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POLYNOMIAL RINGS AND H_i-LOCAL RINGS (II)

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ABSTRACT Four theorems concerning when $D_k = R[X_1,\ldots,X_k]_{(M,X_1,\ldots,X_k)}$ $(k > 0)$ is an H_i-local ring are proved, where (R,M) is a local ring. Many corollaries of the theorems are given, and two of the theorems are generalized to Rees localities. Finally, a condition for certain localities of D_k to satisfy the first chain condition, when D_k is H_i, is proved.

1. **Introduction.** All rings in this paper are assumed to be commutative with an identity element. The undefined terminology is the same as that in [4].

Let (R,M) and D_k be as above, and let a = altitude R. In [11] a number of results concerning when D_k is an H_i-local ring (see (2.3) for the definition) were proved. In this paper we add four new theorems and many corollaries concerning this, and we extend two of the new theorems to Rees localities. Then a result concerning when certain localities of D_k satisfy the first chain condition (f.c.c.) is proved, when it is assumed that D_k is H_i.

In Section 2 we prove our first four theorems (2.6), (2.14), (2.24), and (2.27). (2.6) shows that D_1 is H_i if and only if R is H_i and $i+1 \notin s(D_1) - \{a+1\}$, where $s(D_1) = \{n$; there exists a maximal chain of prime ideals in D_1 of length $n\}$. (2.14) extends (2.6) to D_k $(k > 1)$. One corollary of (2.14) shows that if ℓ is the least element in $s(D_1)$ and if D_1 is H_i, for some $i < \ell - 1$, then, for all $h \geq 0$, D_h is $H_0,\ldots,H_i$ (2.20). A closely related result shows that if there exists $k > 0$ such that D_k is $H_1,\ldots,H_i$, then, for all $h \geq 0$, D_h is $H_o,\ldots,H_i$. On the other extreme, if g is the greatest element in $s(D_1) - \{a+1\}$, then, for all $h \geq 0$, D_h is $H_{g-1+h},\ldots,H_{a+h}$ (2.24). Also, if there exists $k>0$ such that D_k is $H_i,\ldots,H_{a+k-1}$, then, for all $h \geq -k$, D_{k+h} is $H_{i+h},\ldots,H_{a+k+h}$ (2.30). (Thus a sort of symmetry of results is established.) The final theorem in Section 2 shows that

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if D_k is H_i and H_{i+j} (for some $k > 0$, $i \geqslant 0$, and $1 \leqslant j \leqslant k$), then D_k is $H_i,\ldots,H_{i+j}$ (2.27). Numerous corollaries of all four theorems are given in Section 2.

In Section 3 we generalize (2.6) to principal Rees localities $\mathcal{L}(R,bR)$ (see((3.1)) in (3.3). The theorem shows that if D_1 is H_i and R is a local domain, then every principal Rees locality of R is H_i (3.4). (3.7) shows that if R has the property that $n + 1 \in s(D_1)$ if and only if $n \in s(R)$, then a strong converse of (3.3.1) holds, and so a good generalization of (2.6) to principal Rees localities of R holds. (3.8) gives an application of (2.14) to more general Rees localities.

In Section 4 it is first shown that if R is H_i, then, for all prime ideals p in R such that height $p = i$, R_p is H_j, for all $j \geqslant 2i - a$ (4.1). Applying this, it is shown that if, for some $i \geqslant 0$ and all $k \geqslant 0$, D_k is $H_o,\ldots,H_i$, then, for all $h \geqslant 0$ and for all height $\leqslant i$ prime ideals P in D_h, $(D_h)_P$ satisfies the f.c.c. (4.4). Seven applications of (4.4), using results in Section 2, are given in (4.5). In (4.7) and (4.8) the other extreme (D_h is $H_{i+h},\ldots,H_{a+h}$, for all $h \geqslant 0$) is considered, and it is shown that, for all prime ideals p in R such that height $p \geqslant i - 1$, R/p satisfies the second chain condition.

In Section 5 two remarks are given which have the effect of greatly extending and generalizing the results in this paper. Namely, the results continue to hold if: (a) D_k is replaced by $(R_k)_N$, where $R_k = R[X_1,\ldots,X_k]$ and N is a maximal ideal in R_k such that $N \cap R = M$; and, (b) R is replaced by a quasi-local ring S which contains and is integral over R and is such that minimal prime ideals in S lie over minimal prime ideals in R.

A few of the results in [11] are quoted in the body of this paper, so as to make it reasonably self-contained.

2. Four theorems. In this section we will prove our first four theorems concerning when certain localities of $R[X_1,\ldots,X_k]$ (R a local ring and $k \geqslant 1$) are H_i. Before proving the first of these (2.6), a number of preliminary results are needed. We begin by fixing some notation which we constantly use.

(2.1) NOTATION. *Throughout this paper, the following notation is fixed:* (R,M) is a local ring, and altitude $R = a > 0$; h, i, j, and k are non-negative integers; $D_k = R[X_1,\ldots,X_k]_{(M,X_1,\ldots,X_k)}$, where the X_i are indeterminates ($D_o = R$);

and, $s(L) = \{n;$ there exists a maximal chain of prime ideals of length n in $L\}$, where L is a local ring.

The above notation is slightly different from that in [11], since we here define $D_0 = R$, and we use the more intuitive $s(R)$, rather than the $c(R)$ of [11].

The following fact is used in the proof of (2.2): If $P \subset P_1 \subset \cdots \subset P_n \subset Q$ is a saturated chain of prime ideals in a Noetherian ring, then there is such a chain $P \subset Q_1 \subset \cdots \subset Q_n \subset Q$ such that height Q_i = height P+i, for i = 1,...,n [3, Lemma 1].

(2.2) LEMMA. *The following statements hold for a local ring (R,M):*

(2.2.1) *$n \in s(R)$ if and only if there exists a prime ideal p in R such that height p = n - 1 and depth p = 1.*

(2.2.2) *If $n \in s(R)$, then $n+k \in s(D_k)$, for each $k > 0$.*

(2.2.3) *$n \in s(D_1)$ if and only if $n+k-1 \in s(D_k)$, for $k > 0$.*

PROOF. (2.2.1) If there exists a prime ideal p in R such that height p = n - 1 and depth p = 1, then clearly $n \in s(R)$. Conversely, if $n \in s(R)$, then it is an immediate consequence of [3, Lemma 1] that there is a prime ideal p in R such that height p = n - 1 and depth p = 1.

(2.2.2) Since height pD_k = height p and depth pD_k = depth p + k, for each prime ideal p in R, (2.2.2) follows from (2.2.1).

(2.2.3) is proved in [11, (2.4)], q.e.d.

To prove (2.6), the following definition, remark, and lemma are needed.

(2.3) DEFINITION. A ring A is said to be an H_i-*ring* (or, A is said to be H_i) in case, for each height i prime ideal p in A, depth p = altitude A - i (that is, height p + depth p = altitude A).

Many properties of H_i-local domains are given in [5] and [6], and most of these results have been generalized to local rings in [13]. Most of the results on these rings which are needed in what follows are summarized in the following remark.

(2.4) REMARK. The following statements hold for a local ring (R,M):

(2.4.1) Clearly R is H_i, for all $i \geqslant a - 1$ (vaculously, for $i > a$).

(2.4.2) R is H_i if and only if, for all height j ($j \leqslant i$) prime ideals p in R, R/p is H_{i-j} and either depth p = a - j or depth $p \leqslant i - j$ [13, (2.4)].

