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THE HOPF EXTENSION THEOREM FOR TOPOLOGICAL SPACES

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1. **Introduction.** Let X be a topological space. Then by the covering dimension of X, denoted by dim X, we mean, as in our previous paper [5], the least integer n such that every finite normal open cover of X is refined by a finite normal open cover of X of order $\leqslant n+1$; in case there is no such an integer n, we define $\dim X = \infty$.

Let A be a subset of X. For each integer $n \geqslant 0$, let $H^n(X,A)$ be the n-th Čech cohomology group of (X,A) with coefficients in the additive group of integers which is defined by using locally finite normal open covers of X.

Let I be the closed unit interval [0,1] in the real line and $\dot{I}^n$ the boundary of I^n. A continuous map $f: (X,A) \to (I^n, \dot{I}^n)$ is called essential if any continuous map $g: (X,A) \to (I^n, \dot{I}^n)$ with $g|A = f|A$ satisfies $g(X) = I^n$; otherwise f is said to be inessential. Hence f is inessential iff there is a continuous map $g: X \to \dot{I}^n$ which is an extension of $f|A$.

Thus, the following theorem may be viewed as a generalization of the Hopf extension theorem.

THEOREM 1. *Let (X,A) be a pair of topological spaces such that* $\dim X/A \leqslant n$. *Then a continuous map* $f: (X,A) \to (I^n, \dot{I}^n)$ *is inessential iff* $f^*: H^n(I^n, \dot{I}^n) \to H^n(X,A)$ *is zero, where* $n \geqslant 2$.

Now, let us consider the case where A is C-embedded in X. Then, by Shapiro [8], for every locally finite, countable, normal open cover **U** of A there is a locally finite, countable, normal open cover **V** of X such that $\mathbf{V} \cap A$ refines **U**. On the other hand, the Čech cohomology groups with coefficients in the additive group of integers are naturally isomorphic to the corresponding Čech cohomology groups defined by using locally finite, countable, normal open covers (cf. [5, Theorem 6.8]). Hence we can define the coboundary operator $\delta: H^{n-1}(A) \to H^n(X,A)$, and the cohomology sequence of (X,A).

$$\cdots \to H^{n-1}(X) \xrightarrow{i^*} H^{n-1}(A) \xrightarrow{\delta} H^n(X,A) \to \cdots$$

is exact, where i: $A \to X$ is the inclusion map.

In the commutative diagram

$$\begin{array}{ccccccc} H^{n-1}(X) & \xrightarrow{\quad i^* \quad} & H^{n-1}(A) & \xrightarrow{\quad \delta \quad} & H^n(X,A) & & \\ & & \uparrow (f \mid A)^* & & \uparrow f^* & & \\ 0 & \longrightarrow & H^{n-1}(\dot{I}^n) & \xrightarrow{\quad \delta \quad} & H^n(I^n, \dot{I}^n) & \longrightarrow & 0 \end{array}$$

the upper and lower sequences are exact.

Therefore from Theorem 1 we obtain the following theorem which has the same form as the usual Hopf extension theorem.

THEOREM 2. *Let X be a topological space and A a subset of X which is C-embedded in X. Let dim* $X/A \leqslant n$ *and* $n \geqslant 2$. *Then a continuous map g:* $A \to \dot{I}^n$ *can be extended to a continuous map from X to* $\dot{I}^n$ *iff* $g^*H^{n-1}(\dot{I}^n) \subset i^*H^{n-1}(X)$.

The Hopf extension theorem has been proved hitherto for the following cases.

I. (X,A) is a relative CW complex and $\dim(X\text{-}A) \leqslant n$ (Spanier [9]).

II. X is a paracompact normal space with A closed and $\dim X \leqslant n$ (Dowker [1]).

In case I the condition "$\dim(X\text{-}A) \leqslant n$" is equivalent to "$\dim X/A \leqslant n$", and, as was proved in [5], $\dim X \leqslant n$ implies $\dim X/A \leqslant n$ for any pair (X,A) of spaces.

In both cases A is C-embedded in X.

