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SOME TYPES OF SIMPLE RING EXTENSIONS[1]

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Throughout the present article, we understand by a ring a commutative ring with identity, and by an over-ring of a ring R a ring containing R and having the same identity with R.

Let T be an over-ring of a ring R. If T is generated by a single element over R and if φ is a homomorphism of T into a ring T′, then φT is generated by a single element over φR. In §1, we discuss some results related to the converse of this fact, and we mainly concern with the case where T is integral over R.

In §2, we deal with a special type of ring extensions which has some algebro-geometric meaning in connection with birational correspondences, and our results include a criterion for such an extension to be generated by a single element.

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§1. Integrally dependent case. Obviously, the following holds:

LEMMA 1.1 *Let A be an ideal of an over-ring T of a ring R. If T is generated by a single element over R, then so is T/A over R/(A ∩ R).*

The converse of this is obviously false and we want to discuss some cases where the converse is true. We shall begin with the case where R is a field and then we shall discuss its generalization to the case where R is a quasi-semi-local ring.

THEOREM 1.2. *Let R be a field and assume that its over-ring T is a noetherian local ring with maximal ideal M and that T is integral over R. Then:*

(1) *Assume that T/M is a separable algebraic extension of a finite degree over R, then T is generated by a single element over R if and only if M is principal.*

[1] The results in this article were published already in the proceedings of the 1974 symposium in algebra (ring theory and number theory) at Matsuyama. But the proceedings were written in Japanese and copies of the proceedings were distributed only in Japan, and the writer is willing to publish an English version of the article.

(2) *Assume that T/M is an algebraic extension of R which is not separable. If T is generated by a single element over R, then T/M is obviously generated by a single element, say a^*, over R and for any representative a $(\in T)$ of a^*, T is generated by a over R.*

PROOF. (1): The only if part is obvious because $R[X]$ is a principal ideal ring. Assume that $M = sT$ and the $T/M = R(c^*)$ with a separable element c^*. Let $f(X) = X^n + a_1X^{n-1} + \dots + a_n$ be the minimal polynomial for c^* over R and let c be a representative of c^*. Then $f(c) \in M$. Let $f'(X)$ be the derivative of $f(X)$. Since c^* is separable, $f'(c)$ is not in M. (i) Assume first that $f(c)$ generates M. Then we see that $R[c]$ is a dense subset of T, [1]. Since natural topology of T is discrete, we see that $R[c] = T$. (ii) Assume now that $f(c)$ does not generate M. Then $f(c) \in M^2$. $f(c+s) \equiv f(c) + sf'(c)$ modulo M^2 and we see that $f(c+s)$ generates M. Thus, by (i) above, we see that T is generated by $c+s$.

(2): Assume that $T = R[b]$ and let b^* be (b modulo M). Obviously $T/M = R[b^*]$. Since $T/M = R[a^*]$, there is a polynomial $g(X) \in R[X]$ such that $g(a^*) = b^*$. If it holds that $R[g(a)] = T$, then we surely have $R[a] = T$. Thus we may assume that $a^* = b^*$. Let $f(X)$ be the minimal polynomial for b^* over R. Since b^* is not separable, we have $f(a) \equiv f(b)$ modulo M^2. Since $T = R[b]$, we have $M = f(b)T$. Therefore $f(a) - f(b) \in M^2$ implies that $f(a)T = M$, and therefore we see that $R[a] = T$. q.e.d.

THEOREM 1.3. *Assume that a noetherian semi-local ring $(T, M_1,\dots,M_r)$ is integral over its subfield R. Set $T_i = T_{M_i}$ $(i = 1,\dots,r)$. Then:*

(1) *If T is generated by a single element over R, then each T_i is also generated by a single element over R and T is the direct sum of $T_1,\dots,T_r$.*

(2) *Assume conversely that each T_i is generated by a single element b_i over R. Then T is the direct sum of $T_1,\dots,T_r$.*

(3) *Under the assumption in (2) above, set $b = b_1 + \dots + b_r$, $b^*_i = (b_i$ modulo $M_iT_i)$. Let $f_i(X)$ be the minimal polynomial for b^*_i over R. Then $T = R[b]$ if and only if $f_1(X),\dots,f_r(X)$ are mutually different from each other.*

PROOF. (1) and (2): Under the assumption, T is an Artin ring and the assertions follow easily.