(2.4.3) D_1 is H_i if and only if R is H_{i-1} and H_i and, for each height i - 1 prime ideal p in R, all maximal ideals in the integral closure of R/p have the same height (= altitude $R/p = a - i + 1$) [13, (3.7)].

(2.4.4) For $k > 0$, D_{k+1} is H_i if and only if D_k is H_{i-1} and H_i [11, (2.5)].

We adopt the convention that the statement that a local ring is H_g with $g < 0$ says nothing about the ring (it is vacuously true).

(2.5) LEMMA. *The following statements hold for a local ring (R,M):*

(2.5.1) *If $n \in s(R)$, then either R isn't H_{n-1} or $n = a$.*

(2.5.2) *If R is H_i and isn't H_{i-1}, then $i \in s(R)$.*

(2.5.3) *Assume R is H_i and $0 < i < a$. Then R is H_{i-1} if and only if $i \notin s(R)$.*

PROOF. (2.5.1) Assume that $n \in s(R)$. Then, by (2.2.1), there exists a prime ideal p in R such that height $p = n - 1$ and depth $p = 1$. Therefore, if $n \neq a$, then height p + depth $p = n < a$, so R isn't H_{n-1}.

(2.5.2) Assume that R is H_i and isn't H_{i-1}. Then, by (2.4.2), there exists a prime ideal p in R such that height $p = i - 1$ and depth $p = 1$, hence $i \in s(R)$.

(2.5.3) If R isn't H_{i-1}, then $i \in s(R)$ (2.5.2). Conversely, if $i \in s(R)$, then R isn't H_{i-1} (2.5.1), since $i < a$, q.e.d.

We can now prove the first main result in this paper.

(2.6) THEOREM. *D_1 is H_i if and only if R is H_i and $i + 1 \notin s(D_1) - \{a + 1\}$.*

PROOF. Assume first that D_1 is H_i. Then R is H_i (2.4.3). Suppose that $i + 1 \in s(D_1) - \{a + 1\}$. Then, by (2.5.1) applied to D_1, $i + 1 = a + 1$ (since D_1 is H_i and altitude $D_1 = a + 1$); contradiction. Therefore $i + 1 \notin s(D_1) - \{a + 1\}$.

Conversely, assume that R is H_i and $i + 1 \notin s(D_1) - \{a + 1\}$. To show that D_1 is H_i, it may be assumed by (2.4.1), that $i < a$. Consider a height i prime ideal P in R[X] with $P \subset N = (M,X)R[X]$. We must show that height $N/P = a+1-i$. Let height $N/P = d$, so $i+d \in s(D_1)$. Then $d \neq 1$, since $i+1 \in s(D_1)$, $\notin s(D_1) - \{a+1\}$, and $i<a$. Let $p = P \cap R$. If $P = pR[X]$, then height p = height $P = i$, so, since R is H_i, d = height N/P = depth $p + 1 = a - i + 1$, as desired. On the other hand, if $P \neq pR[X]$, then height p = height $P - 1 = i - 1$. Since $d > 1$, by [1, Theorem 3] applied to R/p, there are infinitely many prime ideals q in R such that $p \subset q$, height $q/p = 1$, and depth $q = d - 1$. Then, for some such q, height

q = height $p+1 = i$, by [1, Theorem 1]. Therefore, since R is H_i, $d - 1$ = depth $q = a - i$, and so $d = a - i + 1$, q.e.d.

(2.6) will be generalized in (2.14).

(2.7) REMARK. D_1 is H_i if and only if R is H_i and $i + k \notin s(D_k) - \{a + k\}$, for some $k > 0$ (respectively, for all $k \geq 0$).

PROOF. Clear by (2.6) and (2.2.3), q.e.d.

To prove another corollary to (2.6), we recall the following definitions.

(2.8) DEFINITION. Let A be a ring.

(2.8.1) A satisfies the *first chain condition for prime ideals (f.c.c.)* in case every maximal chain of prime ideals in A has length equal to altitude A.

(2.8.2) A satisfies the *second chain condition for prime ideals (s.c.c.)* in case, for each minimal prime ideal z in A, depth z = altitude A and every integral extension domain of A/z satisfies the f.c.c.

(2.8.3) A satisfies the *chain condition for prime ideals (c.c.)* in case, for each pair of prime ideals $P \subset Q$ in A, $(A/P)_{Q/P}$ satisfies the s.c.c.

(2.9) COROLLARY. *If R satisfies the f.c.c., then* $\{i \,;\, i+1 \in s(D_1)\} = \{i \,;\, D_1$ *isn't* $H_i\} \cup \{a\}$.

PROOF. Since R is H_i, for all $i \geq 0$, (2.6) says that i is in the set on the left side of the equation exactly when it is in the set on the right side, q.e.d.

For another corollary to (2.6), let C be the class of local rings R which satisfy the condition: $n \in s(R)$ if and only if $n + 1 \in s(D_1)$. This is an important class, since it is known [12, (4.1)] that $R \in C$ if any of the following hold: R is complete; R is Henselian; R satisfies the s.c.c.; or, $R = L[X]_{(N,X)}$, where (L,N) is a local ring and X is an indeterminate. (Actually, [12, (4.1)] only says that such local *domains* are in C. But if R is such a local ring (complete, Henselian, etc.), then, for each minimal prime ideal z in R, R/z is also such a ring, hence $R/z \in C$; and $n \in s(R)$ if and only if $n \in s(R/z)$, for some such z. Therefore such local rings are also in C.) (It should also be noted that the definition of C given above is equivalent to the definition of C given in [11] preceding (2.19). This follows from (2.2.1), (2.2.2), and the fact that $n \in c(R)$ (with $c(R)$ as in [11, (2.3.1)]) if and only if $n + 1 \in s(D_1)$ [12, (a) $\Leftrightarrow$ (f)]. (Again, [12, (a) $\Leftrightarrow$ (f)] only says the equality holds for local domains, but $n \in c(R)$ if and only if $n \in c(R/z)$, for some minimal prime ideal z in R [11, (2.3.1)]; and, as already noted, this also holds for $s(D_1)$. Thus [12, (a) ⇔ (f)] holds for the local ring case.))

(2.10) COROLLARY. *If $R \in C$, then D_1 is H_i if and only if R is H_i and $i \notin s(R) - \{a\}$.*

PROOF. Clear by (2.6) and the definition of C, q.e.d.

(2.11) COROLLARY. *If D_1 is H_{i-1} and H_{i+1} but not H_i, then $i+1 \in s(D_1)$ but $i \notin s(R)$.*

PROOF. By (2.4.3), R is H_{i-1} and H_i. Therefore, since R is H_i and D_1 isn't H_i, $i+1 \in s(D_1)$ (2.6). Further, $i<a$, since D_1 isn't H_i, so, since R is H_{i-1}, $i \notin s(R)$ (2.5.1), q.e.d.

(2.12) REMARK. (2.12.1) If the hypothesis of (2.11) holds for some local ring R, and if the Upper Conjecture (that is, $\{n+1\ ; n \in s(R)\} \subseteq \{m; m \in s(D_1)\} \subseteq \{n+1\ ; n \in s(R)\} \cup \{2\}$) holds, then $i = 1$. (See [2, Propositions 3.3 and 3.7] for more information on the Upper Conjecture.)

(2.12.2) If R is as in [4, Example 2, pp. 203-205] in the case $m = 0$, then D_1 is H_0 and H_2, but isn't H_1.

(2.12.3) Since R is H_{i-2} and H_i in (2.11), by (2.4.3), (2.2.1) implies that $i-1$ and $i+1$ are not in $s(R) - \{a\}$.