In case I the singular cohomology groups are used. But, if (X,A) is a relative CW complex, then the Čech cohomology group $H^n(X,A)$ is naturally isomorphic to the corresponding singular cohomology group of (X,A) by virtue of [5, Theorem 6.1] and [9, P. 428], and the cohomology sequence of (X,A) is exact for both cohomology groups, and hence the condition "$g^*H^{n-1}(\dot{I}^n) \subset i^*H^{n-1}(X)$" remains to be equivalent if $H^{n-1}(A)$ and $H^{n-1}(X)$ are replaced by the corresponding singular cohomology groups.

Thus, our Theorem 2 contains the Hopf extension theorem for the cases I and II above.

In §4, Theorem 1 and the arguments in its proof will be applied to covering dimension.

Throughout the paper, N denotes the set of positive integers.

2. **Proof of Theorem 1.** Since the "only if" part is obvious, we have only to

prove the "if" part.

Since $\dim X/A \leqslant n$, for any normal open cover **G** of X there exists a locally finite normal open cover **H** of X such that **H** is a refinement of **G** and the order of $\{H, \mathrm{St}(A,\mathbf{H}) \mid H \in \mathbf{H}, H \cap A = \emptyset\}$ does not exceed n+1.

To prove this, let $\mathbf{G} = \{G_\lambda \mid \lambda \in \Lambda\}$ be a locally finite cozero-set cover of X and let $\Lambda_0 = \{\lambda \in \Lambda \mid G_\lambda \cap A \neq \emptyset\}$. Let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a locally finite zero-set cover of X such that $F_\lambda \subset G_\lambda$ for each $\lambda \in \Lambda$. Let us put

$$F_0 = \cup\{F_\lambda \mid \lambda \in \Lambda_0\}, \quad G_0 = \cup\{G_\lambda \mid \lambda \in \Lambda_0\}.$$

Then by [6, Lemma 2.3] F_0 is a zero-set and G_0 is a cozero-set. Hence there is a continuous map $h: X \to I$ such that $h^{-1}(0) = F_0$, $h^{-1}(1) = X - G_0$. Let us put $L = \{x \in X \mid h(x) \leqslant ½\}$. Then $A \subset F_0 \subset \mathrm{Int}\, L$. Hence $\{G_\lambda - L, G_0 \mid \lambda \in \Lambda - \Lambda_0\}$ is the inverse image of a locally finite cozero-set cover of X/A under the quotient map of X onto X/A. Hence there is a locally finite cozero-set cover $\{H_\lambda, H_0 \mid \lambda \in \Lambda - \Lambda_0\}$ of X such that its order does not exceed n+1 and

$$H_\lambda \subset G_\lambda - L \text{ for each } \lambda \in \Lambda - \Lambda_0;\ A \subset H_0 \subset G_0.$$

Then $\mathbf{H} = \{H_\lambda, H_0 \cap G_\mu \mid \lambda \in \Lambda - \Lambda_0, \mu \in \Lambda_0\}$ is a locally finite cozero-set cover of X and $H_0 = \mathrm{St}(A,\mathbf{H})$. Thus **H** is a desired cover of X.

Let $\{\mathbf{W}_i \mid i \in N\}$ be a normal sequence of open covers of I^n such that each set belonging to $\mathbf{W}_i$ has diameter $\leqslant 1/i$. Then there is a normal sequence $\Phi = \{\mathbf{U}_i \mid i \in N\}$ of open covers of X satisfying the following conditions:

(1) $\mathbf{U}_i$ is a refinement of $f^{-1}(\mathbf{W}_i)$ for $i \in N$,

(2) order $\{U, \mathrm{St}(A,\mathbf{U}_i) \mid U \in \mathbf{U}_i, U \cap A = \emptyset\} \leqslant n+1$.

Let (X,Φ) be a topological space obtained from X by taking $\{\mathrm{St}(x,\mathbf{U}_i) \mid i \in N\}$ as a local base at each point x of X, and X/Φ the quotient space obtained from (X,Φ) by identifying two points x and y such that $y \in \mathrm{St}(x,\mathbf{U}_i)$ for each $i \in N$. Let us denote by Φ the composite of the identity map from X to (X,Φ) and the quotient map from (X,Φ) to X/Φ. Then $\phi: X \to X/\Phi$ is a continuous map and the space X/Φ is metrizable. This fact is proved in [4].