(3): Let e_i be the identity of T_i. Then $1 = e_1 + \dots + e_r$. Since the maximal ideal of T_i is nilpotent, there is a natural number n such that $e_if_i(b_i)^n = 0$ for all $i = 1,\dots,r$. Then, setting $g(X) = (\Pi_i f_i(X))^n$, we have $g(b) = 0$, which shows that (the number of the maximal ideals of $R[b]$) $\leqslant$ (the number of mutually distinct polynomials among $f_1(X),\dots,f_r(X)$). Thus the only if part is proved. Conversely, assume that $f_1(X),\dots,f_r(X)$ are mutually different from each other. Set $g_i(X) = (\Pi_{j\neq i} f_j(X))^n$. Then we have $e_jg_i(b_j) = 0$ for $j \neq i$, and $e_ig_i(b_i) \notin M_iT_i$. Thus $g_i(b) = e_ig_i(b_i)$ which is in T_i outside of M_iT_i. Then there is an $h_i(X) \in R[X]$ such that $e_ih_i(b_i)$ is the inverse of $e_ig_i(b_i) = g_i(b)$ in T_i. Then, with $k_i(X) = g_i(X)h_i(X)$, we see that $k_i(b) = e_i$. Thus $R[b]$ contains all of e_i and we see that $R[b] = T$. q.e.d.

COROLLARY 1.4 *Assume that R is a field containing infinitely many elements and that an over-ring T is the direct sum of local rings $T_1,\dots,T_r$ and is integral over R. Then T is generated by a single element over R if and only if each T_i ($i = 1,\dots,r$) is generated by a single element over R.*

Note that these results 1.2 $\sim$ 1.4 can be generalized easily to the case where R is the direct sum of a finite number of fields in view of the following easy fact:

LEMMA 1.5 *Assume that a ring R is the direct sum of rings $R_1,\dots,R_m$ with identities $e_1,\dots,e_m$ respectively. Let T be an over-ring of R. Then T is generated by a single element over R if and only if every e_iT ($i=1,\dots,m$) is generated by a single element over R_i.*

In view of the lemma of Krull-Azumaya (which may be called the lemma of Nakayama), we see immediately the following

LEMMA 1.6. *Assume that R is a ring with Jacobson radical J and that T is an over-ring which is finitely generated as an R-module. Then T is generated by a single element over R if and only if T/JT is generated by a single element over R/J.*

Note here that our assumption that T is integral over R and is a finitely generated module over R is important in view of the fact that there is a discrete valuation ring V of a field such that (i) V is not complete and (ii) the completion of V is integral over V (cf. [1], pp. 205-207).

Anyway, this Lemma 1.6 enables us to generalize our results 1.2 $\sim$ 1.4 to the case where R is a quasi-semi-local ring.

§2. **Some results on UFD.** By a prime element of a ring, we understand an element which generates a non-zero prime ideal. The term UFD stands for unique factorization domain. The following result is well known and sometimes called a theorem of Nagata:

LEMMA 2.1. *Let S be a multiplicatively closed subset generated by some prime elements in an integral domain R, satisfying the ascending chain condition for principal ideals. If the ring R_S is a UFD, then R is also a UFD.*

We review here a well known fact on monoidal dilations as follows:

LEMMA 2.2. *Assume that a prime ideal P of a noetherian integral domain R is generated by s elements $a = a_1,\dots,a_s$ with s = height P. Set $T = R[a_2/a,\dots,a_s/a]$. Then (i) there is one and only one prime ideal P′ of height one in T so that $P' \cap R = P$, (ii) $V = T_{P'}$ is a discrete valuation ring and $P'V = aV$, (iii) V is uniquely determined, independently of the choice of the a_i and (iv) if W is a valuation ring of the field K of fractions of R such that the maximal ideal M of W lies over $P (R \subseteq W)$ and such that $a_2/a,\dots,a_s/a$ modulo M are algebraically independent over $R/(M \cap R)$, then W coincides with V.*

The valuation ring V given above is called the divisor of K obtained by the monoidal dilatation with center P.

PROOF. We may employ R_P instead of R hence we may assume that R is a regular local ring. Then the proof is easy and we omit the detail (cf. [1], §38). q.e.d.

From now on, we assume

HYPOTHESIS 2.3. R is a UFD, K is its field of fractions and $p_1,\dots,p_n$ are prime elements of R which generate mutually different ideals. Set $f = p_1,\dots,p_n$. $V_1,\dots,V_n$ are valuation rings of K satisfying the following four conditions.