The following lemma, which is needed for the proof of (2.14), is an easy corollary of (2.5.2).

(2.13) LEMMA. *Assume that R is H_i and $h < i$. If none of $h+1,\dots,i$ are in $s(R) - \{a\}$, then R is $H_h, H_{h+1},\dots,H_i$.*

PROOF. This follows immediately from repeated applications of (2.5.2), q.e.d.

A characterization of when D_k is H_i can be given using (2.4.3) and (2.4.4). The following result, which generalizes (2.6), gives another characterization.

(2.14) THEOREM. *For $k \geq 1$, D_k is H_i if and only if R is H_i and none of $i-k+2, i-k+3,\dots,i+1$ are in $s(D_1) - \{a+1\}$.*

PROOF. For $k = 1$, this is (2.6).

For $k > 1$, by repeated applications of (2.4.4), D_k is H_i if and only if D_1 is $H_{i-k+1},\dots,H_i$. By (2.6), this happens exactly when R is H_j and $j+1$ is not in $s(D_1) - \{a+1\}$, for $j = i-k+1, i-k+2,\dots,i$. Of course, this implies that R is H_i and none of $i-k+2,\dots,i+1$ are in $s(D_1) - \{a+1\}$.

For the converse, assume that R is H_i and that none of $i-k+2,\dots,i+1$ are in $s(D_1)-\{a+1\}$. Then, by (2.6), D_1 is H_i. Also, none of $i-k+2,\dots,i+1$ are in $s(D_1)-\{a+1\}$, so D_1 is $H_{i-k+1},\dots,H_i$ (2.13). Therefore, by (2.4.4), D_k is H_i, q.e.d.

The following result generalizes (2.10).

(2.15) COROLLARY. *Assume that* $R \in C$. *Then, for each* $k \geqslant 1$, D_k *is* H_i *if and only if* R *is* H_i *and* $i-k+1,\dots,i \notin s(R)-\{a\}$.

PROOF. This is clear by (2.14) and the definition of C, q.e.d.

(2.14) affords an alternate proof to the following known result.

(2.16) COROLLARY. *(cf. [13, (3.10)].) If* D_{a-1} *is* H_{a-1}, *then R satisfies the s.c.c. (and conversely).*

PROOF. If D_{a-1} is H_{a-1}, then $2,\dots,a \notin s(D_1)$, by (2.14). Therefore, since $0,1 \notin s(D_1)$, $s(D_1)=\{a+1\}$, hence D_1 satisfies the f.c.c., so R satisfies the s.c.c. [7, Theorem 2.21]. The converse follows from [7, Theorem 2.6], q.e.d.

The following corollary sharpens [11, (3.1) and (3.2)].

(2.17) COROLLARY. *If, for some* $k \geqslant 1$ *and* $i \leqslant k$, D_k *is* H_i, *then, for all* $h \geqslant 0$, D_h *is* $H_o,\dots,H_i$.

PROOF. By (2.14), R is H_i and none if i-k+$2,\dots,i+1$ are in $s(D_1)-\{a+1\}$. Since $i-k+2 \leqslant 2$, and since $j \notin s(D_1)$, for $j \leqslant 1$ (since $a > 0$), we have $j \notin s(D_1)-\{a+1\}$, for all $j \leqslant i+1$. Therefore, by (2.2.2), $j \notin s(R)-\{a\}$, for all $j \leqslant i$. Hence, by (2.13), R is H_j, for all $j \leqslant i$, so the corollary holds for $h=0$. For $h \geqslant 1$ and $j \leqslant i$, we have that R is H_j and $j+1 \leqslant i+1$, so by what was noted above, none of $j-h+2,\dots,j+1$ are in $s(D_1)-\{a+1\}$. Therefore D_h is H_j (2.14), q.e.d.

(2.18) REMARK. Since $a > 0$, $0,1,\dots,k \notin s(D_k)$, by (2.2.3), since $1 \notin s(D_1)$. This, together with (2.13), (2.4.3), and (2.4.4), affords an alternate proof of (2.17).

(2.19) DEFINITION. Let ℓ and g denote, respectively, the least and the greatest element in $s(D_1)-\{a+1\}$. (If $s(D_1)=\{a+1\}$, let $\ell = g = \infty$.)

If $s(D_1)=\{a+1\}$, then D_1 satisfies the f.c.c., so R satisfies the s.c.c. [7, Theorem 2.21], hence, for all $k \geqslant 0$ and $i \geqslant 0$, D_k is H_i [7, Theorem 2.6]. Therefore, in what follows it will be assumed that $\ell \neq \infty \neq g$; hence $1 < \ell \leqslant g \leqslant a$ (2.18).

(2.20) COROLLARY. *With* $\ell \leqslant a$ *as in (2.19), if* $i < \ell - 1$ *and if* D_1 *is* H_i, *then, for all* $h \geqslant 0$, D_h *is* $H_o,\ldots,H_i$.

PROOF. By (2.14), R is H_i. If $j \leqslant i$, then $j+1 < \ell$, so $j+1 \notin s(D_1) - \{a+1\}$. Therefore, by (2.2.2), $j \notin s(R) - \{a\}$, for $j \leqslant i$. Thus, by (2.13), R is H_j, for all $j \leqslant i$. For $h \geqslant 1$ and $j \leqslant i$, since R is H_j and $j-h+2,\ldots,j+1$ are all less than $i+1$ (and hence not in $s(D_1) - \{a+1\}$), D_h is H_j (2.14), q.e.d.

(2.21) REMARK. Let $\ell \leqslant a$ be as in (2.19), and assume that there exist $k \geqslant 1$ and $i < \ell+k-2$ such that D_k is H_i. Then the following statements hold:

(2.21.1) For all $h \geqslant 0$, D_h is $H_o,\ldots,H_i$.

(2.21.2) For all $h \geqslant 1$, D_h isn't $H_{\ell+h-2}$.

(2.21.3) If $i > 0$, then the H-Conjecture (that is, if R is an H_1-local domain, then R satisfies the f.c.c.) fails.

(2.21.4) If $s(D_1) = \{\ell, a+1\}$ (so $\ell = g$) and D_1 is $H_{\ell-2}$, then, for all $h \geqslant 1$, D_h is $H_o,\ldots,H_{\ell-2}, H_{\ell+h-1},\ldots,H_{a+h}$ and isn't $H_{\ell-1},\ldots,H_{\ell+h-2}$.

PROOF. (2.21.1) If $k = 1$, then the conclusion follows from (2.20), so assume that $k > 1$. Then D_1 is $H_{i-k+1},\ldots,H_i$ (2.4.4) and $i-k+1 < \ell-1$. Therefore D_1 is $H_o,\ldots,H_{i-k+1},\ldots,H_i$ (2.20). Therefore, by (2.23.1) below, for all $h \geqslant 0$, D_h is $H_o,\ldots,H_i$.

(2.21.2) By (2.2.3), $\ell+h-1$ is the least element in $s(D_h) - \{a+h\}$. Therefore, since $\ell+h-1 < a+h$, D_h isn't $H_{\ell+h-2}$ (2.5.1).

(2.21.3) By (2.21.1) and (2.21.2), D_2 is H_1 and isn't H_ℓ. Now there exists a minimal prime ideal z in D_2 such that D_2/z is H_1 and isn't H_ℓ (by (2.4.2), since D_2 is also H_o). Hence, since $\ell \geqslant 2$ (2.18), the H-Conjecture fails.