For any subset K of X let us put

$$\mathrm{Int}(K;\Phi) = \{x \in X \mid \exists\, i \in N:\ \mathrm{St}(x,\mathbf{U}_i) \subset K\}.$$

Then $\mathrm{Int}(K;\Phi)$ is an open set of (X,Φ) and $\phi^{-1}\phi(\mathrm{Int}(K;\Phi)) = \mathrm{Int}(K;\Phi)$. Let us put further

(3) $B_i = \mathrm{Int}(\mathrm{St}(A,U_i);\Phi)$,

(4) $V_i = \{\mathrm{Int}(U;\Phi) \mid U \in U_i\}$.

Then V_i is an open cover of X such that $V_{i+1} > U_{i+1} > V_i$ where ">" means "is a refinement of". Moreover, $\{\phi(V_i) \mid i \in N\}$ is a normal sequence of open covers of X/Φ such that $\{\mathrm{St}(y,\phi(V_i)) \mid i \in N\}$ is a local base at any point y of X/Φ. This fact is proved in [4].

Since

(5) $\mathrm{St}(A,U_{i+1}) \subset B_i \subset \mathrm{St}(A,U_i)$

we have

$$\mathrm{Cl}\ \phi(A) \subset \mathrm{St}(\phi(A), \phi(V_{i+1})) \subset \phi(B_i) \subset \mathrm{St}(\phi(A), \phi(V_{i-1}))$$

and hence

(6) $\mathrm{Cl}\ \phi(A) = \cap\{\phi(B_i) \mid i \in N\}$.

On the other hand, for $j \geqslant i+1$ we have by (5)

(7) $(\phi(X) - \phi(B_i)) \cap \phi(U) = \emptyset$ for $U \in U_j$ with $U \cap A \neq \emptyset$.

Since V_j is a cover of X, we have

(8) $\phi(X) - \phi(B_i) \subset \cup\ \{\phi(\mathrm{Int}(U;\Phi) \mid U \in U_j, U \cap A = \emptyset\}$.

From (2), (7) and (8) it follows that the order of $\phi(V_j)$ on $\phi(X) - \phi(B_i)$ does not exceed n+1 for $j > i$. Hence by Nagata [7]

(9) $\dim(\phi(X) - \phi(B_i)) \leqslant n$ for $i \in N$.

Therefore, by (6) and by the sum theorem on dimension we have

(10) $\dim(X/\Phi - \mathrm{Cl}\ \phi(A)) \leqslant n$.

By (1) there is a continuous map $g: (X/\Phi, \phi(A)) \to (I^n, \dot{I}^n)$ such that $f = g \circ \phi$.

Let $\{\Phi_\alpha \mid \alpha \in \Omega\}$ be the set of all normal sequences of open covers of X satisfying (1), (2) and (3). If each cover of Φ_α is refined by some cover of Φ_β we write $\Phi_\alpha < \Phi_\beta$; in this case there is a canonical map $\phi_\alpha^\beta: X/\Phi_\beta \to X/\Phi_\alpha$. If $\Phi_\alpha < \Phi_\beta < \Phi_\gamma$, then $\phi_\alpha^\beta \circ \phi_\beta^\gamma = \phi_\alpha^\gamma$ and if ϕ_α denotes the map from X to X/Φ_α defined above then $\phi_\alpha = \phi_\alpha^\beta \circ \phi_\beta$ when $\Phi_\alpha < \Phi_\beta$ (cf. [4]). As is proved in [5], $\{\phi_\alpha^* \mid \alpha \in \Omega\}$ defines an isomorphism

$$\varinjlim \{H^n(X/\Phi_\alpha, \phi_\alpha(A)), (\phi_\alpha^\beta)^*\} \cong H^n(X,A).$$

Let $g_\alpha: (X/\Phi_\alpha, \phi_\alpha(A)) \to (I^n, \dot{I}^n)$ be a continuous map defined above such that $f = g_\alpha \circ \phi_\alpha$. Then, if $\Phi_\alpha < \Phi_\beta$ we have $g_\beta = g_\alpha \circ \phi_\alpha^\beta$.

