \to (E\,\mathrm{Mod})^{op}$ *induces a duality between discrete Artinian R-modules which are I-torsionfree divisible and finitely generated E-modules which are cogenerated by* ${}_EI$. *Moreover, for any A in Cont R, the following statements are equivalent:*

(1) $A \in \mathrm{Fix}(UF,\eta)$

(2) *A is a limit of discrete Artinian modules which are I-torsionfree divisible,*

(3) *A is a filtered limit of discrete Artinian modules which are I-torsionfree divisible.*

We shall call an object of Cont R *pro-Artinian* if it is a filtered limit of discrete Artinian modules.

PROOF. Assume A is discrete Artinian. Then $\mathrm{Cont}_R(A,I)$ is easily seen to be a finitely generated left E-module; moreover, it follows that UF(A) has the discrete topology and coincides with the module of quotients of A with respect to I [9, Proposition 5.5]. In particular, if A is I-torsionfree divisible, then A is in $\mathrm{Fix}(UF,\eta)$. Moreover, its image F(A) in E-Mod is cogenerated by ${}_EI$, by Theorem 3.3.

Conversely, assume 3 is a finitely generated left E-module. Then there is an exact sequence

$$0 \to K \to E^n \to B \to 0$$

in E Mod. Applying the functor $\mathrm{Hom}_E(-,I)$ to this and observing that

$$\mathrm{Hom}_E(E^n,I) \cong I^n,$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}_E(B,I) \to I^n.$$

Since I is Artinian, so will be $\mathrm{Hom}_E(B,I)$. Moreover, $\mathrm{Hom}_E(B,I)$ is I-torsionfree. Since I is injective, we actually have an exact sequence

$$0 \to \mathrm{Hom}_E(B,I) \to I^n \to \mathrm{Hom}_E(K,I) \to 0.$$

Since $\mathrm{Hom}_E(K,I) \subseteq I^K$ is I-torsionfree, $\mathrm{Hom}_E(B,I)$ is I-divisible [6, Proposition 0.6].

If also B is cogenerated by ${}_E I$ then B is in Fix(FU,ϵ) by Theorem 3.3.

Finally, we shall prove that $(3) \Rightarrow (2) \Rightarrow (1) \Rightarrow (3)$. Clearly, $(3) \Rightarrow (2)$, and $(2) \Rightarrow (1)$ since Fix(UF,η) is closed under limits and contains all discrete Artinian modules. It remains to show that $(1) \Rightarrow (3)$.

We know that any $A \in$ Fix(UF,η) has the form $\mathrm{Hom}_E(B,I)$, where B is a submodule of some power of I. We may represent any E-module B as a filtered colimit of finitely generated submodules B_x of B, where $x \in X$, say. Now $U = \mathrm{Hom}_E(-,I)$ is a right adjoint, hence it converts colimits of E Mod to limits of Cont R. Thus A is a filtered limit of the $\mathrm{Hom}_E(B_x,I)$, and these modules are discrete Artinian as well as I-torsionfree divisible, by the part of the proposition already proved.

EXAMPLE 4.5 Let R be a commutative Noetherian ring, P a prime ideal, I the injective hull of R/P, E its ring of endomorphisms. Matlis [16] (see also [18]) has shown that I is Artinian and that $E \cong \widetilde{R}_P$, the P-adic completion of the localization R_P of R, so we may identify E Mod with Mod $\widetilde{R}_P$.

In view of [7,8], $\widetilde{R}_P = \mathrm{UF}(R)$, and so the I-torsionfree divisible R-modules are precisely the R_P-modules. Moreover, I is a cogenerator of Mod $\widetilde{R}_P$.

Thus Proposition 4.4 allows us to recapture Matlis duality between discrete Artinian R_P-modules and abstract finitely generated $\widetilde{R}_P$-modules. More generally, it yields a duality between pro-Artinian R_P-modules and abstract $\widetilde{R}_P$-modules. Here the pro-Artinian R_P-modules have the inverse limit topology, in fact a sub-I-adic topology, and they are Hausdorff and complete.

REMARK 4.6. When R is a discrete rank one valuation ring, this last duality was first noticed by Kaplansky [4], who used "linearly compact" in place of our "pro-Artinian". The relation between linearly compact modules and inverse limits of Artinian modules is discussed by MacDonald [13].

While Theorem 3.3 and its consequences are in line with Kaplansky's program (see the last two pages of his book [5]), Leptin, MacDonald and Müller have gone off in a different direction, replacing abstract E-modules by linearly topologized E-modules (see the summary at the beginning of [17]).

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