(2.21.4) By (2.20), D_h is $H_o,\ldots,H_{\ell-2}$. By (2.24) below, D_h is $H_{\ell+h-1},\ldots,H_{a+h}$. By (2.21.2) D_1 isn't $H_{\ell-1}$. Therefore, by (2.4.4), D_h isn't $H_{\ell-1},\ldots,H_{\ell+h-2}$, q.e.d.

(2.23.1) below is closely related to (2.20). The following remark will be used in the proof of (2.23).

(2.22) REMARK. The following statements are equivalent: R is H_o; D_k is H_o, for some $k \geqslant 1$; D_k is H_o, for all $k \geqslant 0$.

PROOF. Since the minimal prime ideals in D_k are the ideals zD_k, where z is a minimal prime ideal in R , and since depth zD_k = depth z + k , the conclusion easily follows, q.e.d.

The following result gives some further information which is closely related to (2.17), (2.20), and (2.21.1).

(2.23) PROPOSITION. (2.23.1) *If there exist* $k \geq 1$ *and* $i \geq 1$ *such that* D_k *is* $H_1,\ldots,H_i$ *, then, for all* $h \geq 0$ *,* D_h *is* $H_o,H_1,\ldots,H_i$.

(2.23.2) *If there exist* $k \geq 1$ *and* $i \geq 1$ *such that* D_k *is* $H_k,\ldots,H_{k+i}$ *, then, for all* $h \geq 0$ *,* D_h *is* $H_o,\ldots,H_{k+i}$.

PROOF. (2.23.1) If D_k is H_1, then D_{k-1} is H_o and H_1 (2.4.4), so D_h is H_o, for all $h \geq 0$ (2.22). Therefore, if D_k is $H_1,\ldots,H_i$, then D_k is $H_o,\ldots,H_i$. Thus, if $0 \leq h < k$, then D_h is $H_o,\ldots,H_i$ (2.4.4). Also, for h = k+1, D_h is $H_1,\ldots,H_i$ (2.4.4), hence D_h is $H_o,\ldots,H_i$. Therefore the conclusion follows from repetitions of this last step.

(2.23.2) By (2.17), if D_k is H_k , then D_k is $H_o,\ldots,H_k$, so the conclusion follows from (2.23.1), q.e.d.

In (2.20) and (2.21) we considered what can be said when ℓ is the least element in $s(D_1) - \{a+1\}$. In the next theorem we will now consider what can be said when g is the greatest element in $s(D_1) - \{a+1\}$.

(2.24) THEOREM. *(cf. [12, (5.6)].) If* $g \leq a$ *is as in (2.19), then, for all* $k \geq 0$ *,* D_k *is* $H_{g-1+k},\ldots,H_{a+k}$ *and isn't* H_{g-2+k}. *Conversely, if there exists* $k \geq 1$ *such that* D_k *is* $H_{n+k},\ldots,H_{a+k}$ *and isn't* H_{n-1+k} *, then n+1 is the largest element in* $s(D_1) - \{a+1\}$.

PROOF. By [12, (5.6)], if g is the largest element in $s(D_1) - \{a+1\}$, then D_1 is $H_g,\ldots,H_{a+1}$. Therefore D_2 is $H_{g+1},\ldots,H_{a+1}$ (2.4.4), and D_2 is H_{a+2} . Repeating this, it follows that, for all $k \geq 1$, D_k is $H_{g-1+k},\ldots,H_{a+k}$. Finally, R is $H_{g-1},\ldots,H_a$ (2.4.3), and D_k isn't H_{g-2+k} (2.2).

Conversely, D_1 is $H_{n+1},\ldots,H_{a+1}$ and isn't H_n , by repeated use of (2.4.4). Therefore, by (2.5.2), $n+1 \in s(D_1)$. Also, since D_1 is $H_{n+1},\ldots,H_{a+1}$, n+1 is the largest element in $s(D_1) - \{a+1\}$, by (2.2.1), q.e.d.

The following corollary can be extended (with suitable assumptions) to local rings, much as in [13, (3.14)]. It can also be extended to quasi-local rings which contain and are integral over a local ring, much as in [13, (2.17) and (3.20)]. However, we content ourselves with the domain case here. Before stating the corollary, we first recall a definition.

(2.25) DEFINITION. A ring A is said to be a C_i-ring (or, A is said to be C_i) in case A is H_i, H_{i+1} and, for each height i prime ideal p in A, all maximal ideals in the integral closure of A/p have the same height (= altitude A/p = altitude A - i).

(2.26) COROLLARY. *Assume that (R,M) is a local domain with quotient field F. If* $g \leqslant a$ *is as in (2.19), then, for all* $k \geqslant 1$ *and* $x_1, \dots, x_k$ *in F such that* $N = (M, x_1, \dots, x_k)A$ *is proper, where* $A = R[x_1, \dots, x_k]$, A_N *is* $C_{g-1}, \dots, C_a$.

PROOF. By (2.24), D_k is $H_{g-1+k}, \dots, H_{a+k}$, so D_k is $C_{g-1+k}, \dots, C_{a+k-1}$ [11, (2.6)]; and clearly D_k is C_{a+k}. Also, $A_N = D_k/K$, for some prime ideal K in D_k. Then height K = k, since K is maximal with respect to the property of contracting to (0) in R. Therefore $A_N = D_k/K$ is $C_{g-1}, \dots, C_a$ [13, (3.3)], q.e.d.

We come now to the fourth and final theorem in this section.

(2.27) THEOREM. *Assume that there exist non-negative integers i,j,k such that* $1 \leqslant j \leqslant k$ *and* D_k *is* H_i *and* H_{i+j}. *Then* D_k *is* $H_i, \dots, H_{i+j}$.

PROOF. Suppose that D_k isn't H_{i+r}, for some r $(0 < r < j)$, and take the largest such r. Then $i+r+1 \in s(D_k)$ (2.5.2), so $i+r-k+2 \in s(D_1)$ (2.2.3). Now $0 < r$ (so $i-k+1 < i+r-k+2$) and $r < j \leqslant k$ (so $i+r-k+2 \leqslant i+1$). Also, D_k is H_i, so D_1 is $H_{i-k+1}, \dots, H_i$ (2.4.4). Therefore, since $i-k+1 \leqslant i+r-k+1 \leqslant i$, D_1 is $H_{i+r-k+1}$, hence $i+r-k+2 = a+1$ (2.5.1). But $i+r-k+1 < a$, since D_k isn't H_{i+r} (2.4.1); contradiction. Therefore, D_k is $H_i, \dots, H_{i+j}$, q.e.d.

It is clear from (2.27) that if D_k is H_i and H_{i+j} and isn't H_{i+1}, then $j > k$.

The first part of the following corollary to (2.27) holds without the assumption that R is H_o (2.17).

(2.28) COROLLARY. (2.28.1) *With j,k as in (2.27), assume that R is* H_o *and* D_k *is* H_j. *Then, for all* $h \geqslant 0$, D_h *is* $H_o, \dots, H_j$.

(2.28.2) *If there exist* $1 \leqslant k < i$ *such that* D_k *is* H_i, *then, for all prime ideals p in R such that height* $p = i-j$ $(0 \leqslant j \leqslant k)$ *and for all* $h \geqslant 0$, D_h/pD_h *is* $H_o, \dots, H_j$.

PROOF. (2.28.1) Since R is H_o, D_k is H_o (2.22). Therefore D_k is $H_o, \dots, H_j$ (2.27), hence the conclusion follows from (2.23.1).