Therefore, if $f^*: H^n(I^n,\dot{I}^n) \to H^n(X,A)$ is zero, then there is some $\alpha \in \Omega$ such that

g_α^*: $H^n(I^n, \dot{I}^n) \to H^n(X/\Phi_\alpha, \phi_\alpha(A))$ is zero.

Since $g_\alpha(\mathrm{Cl}\, \phi_\alpha(A)) \subset \dot{I}^n$, g_α is viewed as a continuous map h_α: $(X/\Phi_\alpha, \mathrm{Cl}\, \phi_\alpha(A)) \to (I^n, \dot{I}^n)$ and we have $g_\alpha = h_\alpha \circ \psi_\alpha$ where ψ_α: $(X/\Phi_\alpha, \phi_\alpha(A)) \to (X/\Phi_\alpha, \mathrm{Cl}\, \phi_\alpha(A))$ is the inclusion map. Since ψ_α^*: $H^n(X/\Phi_\alpha, \mathrm{Cl}\, \phi_\alpha(A)) \to H^n(X/\Phi_\alpha, \phi_\alpha(A))$ is an isomorphism, h_α^*: $H^n(I^n, \dot{I}^n) \to H^n(X/\Phi_\alpha, \mathrm{Cl}\, \phi_\alpha(A))$ is also zero. Moreover, by (10) $\dim(X/\Phi_\alpha)/\mathrm{Cl}\, \phi_\alpha(A) \leqslant n$.

Therefore, if the "if" part of Theorem 1 is proved for the case where X is metrizable and A is closed in X, then the map h_α, and hence g_α, is inessential and consequently f is inessential.

Thus, as for the "if" part of Theorem 1 we have only to prove it for the case where X is metrizable and A is closed in X.

3. **Proof of Theorem 1 for the case of metric spaces.** Let X be a metric space and A a closed subset of X such that $\dim X/A \leqslant n$. Let $\mathbf{G} = \{ G_\lambda \mid \lambda \in \Lambda \}$ be any locally finite normal open cover of X. The following argument is given in [2, the proof of Theorem 2.2].

There exist two collections $\{ P_\lambda \mid \lambda \in \Lambda_o \}$ and $\{ H_\lambda \mid \lambda \in \Lambda_o \}$ of open subsets of X such that $\Lambda_o \subset \Lambda$ and

(11) $\mathrm{Cl}\, P_\lambda \subset H_\lambda \subset G_\lambda$, $A \cap P_\lambda \neq \emptyset$ for $\lambda \in \Lambda_o$; $A \subset \cup P_\lambda$.

(12) $\{A \cap P_\lambda \mid \lambda \in \Lambda_o\}$ is similar to $\{H_\lambda \mid \lambda \in \Lambda_o\}$.

Since $\dim X/A \leqslant n$, we have $\dim(X - \cup P_\lambda) \leqslant n$ and hence by [3, Theorem 1.2] there is a locally finite collection $\mathbf{V}$ of open subsets of X such that

(13) $\mathbf{V}$ is a refinement of $\{ H_\lambda, X - \mathrm{Cl}\, P_\lambda \}$ for each $\lambda \in \Lambda_o$ and also a refinement of $\mathbf{G}$,

(14) $X - \cup P_\lambda \subset \cup \{V \mid V \in \mathbf{V}\}$,

(15) order $\mathbf{V} \leqslant n+1$.

Let us well-order Λ_o such that $\Lambda_o = \{\lambda \mid \lambda < \alpha_o\}$ for some ordinal α_o and let Q_λ be the union of all $V \in \mathbf{V}$ such that $V \cap P_\lambda \neq \emptyset$ and $V \cap P_\alpha = \emptyset$ for each α with $\alpha < \lambda$. The sets $V \in \mathbf{V}$ for which $V \cap P_\lambda = \emptyset$ for all $\lambda < \alpha_o$ shall be denoted by $\{ V_\mu \mid \mu < \beta_o \}$. Let us put

$$W_\lambda = P_\lambda \cup Q_\lambda \text{ for } \lambda < \alpha_o.$$

Then by (13) we have $W_\lambda \subset H_\lambda$ for $\lambda < \alpha_o$.