(2.28.2) If D_k is H_i and height $p = i-j$, then D_k/pD_k is H_j (2.4.2.) and $0 \leqslant j \leqslant k$. Therefore, since, for all $h \geqslant 0$, $D_h/pD_h \cong (R/p)[X_1, \dots, X_h]$

$(M/p,X_1,...,X_h)$, the conclusion follows from (2.28.1), q.e.d.

(2.29) COROLLARY. *If there exist i,j,k such that* $k \geqslant max\{1,i,j\}$ *and* D_k *is* H_i *and* H_{i+j}, *then, for all* $h \geqslant 0$, D_h *is* $H_o,...,H_{i+j}$.

PROOF. By (2.23.1), it suffices to prove that D_k is $H_o,...,H_{i+j}$. For this, since D_k is H_i and $i \leqslant k$, D_k is $H_o,...,H_i$ [11, (3.1)]. Therefore, since $j \leqslant k$ and D_k is H_{i+j}, D_k is $H_o,...,H_{i+j}$ (2.27), q.e.d.

The following lemma, which will be used to derive some further corollaries of (2.27), is analogous to (2.23.1).

(2.30) LEMMA. *Assume that there exist* $k \geqslant 1$ *and* $i \geqslant 0$ *such that* D_k *is* $H_i,...,H_{a+k-1}$. *Then, for all* $h \geqslant -k$, D_{k+h} *is* $H_{i+h},...,H_{a+k+h}$.

PROOF. D_k is $H_i,...,H_{a+k}$ (2.4.1). Therefore, by (2.4.4), D_{k+1} is $H_{i+1},...,H_{a+k}$, and is also H_{a+k+1} (2.4.1). Repeating this D_{k+h} is $H_{i+h},...,H_{a+k+h}$, for all $h \geqslant 0$. Also D_{k-1} is $H_{i-1},...,H_{a+k-1}$ (2.4.4), so repetitions show that D_{k+h} is $H_{i+h},...,H_{a+k+h}$, for $h = -k,...,-1$, q.e.d.

(2.31) COROLLARY. *If there exists* $k \geqslant 1$ *such that* D_k *is* H_{a-1}, *then, for all* $h \geqslant -k$, D_{k+h} *is* $H_{a-1+h},...,H_{a+k+h}$.

PROOF. D_k is H_{a+k-1} (2.4.1). Therefore, if D_k is H_{a-1}, then D_k is $H_{a-1},...,H_{a+k-1}$ (2.27). Thus the conclusion follows from (2.30), q.e.d.

As has already been pointed out, if R satisfies the s.c.c., then, for all $i \geqslant 0$ and $k \geqslant 0$, D_k is H_i. In (2.32) the converse is considered. (See also (2.16).)

(2.32) REMARK. (2.32.1) If $k \geqslant a-1$ in (2.31), then R satisfies the s.c.c.

(2.32.2) If $k < a-1$ in (2.31), then let h be such that $k+h = a-1$. Then $h+1$ is $\geqslant$ the greatest element in $s(D_1) - \{a+1\}$.

(2.32.3) (cf. [11, (2.12)].) If there exist $h \geqslant 0$ and $0 \leqslant i \leqslant h$ such that D_{a-1+h} is H_{a-1+i}, then R satisfies the s.c.c.

(2.32.4) (cf. [11, (2.13)].) If there exist $h \geqslant 0$ and $0 \leqslant i \leqslant h$ such that D_{a-2+h} is H_{a-1+i}, then, for all minimal prime ideals z in R, the integral closure of R/z satisfies the c.c.

PROOF. (2.32.1) By (2.31), D_1 is $H_o,...,H_{a+1}$, so D_1 satisfies the f.c.c. [3, Proposition 7], hence R satisfies the s.c.c. [7, Theorem 2.21].

(2.32.2) By (2.31), D_k is $H_{h+k},...,H_{a+k}$, so the conclusion follows from (2.24).

(2.32.3) If D_{a-1+h} is H_{a-1+i} and $0 \leqslant i \leqslant h$, then $D_{a-1+h-i}$ is H_{a-1} (2.4.4), so the conclusion follows from (2.32.1).

(2.32.4) If D_{a-2+h} is H_{a-1+i} and $0 \leqslant i \leqslant h$, then $D_{a-2+h-i}$ is H_{a-1} (2.4.4), hence the conclusion follows from [11, (2.13)], q.e.d.

(2.33.1) is somewhat analogous to (2.23.2), and (2.33.2) is in a similar relationship to (2.29).

(2.33) COROLLARY. (2.33.1) *If there exist* $k \geqslant 1$ *and* $i \geqslant 0$ *such that* D_k *is* $H_i,\ldots,H_{a-1}$, *then, for all* $h \geqslant -k$, D_{k+h} *is* $H_{i+h},\ldots,H_{a+k+h}$.

(2.33.2) *If there exist* $k \geqslant 1$ *and* $i \geqslant a-1-k$ *such that* D_k *is* H_i *and* H_{a-1}, *then, for all* $h \geqslant -k$, D_{k+h} *is* $H_{i+h},\ldots,H_{a+k+h}$.

PROOF. (2.33.1) If D_k is $H_i,\ldots,H_{a-1}$, then D_k is $H_i,\ldots,H_{a+k}$ (2.31), so the conclusion follows from (2.30).

(2.33.2) If D_k is H_i and H_{a-1} and $i \geqslant a-1-k$, then D_k is $H_i,\ldots,H_{a-1}$ (2.27), so the conclusion follows from (2.33.1), q.e.d.

(2.34) COROLLARY. *Assume that R is an integrally closed local domain which is* H_1 *and that* D_k *is* H_{k+1}, *for some* $k \geqslant 1$. *Then, for all* $h \geqslant 0$, D_h *is* $H_o,\ldots,H_{k+1}$.

PROOF. The hypotheses on R imply that D_k is H_o and H_1 [9, (3.3)]. Therefore, since D_k is H_{k+1}, the conclusion follows from (2.27) and (2.23.1), q.e.d.

(2.35) COROLLARY. *If there exist* $k \geqslant 1$ *and* $i \geqslant 0$ *such that* D_k *is* H_i *and* H_{i+k+1}, *then* D_k *is* H_{i+1} *if and only if* D_k *is* H_{i+2} *if and only if* $\cdots$ *if and only if* D_k *is* H_{i+k}. *If* D_k *isn't* H_{i+1}, *then, for all height* j $(i+1 \leqslant j \leqslant i+k)$ *prime ideals* p *in* D_k, *depth* $p \in \{a+k-j, i+1+k-j\}$.

PROOF. Assume that D_k is H_i and H_{i+k+1}. Then, if D_k is H_h, for some h $(1 \leqslant h \leqslant k)$, then D_k is H_i,H_{i+h},H_{i+k+1}, so D_k is $H_i,\ldots,H_{i+k+1}$ (2.27).

Now assume that D_k isn't H_{i+1}, and let p be a prime ideal in D_k such that height $p = j$ $(i+1 \leqslant j \leqslant i+k)$ and $d =$ depth $p < a+k-j$ (D_k isn't H_j, by the preceding paragraph). Then $j+d \leqslant i+k+1$ (2.4.2), and, clearly, $j+d \in s(D_k)$, so $j+d-k+1 \in s(D_1)$ (2.2.3). Therefore either D_1 isn't H_{j+d-k} or $j+d-k = a$ (2.5.1). Now, by hypothesis, $d<a+k-j$, so D_1 isn't H_{j+d-k}. Also, since D_k is H_i and H_{i+k+1}, D_1 is $H_{i-k+1},\ldots,H_i,H_{i+2},\ldots,H_{i+k+1}$ (2.4.4). Therefore, $j+d-k \notin \{i-k+1,\ldots,i,i+2,\ldots,i+k+1\}$. But, by the above inequalities, $i+1+d-k \leqslant j+d-k \leqslant i+1$. Therefore $j+d-k = i+1$,

so $d = i+1+k-j$, q.e.d.

To obtain one more corollary to (2.27), the following lemma is needed. For the lemma, recall that R is C_i (2.25) if and only if, for all height j ($j \leqslant i$) prime ideals p in R, R/p is C_{i-j} and either altitude $R/p = a-j$ or $\leqslant i-j$ [13, (3.3)].

(2.36) LEMMA. *Assume that there exist $i \geqslant 0, j \geqslant 0$, and $k \geqslant 1$ such that, for all height j prime ideals p in R, D_k/pD_k is H_i and either altitude $D_k/pD_k = a+k-j$ or $\leqslant i$. Then the following statements hold:*

(2.36.1) *If $i \geqslant k$, then D_k is H_{i+j}.*

(2.36.2) *If $k \geqslant i$, then D_i is H_{i+j}.*

PROOF. If there does not exist a prime ideal p in R such that height $p = j$, then $j > a$, so $i+j > i+a \geqslant k+a$ and so D_k is H_{i+j} (2.4.1) and (2.36.1) holds (respectively, $i+j > i+a$ and so D_i is H_{i+j} and (2.36.2) holds). Therefore assume that $j \leqslant a$. If $i \geqslant a+k-j-1$, then D_k/pD_k is H_i and D_k is H_i and D_k is H_{i+j} (2.4.1). Therefore assume that $i < a+k-j-1$. Then, by hypothesis and (2.4.4), D_1/pD_1 is $H_{i-k+1},\ldots,H_i$ and either altitude $D_1/pD_1 = a+1-j$ or $\leqslant i-k+1$. Therefore, by (2.4.3), R/p is $C_{i-k},\ldots,C_{i-1}$ and either altitude $R/p = a-j$ or $\leqslant i-k$. Thus, since p was an arbitrary height j prime ideal in R, if $i \geqslant k$, then R is $C_{i-k+j},\ldots,C_{i+j-1}$ [13, (3.3)], so D_1 is $H_{i-k+j+1},\ldots,H_{i+j}$ (2.4.3), and so D_k is H_{i+j} (2.4.4); and, if $k \geqslant i$, then R is $C_j,\ldots,C_{i-1+j}$ (by [13, (3.3)] and since the statement that R/p is C_{-h} ($h > 0$) says only that R/p is H_o), so D_1 is $H_{j+1},\ldots,H_{i+j}$ (2.4.3), and so D_i is H_{i+j} (2.4.4), q.e.d.

We close this section with the following corollary of (2.27).

(2.37) COROLLARY. *If there exist integers $0 \leqslant h \leqslant k \leqslant i$ ($k \geqslant 1$) such that, for all height j prime ideals p in R, D_k/pD_k is H_i and H_{i+h} and either altitude $D_k/pD_k = a+k-j$ or $\leqslant i$, then D_k is $H_{i+j},\ldots,H_{i+h+j}$.*

PROOF. If D_k/pD_k is H_i and H_{i+h} and $k \geqslant h$, then D_k/pD_k is $H_i,\ldots,H_{i+h}$ (2.27). Therefore the conclusion follows from (2.36.1), q.e.d.

3. A generalization to principal Rees localities. In this section we will generalize (2.6) to principal Rees localities (3.3).

(3.1) DEFINITION. Let (R,M) be a local ring, let $B = (b_1,\ldots,b_k)R$ be an ideal in R, let t be an indeterminate, and let $u = 1/t$. Then the ring

$\mathcal{L}(R,B) = R[tb_1,\ldots,tb_k,u]_{(M,tb_1,\ldots,tb_k,u)}$ is called the *Rees locality of* R *with respect to* B .

It should be noted that $\mathcal{L}(R,(0)) \cong D_1$.

To prove (3.3), the following two facts concerning Rees localities are needed.

(3.2) REMARK. With the notation of (3.1), fix an ideal B in R and let $\mathcal{L} = \mathcal{L}(R,B)$. Then the following statements hold:

(3.2.1) Altitude $\mathcal{L} = a+1$ (a = altitude R) [7, Remark 3.7].

(3.2.2) For a prime ideal p in R, let $p^+ = (pR[t,u] \cap R[tb,u]) \mathcal{L}$. Then height p^+ = height p and depth p^+ = depth p + 1 . (This follows easily from [7, Remarks 3.6(ii) and 3.7].)

Also, in the proof of (3.3), the following fact will be used: If there exists a prime ideal p in R such that there is a height one maximal ideal N in the integral closure S of R/p , then there exists a prime ideal P in D_1 such that $P \cap R = p$, depth P = 1 , height P = height p + 1 , and X_1+P is integral over R/p and is in the quotient field of R/p. (To see this, let x be an element in N such that 1-x is in all other maximal ideals in S . Then altitude $(R/p)[x]_{(M/p,x)} = 1$, and the existence of P easily follows from this.)

It is known [10, (2.10)] that R is C_{i-1} (2.25) if and only if, for all $b \in E = \{b \in M\,;\ \text{height } bR = 1\} \cup \{0\}$, $\mathcal{L}(R,bR)$ is H_i . (3.3) gives a variation of this and, at the same time, a generalization of (2.6) (since $\mathcal{L}(R,(0)) \cong D_1$) .

(3.3) THEOREM. (3.3.1) *If there exists* $b \in M$ *such that* $\mathcal{L} = \mathcal{L}(R,bR)$ *is* H_i , *then* R *is* H_i *and* $i+1 \notin s(\mathcal{L}) - \{a+1\}$.

(3.3.2) *If* R *is* H_i *and there exists* $b \in M$ *such that, for all but finitely many* k, $i+1 \notin s(\mathcal{L}(R,b^kR)) - \{a+1\}$, *then, for all* $c \in E = \{b \in M; \text{height } bR = 1\} \cup \{0\}$, $\mathcal{L}(R,cR)$ *is* H_i.

PROOF. (3.3.1) If $\mathcal{L}$ is H_i , then it follows easily from (3.2.2) that R is H_i . Also, it is clear that $i+1 \notin s(\mathcal{L}) - \{a+1\}$.

(3.3.2) By (2.4.1), it may be assumed that $i < a$. Also, by [10, (2.10)], it suffices to prove that R is C_{i-1} . That is, since R is H_i , it suffices to prove that there does not exist a height i-1 prime ideal p in R such that there exists a height one maximal ideal in the integral closure of R/p. Suppose there exist such p . If depth p = 1 , then, for each fixed k , height $p^+ = i-1$ and depth $p^+ = 2$ (3.2.2), so, since $i+1 \leqslant a$, $i+1 \in s(\mathcal{L}(R,b^kR)) - \{a+1\}$, for all $k > 0$; contradiction. Therefore depth $p > 1$, so, by the comment preceding this theorem, there exists a prime ideal P in $D = R[u]_{(M,u)}$ such that $P \cap R = p$, depth $P = 1$, height $P =$ height $p+1$, and $u+P$ is integral over R/p and is in the quotient field of R/p. Now $u \notin P$, since altitude $D/P = 1 <$ altitude R/p. Let $*$ denote residue class modulo P. Then, in $D^* = D/P$, $b^{*k} \in u^*D^*$, for all large k (since altitude $D^* = 1$). (Possibly $b^* = 0^*$.) Therefore, since $D[1/u]/PD[1/u] = D^*[1/u^*]$ is the quotient field of R/p and $tb^k = b^k/u$, $D^*[(tb^k)^*] = D^*$, for all large k. Fix a large k, let $\mathcal{L} = \mathcal{L}(R,b^kR)$, and let $P' = (PD[1/u] \cap D[tb^k])\mathcal{L}$. (Since u isn't a zero divisor, it is clear that $\mathcal{L}$ is a quotient ring of $D[tb^k]$.) Then $\mathcal{L}/P' = D/P = D^*$, so depth $P' = 1$. Also, height $P' =$ height $P =$ height $p + 1 = i$. Therefore $i+1 \in s(\mathcal{L})$; contradiction. Therefore R is C_{i-1}, q.e.d.