If $\lambda_1 < \cdots < \lambda_r < \alpha_o$, $\nu_1 < \cdots < \nu_s < \beta_o$ and

$$(\bigcap_{i=1}^{r} W_{\lambda_i}) \cap (\bigcap_{i=1}^{s} V_{\nu_i}) \neq \emptyset ,$$

and if this intersection does not meet A, then $r+s \leqslant n+1$. Indeed, we have then $s \geqslant 1$ by (12) and hence

$$(\bigcap_{i=1}^{r} W_{\lambda_i}) \cap (\bigcap_{i=1}^{s} V_{\nu_i}) = (\bigcap_{i=1}^{r} Q_{\lambda_i}) \cap (\bigcap_{i=1}^{s} V_{\nu_i})$$

and hence $r+s \leqslant n+1$ by (15). Let us put

$$\mathbf{U} = \{ W_\lambda, V_\mu \mid \lambda < \alpha_0, \mu < \beta_0 \} .$$

Then by (14) **U** is a cover of X and is a refinement of **G**.

Let N(**U**) (resp. N(**U** $\cap$ A)) be the nerve of **U** (resp. **U** $\cap$ A) with the weak topology. Then by the above consideration each simplex of N(**U**) which is not contained in N(**U** $\cap$ A) has dimension $\leqslant n$ (that is, $\dim(N(\mathbf{U}) - N(\mathbf{U} \cap A)) \leqslant n$).

Let $f: (X,A) \to (I^n, \dot{I}^n)$ be a continuous map such that $f^*: H^n(I^n, \dot{I}^n) \to H^n(X,A)$ is zero. Then there are a locally finite normal open cover **U** of X and a continuous map $g: (N(\mathbf{U}), N(\mathbf{U} \cap A)) \to (I^n, \dot{I}^n)$ satisfying the following conditions, where $\phi: (X,A) \to (N(\mathbf{U}), N(\mathbf{U} \cap A))$ is a canonical map (cf. [5, §4]).

(15) $f \cong g \circ \phi: (X,A) \to (I^n, \dot{I}^n)$,

(16) $g^* = 0: H^n(I^n, \dot{I}^n) \to H^n(N(\mathbf{U}), N(\mathbf{U} \cap A))$,

(17) $\dim(N(\mathbf{U}) - N(\mathbf{U} \cap A)) \leqslant n$.

Hence, by Theorem 2 for the case of complexes, which is equivalent to Theorem 1 in this case as was shown in the introduction and has been proved already (cf. [9]), g is inessential. Hence $g \circ \phi$ is inessential. Therefore, by the homotopy extension theorem of Borsuk f is also inessential. Thus, the proof of the "if" part of Theorem 1 for the special case of X being a metric space with A closed is completed. Hence our Theorem 1 is completely proved.

4. Some theorems on covering dimension. A combination of the arguments in §2 with those in §3 yields the "only if" part of the following theorem.

THEOREM 3. *Let (X,A) be a pair of topological spaces. Then* $\dim X/A \leqslant n$ *iff for any finite normal open cover* **G** *of X there is a locally finite normal open cover* **U** *of X such that* **U** *is a refinement of* **G** *and* $\dim N(\mathbf{U})/N(\mathbf{U} \cap A) \leqslant n$.

To prove the "if" part, let $\mathbf{G} = \{ G_0, G_1, \cdots, G_s \}$ be any finite cozero-set cover of X such that it is the inverse image of a finite cozero-set cover of X/A under the quotient map of X onto X/A. Without loss of generality we may assume that $A \subset G_0$ and $A \cap G_i = \emptyset$ for $i \geqslant 1$. Then by assumption there is a locally finite cozero-set cover **U** of X such that it refines **G** and dim N(**U**)/ N(**U** $\cap$ A) $\leqslant$ n. Then from **U** we can construct a finite cozero-set cover **V** = $\{V_0, V_1, \cdots, V_s\}$ of X such that $V_i \subset G_i$ for $i = 0,1,\cdots,s$ and that dim N(**V**) $\leqslant$ n, by forming the unions of suitable members of **U**. This proves the "if" part of Theorem 3.