(3.4) COROLLARY. *If D_1 is H_i, then, for all $b \in E$ (E as in (3.3.2)), $\mathcal{L}(R,bR)$ is H_i.*

PROOF. If D_1 is H_i, then R is H_i and $i+1 \notin s(D_1)$ (2.6), so the conclusion follows from (3.3.2) (since $0 \in E$), q.e.d.

(3.5) REMARK. If R is a local domain and $i+1 \notin s(D_1)$, then, for all $b \in M$, $i+1 \notin s(\mathcal{L}(R,bR))$, even if R isn't H_i. (This follows easily from [12, (2.5)] and the fact that $D_1 \in C$.)

(3.6) COROLLARY. *Assume that R satisfies the f.c.c. and fix $b \in E$ (E as in (3.3.2)). Then $\{i\,; i+1 \notin s(\mathcal{L}(R,b^kR)) - \{a+1\}$, for all large $k\} = \{i\,;$ for all $c \in E$, $\mathcal{L}(R,cR)$ is $H_i\}$.*

PROOF. Let A and B denote the sets on the left and right side of the equation, respectively. Then $A \subseteq B$, by (3.3.2). Conversely, if $i \in B$, then, since $0 \in E$, D_1 is H_i. Therefore $\mathcal{L}(R,b^kR)$ is H_i, for all $k \geqslant 1$ (3.4), so $i \in A$ (2.5.1), q.e.d.

The authors don't know if, for $b \in M$, (3.3.2) can be proved under the simpler assumption: (+) R is H_i and $i+1 \notin s(\mathcal{L}(R,bR)) - \{a+1\}$. (Of course, for $b = 0$, (+) says that D_1 is H_i (2.6), so R is C_{i-1} (2.4.3), hence, for all $c \in E$, $\mathcal{L}(R,cR)$ is H_i and $i+1 \notin s(\mathcal{L}(R,cR)) - \{a+1\}$, by [10, (2.10)].) However, if $b \in p$ (p as in the proof of (3.3.2)), then (+) does suffice (by the proof of (3.3.2)). The following theorem shows that (+) always suffices, if $R \in C$.

(3.7) THEOREM. *(cf. [10, (2.12)].) Assume that* $R \in C$. *If* R *is* H_i *and there exists* $b \in M$ *such that* $i+1 \notin s(\mathcal{L}(R,bR)) - \{a+1\}$, *then, for all* $c \in E$ *(E as in (3.3.2)),* $\mathcal{L}(R,cR)$ *is* H_i.

PROOF. By [10, (2.10)], it suffices to prove that R is C_{i-1}. Therefore, since $R \in C$, it suffices to prove that R is H_{i-1} (since R is C_{i-1} if and only if R is H_{i-1} and H_i [11, (2.19)] (since $R \in C$) and since R is H_i, by hypothesis). Now it may clearly be assumed that $0 < i < a$, so, by (2.5.3), R is H_{i-1} if and only if $i \notin s(R)$. Therefore, assume that $i \in s(R)$, and let p be a height i-1 prime ideal in R such that depth $p = 1$ (2.2.1). Then, for each fixed $c \in M$, height $p^+ = i-1$ and depth $p^+ = 2$ (3.2.2), hence $i+1 \in s(\mathcal{L}(R,cR)) - \{a+1\}$. Therefore, if there exists $b \in M$ such that $i+1 \in s(\mathcal{L}(R,bR)) - \{a+1\}$, then $i \notin s(R)$, so R is H_{i-1}, hence R is C_{i-1}, q.e.d.

(3.7) is, clearly, a strong converse of (3.3.1), so, if $R \in C$, then there exists $b \in E$ such that $\mathcal{L}(R,bR)$ is H_i if and only if R is H_i and there exists $c \in E$ such that $i+1 \notin s(\mathcal{L}(R,cR)) - \{a+1\}$. In this form we have a generalization of (2.6) (since $D_1 \cong \mathcal{L}(R,(0))$).

This section will be closed with an application of (2.14). To prove (3.8), the following known result is needed: if there exist $k \geqslant 1$ and $i \geqslant 1$ such that D_k is H_{i+k-1}, then, for all ideals $B = (b_1,\ldots,b_k)R$ such that height $B \geqslant 1$, $\mathcal{L}(R,B)$ is H_i [10, (4.2)].

(3.8) PROPOSITION. *If R is* H_{i+k-1} *and* $i+1,\ldots,i+k \notin s(D_l) - \{a+1\}$, *then, for all ideals* $B = (b_1,\ldots,b_k)R$ *such that height* $B \geqslant 1$, $\mathcal{L}(R,B)$ *is* H_i.

PROOF. This follows immediately from (2.14) and [10, (4.2)], q.e.d.

4. A theorem on the f.c.c. In this section we prove, as one application of (4.4), that if D_1 is H_i and $i < \ell-1$ (with ℓ as in (2.19)), then $(D_k)_P$ satisfies the f.c.c., for all $k \geqslant 0$ and for all prime ideals P in D_k such that height $P \leqslant i$ (4.5.2).

To prove (4.4), we need the following result:

(4.1) PROPOSITION. *Assume that R is* H_i. *Then, for all height i prime ideals p in R,* R_p *is* H_j, *for all* $j \geqslant 2i-a$.

PROOF. By (2.4.1), it may be assumed that $i < a-1$. Let p be a height i prime ideal in R, and assume that there exists a prime ideal $q \subset p$ such that $j = \text{height } q \geqslant 2i-a$. Let $d = \text{height } p/q$. Then it suffices to show that $j+d = i$.

Suppose that $j+d < i$. Then, since $p \neq M$, by repeated use of [1, Theorem 1] and [8, (2.2)], there exists a prime ideal P in R such that $q \subset P$, height $P = j+d$, and depth P = depth p = a-i (since R is H_i). Now consider a saturated chain of prime ideals $P \subset p_1 \subset \cdots \subset p_{a-i} = M$. Since $2i \leqslant a+j$, $a-i \geqslant i-j > i-(j+d) > 0$ (it is clear that $j+d > 0$). Thus this chain is $P \subset \cdots \subset p_{i-(j+d)} \subset \cdots p_{a-i} = M$. By [3, Lemma 1], we may assume that height $p_{i-(j+d)}$ = height $P+i-(j+d) = i$. However, since depth $P = a-i$, clearly depth $p_{i-(j+d)} = a-i-(i-(j+d))$. But R is H_i, so depth $p_{i-(j+d)} = a-i$. Therefore $a-2i+j+d = a-i$, so $j+d = i$; contradiction. Therefore R_p is H_j, for all $j \geqslant 2i-a$, q.e.d.

(4.1) allows us to prove the following interesting result:

(4.2) COROLLARY. *If $0 \leqslant i \leqslant a/2$ and if R is H_i, then, for all height i prime ideals p in R, R_p satisfies the f.c.c.*

PROOF. By (4.1), R_p is H_j, for all $j \geqslant 2i-a$. Now, by hypothesis, $2i \leqslant a$, so R_p is H_j, for all $j \geqslant 0$. Therefore R_p satisfies the f.c.c. [3, Proposition 7], q.e.d.

From (4.2) we get yet another variation of (2.16) and (2.32.3).

(4.3) COROLLARY. *If D_{a+2} is H_{a+2}, then R satisfies the s.c.c.*

PROOF. $(D_{a+2})_{(M,X_1)}$ satisfies the f.c.c. (4.2), so D_1 satisfies the f.c.c. [8, Theorem 4.11], hence R satisfies the s.c.c. [7, Theorem 2.21], q.e.d.

We now come to the main result in this section. It is the application of this result to the results in Section 2 which make it particularly interesting.

(4.4) THEOREM. *If there exists $i \geqslant 0$ such that, for all $h \geqslant 0$, D_h is $H_0, \ldots, H_i$, then, for all $n \geqslant 0$ and for all prime ideals P in D_n such that height $P \leqslant i$, $(D_n)_P$ satisfies the f.c.c.*

PROOF. Let $n \geqslant 0$, and let P be a prime ideal in D_n such that j = height $P \leqslant i$. Let $m = \max\{n, 2j-a\}$. By [8, Theorem 4.11], it is enough to show that $(D_m)_{PD_m}$ satisfies the f.c.c. Now height PD_m = height $p = j \leqslant i$ and D_m is H_j, by assumption. Also, $m \geqslant 2j-a$, so $j \leqslant (m+a)/2$ = (altitude D_m)/2, hence $(D_m)_{PD_m}$ satisfies the f.c.c. (4.2), q.e.d.

(4.5) REMARK. The hypothesis of (4.4) is satisfied in the following cases:

(4.5.1) There exist $k \geqslant 1$ and $i \leqslant k$ such that D_k is H_i, by (2.17).

(4.5.2) D_1 is H_i, for some $i < \ell-1$ (with ℓ as in (2.19)), by (2.20).

(4.5.3) There exist $k \geqslant 1$ and $i < \ell+k-2$ such that D_k is H_i (with ℓ as in (2.19)), by (2.21.1).

(4.5.4) There exist $k \geqslant 1$ and $i \geqslant 1$ such that D_k is $H_1,\ldots,H_i$, by (2.23.1).

(4.5.5) There exist $k \geqslant 1$ and $i \geqslant k$ such that D_k is $H_k,\ldots,H_i$, by (2.23.2).

(4.5.6) There exists $k \geqslant \max\{1,i,j\}$ such that D_k is H_i and H_{i+j}, by (2.29).

(4.5.7) R is an integrally closed H_1-local domain and there exists $k \geqslant 1$ such that D_k is H_{k+1}, by (2.34).

(4.6) COROLLARY. *With the hypothesis of (4.4), for all $k \geqslant 0$ and for all prime ideals P in D_k such that height $P < i$ and depth $P \geqslant 1$, $(D_k)_P$ satisfies the s.c.c.*

PROOF. Let $k \geqslant 0$ and let P be a prime ideal in D_k such that $h = \text{height } P < i$ and depth $P \geqslant 1$. To prove that $(D_k)_P$ satisfies the s.c.c., it suffices to prove that $(D_{k+1})_{PD_{k+1}}$ satisfies the s.c.c. (the proof is straightforward by the definition). For this, let $P^* = (P,X_{k+1})D_{k+1}$, so height $P^* \leqslant i$, hence $(D_{k+1})_{P^*}$ satisfies the f.c.c. (4.4). Therefore, since depth $P(D_{k+1})_{P^*} = 1$, $(D_{k+1})_{PD_{k+1}}$ satisfies the s.c.c. [8, Theorem 3.9], q.e.d.

In analogy to (4.4) and (4.5), the last two results of this section consider the opposite extreme.

(4.7) THEOREM. *If there exist $i \geqslant 0$ and $k \geqslant 0$ such that, for all $h \geqslant -k$, D_{k+h} is $H_{i+h},\ldots,H_{a+h}$, then, for all height $\geqslant i$ prime ideals p in R and for all $n \geqslant 0$, D_n/pD_n satisfies the f.c.c., so R/p satisfies the s.c.c.*

PROOF. By [7, Theorem 2.6], it suffices to prove that, for all prime ideals p in R such that height $p \geqslant i$, R/p satisfies the s.c.c. For this, it suffices to prove that D_1/pD_1 satisfies the f.c.c. [7, Theorem 2.21]. Now, if p is a prime ideal in R such that $j = \text{height } p \geqslant i$, then D_1/pD_1 is $H_1,\ldots,H_{a+1-j}$ (2.4.2); and D_1/pD_1 is clearly H_o. Also, altitude $D_1/pD_1 \leqslant a+1-j$. Therefore D_1/pD_1 satisfies the f.c.c. [3, Proposition 7], q.e.d.

(4.8) REMARK. The hypothesis of (4.7) is satisfied in the following cases:

(4.8.1) $k = 0$ and $i = g-1$ (with g as in (2.19)), by (2.24).

(4.8.2) There exist $i \geqslant 0$ and $k \geqslant 1$ such that D_k is $H_i,\ldots,H_{a+k-1}$, by (2.30).

(4.8.3) There exists $k \geqslant 1$ such that D_k is H_{a-1}, by (2.31) (let $i = a-1$).

(4.8.4) There exist $i \geqslant 0$ and $k \geqslant 1$ such that D_k is $H_i,\ldots,H_{a-1}$, by (2.33.1).

(4.8.5) There exist $k \geqslant 1$ and $i \geqslant a-1-k$ such that D_k is H_i and H_{a-1}, by

(2.33.2).

5. Concluding remarks. We close this paper with the following two remarks which have the effect of greatly extending and generalizing the results in this paper.

(5.1) REMARK. Throughout this paper attention has been directed at D_k. It is conceivable that if a maximal ideal $N \neq (M,X_1,...,X_k)$ in $R_k = R[X_1,...,X_k]$ had been chosen, then different results would be obtained. This isn't true, if $N \cap R = M$, since in this case, D_k is H_i (respectively, C_i, satisfies the f.c.c. or the s.c.c.) if and only if $(R_k)_N$ is H_i (respectively, C_i, satisfies the f.c.c or the s.c.c.) [14, (5.5)].

(5.2) REMARK. Our base ring throughout has been a local ring (R,M). All the results in this paper can be generalized to the case where R is replaced by a quasi-local ring (S,N) which contains and is integral over R and is such that minimal prime ideals in S lie over minimal prime ideals in R. For: (a) by [13, (2.17) and (3.18)] and [8, Remark 2.24(ii) and (iv), and Theorem 3.2], R is H_i (respectively, C_i, satisfies the f.c.c. or the s.c.c.) if and only if S is H_i (respectively, C_i, satisfies the f.c.c. or the s.c.c.); (b) by [14, (3.4)], $n \in s(R)$ if and only if $n \in s(S)$; (c) it is clear that D_k and $S[X_1,...,X_k]_{(N,X_1,...,X_k)}$ satisfy the hypotheses on R and S; and, (d) it is easily seen that $\mathcal{L}(S,B)$ and $\mathcal{L}(R_1,B_1)$ satisfy the hypotheses on R and S, where $B = (b_1,...,b_k)S$, $R_1 = R[b_1,...,b_k]$, and $B_1 = (b_1,...,b_k)R_1$.

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