COROLLARY 4. *Let A be C*-embedded in a topological space X. Then dim X = Max(dim A, dim X/A).*

PROOF. If dim X $\leqslant$ n, then dim A $\leqslant$ n and dim X/A $\leqslant$ n by [5, Lemmas 5.8 and 5.14]. Conversely, suppose that dim A $\leqslant$ n and dim X/A $\leqslant$ n. Let **G** be any finite normal open cover of X. Then there is a finite normal open cover **H** of X which is a refinement of **G** with dim N(**H** $\cap$ A) $\leqslant$ n. Let $\{ H_i \mid i=1,\cdots,p \}$ be the totality of $H \in \mathbf{H}$ with $H \cap A \neq \emptyset$. By Theorem 3 we can find a finite cozero-set cover **V** of X such that **V** is a refinement of **H** and dim N(**V**)/N(**V** $\cap$ A) $\leqslant$ n. Let us put

$$U_1 = \cup \{V \in \mathbf{V} \mid V \cap A \neq \emptyset, V \subset H_1 \},$$

$$U_i = \cup \{V \in \mathbf{V} \mid V \cap A \neq \emptyset, V \not\subset H_j \text{ for } j < i, V \subset H_i \}$$

for $2 \leqslant i \leqslant p$; let

$$V_1, \cdots, V_q$$

be the totality of $V \in \mathbf{V}$ with $V \cap A = \emptyset$.

Assume that

$$L = \bigcap_{\lambda=1}^{r} U_{i_\lambda} \cap \Big(\bigcap_{\mu=1}^{s} V_{j_\mu} \Big) \neq \emptyset$$

for $1 \leqslant i_1 < \cdots < i_r \leqslant p$ and $1 \leqslant j_1 < \cdots < j_s \leqslant q$. If $L \cap A \neq \emptyset$ then $s = 0$ and $\bigcap_{\lambda=1}^{r} H_{i_\lambda} \cap A \neq \emptyset$ and hence $r+s = r \leqslant n+1$. If $L \cap A = \emptyset$, then there are V'_{i_λ} of **V** for $1 \leqslant \lambda \leqslant r$ such that

$$V'_{i_\lambda} \subset U_{i_\lambda} \text{ for each } \lambda \text{ and } M = \bigcap_{\lambda=1}^{r} V'_{i_\lambda} \cap \Big(\bigcap_{\mu=1}^{s} V_{j_\mu} \Big) \neq \emptyset$$

Since $M \cap A = \emptyset$ and dim N(**V**)/N(**V** $\cap$ A) $\leqslant$ n, we have $r+s \leqslant n+1$. Thus, if we put

$$\mathbf{U} = \{U_1, \cdots, U_p, V_1, \cdots, V_q \},$$

then **U** is a finite cozero-set cover of X and a refinement of **H** and hence of **G**. Moreover dim N(**U**) $\leqslant$ n. This completes the proof of Corollary 4.

The following theorem is obtained as another application of the arguments in §2.

In case $A = \emptyset$, by X/A we mean the disjoint union of X and one point. Hence, if G is open in X, then X/(X-G) = Cl G/Bd G.

THEOREM 5. *Let* **G** *be a normal open cover of a topological space X. If dim Cl G/Bd G* $\leqslant n$ *for each* $G \in \mathbf{G}$, *then dim* $X \leqslant n$.

Roughly speaking, Theorem 5 asserts that if X is uniformly locally at most n-dimensional then X is at most n-dimensional.

Before proceeding to the proof of Theorem 5, we shall prove

LEMMA 6. *If* $\{G_\lambda \mid \lambda \in \Lambda\}$ *is a discrete collection of open subsets of X and if dim Cl* G_λ*/Bd* $G_\lambda \leqslant n$ *for each* $\lambda \in \Lambda$, *then dim Cl G/Bd G* $\leqslant n$ *for* $G = \cup \{G_\lambda \mid \lambda \in \Lambda\}$.

PROOF. The space Cl G/Bd G is homeomorphic to the quotient space obtained from the disjoint union of Cl G_λ/Bd G_λ by identifying all the base points of Cl G_λ/Bd G_λ to a single point, where the base point of X/A is either the point to which all the points of A are identified or the point which is added by definition in case $A = \emptyset$. Hence Lemma 6 is proved easily.

PROOF OF THEOREM 5. If G and G′ are open in X and $G \subset G'$, then dim X/(X-G) $\leqslant$ dim X/(X-G′). Since any normal open cover of X is refined by a σ-discrete cozero-set cover of X, by Lemma 6 we have only to prove the theorem for the case where **G** is a countable normal open cover of X; let $\mathbf{G} = \{G_i \mid i \in N\}$.

Let **V** be any countable normal open cover of X.

Then there is a normal sequence $\Phi = \{\mathbf{U}_i \mid i \in N\}$ of open covers of X satisfying the following conditions.

(1)′ $\mathbf{U}_1$ is a refinement of **G** and of **V**,

(2)′ order $\{U, St(X-G_j, \mathbf{U}_i) \mid U \in \mathbf{U}_i, U \subset G_j\} \leqslant n+1$ for each $i \geqslant 2$ and $j \leqslant i$.

Here X/Φ and $\phi: X \to X/\Phi$ have the same meaning as in §2.

Hence by the arguments in §2 we have

(10)′ $\dim(X/\Phi - Cl\ \phi(X-G_j)) \leqslant n$ for $j \in N$.

If we put $H_j = Int(G_j; \Phi)$ (for the notation, cf.§2), then $\mathbf{H} = \{H_j \mid j \in N\}$ is an open cover of X and $\phi^{-1}\phi(H_j) = H_j$. Since $\phi(X-G_j) \subset \phi(X) - \phi(H_j)$, we have

$$\cap \{Cl\ \phi(X-G_j) \mid j \in N\} = \emptyset.$$

By the sum theorem on dimension, we conclude from (10)′ that dim $X/\Phi \leqslant n$. The theorem follows readily from the last inequality.

Combining the arguments in the proof of [5, Theorem 5.2] with Theorem 1, we have the following theorem, which is an extension of [5, Theorem 5.2].

THEOREM 7. *Let (X,A) be a pair of topological spaces such that $\dim X/A = n$, and let $f: (X,A) \to (I^n, \dot{I}^n)$ be a continuous map. If f is essential, so is the product map $f \times 1: (X,A) \times (I,\dot{I}) \to (I^{n+1}, \dot{I}^{n+1})$, where $f \times 1$ is defined by $(f \times 1)(x,t) = (f(x),t)$ for $x \in X$, $t \in I$ and $I^{n+1} = I^n \times I$.*

THEOREM 8. *If $\dim X/A = n$, then*

$$\dim(X \times I)/(A \times I \cup X \times \dot{I}) = n+1.$$

PROOF. Since $(X/A) \times I \cong (X \times I)/(A \times I)$, by [5, Theorem 5.7 and Lemma 5.14] we have

$$\begin{aligned} \dim(X \times I)/(A \times I \cup X \times \dot{I}) &\leqslant \dim(X \times I)/(A \times I) \\ &= \dim X/A + 1 = n + 1. \end{aligned}$$

On the other hand, by [5, Theorem 5.1] and the remark at the beginning of the proof of [5, Lemma 5.13] there are a closed subset B of X containing A and an essential map $f: (X,B) \to (I^n, \dot{I}^n)$. Hence by Theorem 7, $f \times 1: (X,B) \times (I,\dot{I}) \to (I^{n+1}, \dot{I}^{n+1})$ is essential and consequently $H^{n+1}(X \times I, (B \times I) \cup (X \times \dot{I})) \neq \emptyset$ by Theorem 1. Since $(B \times I) \cup (X \times \dot{I}) \supset (A \times I) \cup (X \times \dot{I})$, this shows that $\dim (X \times I)/(A \times I \cup X \times I) \geqslant n + 1$. Thus, Theorem 8 is proved.

COROLLARY 9 ([5, Lemma 5.13]). *Let (X,x_0) be a pointed space of finite dimension. Then $\dim SX = \dim X+1$, where SX is the reduced suspension of X.*

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