(i) R is contained in every V_i, (ii) p_iV_i is the maximal ideal of V_i, (iii) $p_j \notin p_iV_i$ if $i \neq j$ and (iv) $p_iV_i \cap R \neq p_iR$.

For each $t = 1,\dots,n$, we set $R_t = R[f^{-1}] \cap V_1 \cap \dots \cap V_t$.

THEOREM 2.4. *Each R_t is a UFD and the set U_t of units in R_t is the set of elements of the form $up_{t+1}^{z_{t+1}}\dots p_n^{z_n}$ with a unit u of R and rational integers $z_{t+1},\dots,z_n$. In particular units in R_n are units in R.*

PROOF. $x \in p_iV_i \cap R_t$ implies that $x = p_iy$ with $y \in V_i$. Then $y \in V_j$ ($j = 1,\dots,t$) in view of the condition (iii) in the Hypothesis 2.3. $x \in R[f^{-1}]$ and therefore $fy \in R[f^{-1}]$, i.e., $y \in R[f^{-1}]$. Therefore we have $p_iR_t = p_iV_i \cap R_t$ for $i = 1,\dots,t$. Therefore each p_i ($i = 1,\dots,t$) is a prime element in R_t and R_t is a UFD by virtue of

Lemma 2.1. The remaining assertions are easy. q.e.d.

In order to observe a sufficient condition for R_t to be generated by a single element, we assume furthermore the following

HYPOTHESIS 2.5. R is noetherian and satisfies the condition: (*) If A is an intermediate ring between R and K and if Q is a prime ideal of A, then

$$\text{trans. deg}_{R/(Q \cap R)} A/Q = \text{height } (Q \cap R) - \text{height } Q.$$

Note that this condition (*) is satisfied by an affine ring over a field or a Dedekind domain (cf. [1], (35.5)).

THEOREM 2.6. *Under these hypotheses, we set* $P_i = p_i V_i \cap R$. *Then:*

(1) *There are* $g, h \in R$ $(h \neq 0)$ *such that* $R_t = R[g/h]$ *if and only if there are an element* g *of* R *and natural numbers* c_i $(i=1,\dots,t)$ *such that (i)* $P_i = p_i R + gR$ *and (ii)* $(g/p_i^{c_i}$ *modulo* $p_i V_i)$ *is transcendental over* R/P_i $(i=1,\dots,t)$.

(2) *If the condition in (1) is satisfied and if* R *and the* $R/P_i (i = 1,\dots,t)$ *are regular rings, then* R_t *is also a regular ring.*

PROOF. Considering $R[p_{t+1}^{-1},\dots,p_n^{-1}]$ instead of R, we may assume that $t = n$ and we set $F = R_n$. Assume that $F = R[g/h]$ and we may assume that h is of the form $p_1^{c_1}\dots p_n^{c_n}$ and that g is not divisible by any of the p_i. Since $p_iF = p_iV_i \cap F$ as we saw in the proof of Theorem 2.4, $b \in P_i$ implies $b/p_i \in F$. By (iv) in the Hypothesis 2.3, we see that h must be divisible by p_i. Hence $c_i \geqslant 1$ for all i. Since $b/p_i \in F$, we have $b/p_i = a_0 + a_1(g/h) +\dots+ a_s(g/h)^s$ $(a_i \in R)$. The right hand side is expressed in the form $a_0 + gg^*/(p_1^{e_1}\dots p_n^{e_n})$. Then we have $e_i = 1$, $e_j = 0$ $(j \neq i)$. Therefore $b \in p_iR + gR$. Thus we have $P_i = p_iR + gR$. Then height $P_i = 2$. By our Hypothesis 2.5, F/p_iF is transcendental over R/P_i. Since $F = R[g/h]$, $(g/h$ modulo $p_iV_i)$ hence $(g/p_i^{c_i}$ modulo $p_iV_i)$ too are transcendental over R/P_i. Thus we completed the proof of the only if part of (1). Conversely, assume the existence of g and c_i, and we want to show that $F = R[g/h]$ with $h = p_1^{c_1}\dots p_n^{c_n}$. Set $h' = p_1\dots p_n$ and $g' = g/h'$. By (ii), we have $gV_i = p_i^{c_i}V_i$ and we have $g' \in F$. Let W_i be the divisor obtained by the monoidal dilatation with center P_i. Then we have $R[g'] = R[h^{-1}] \cap W_1 \cap \dots \cap W_n$ (The inclusion $R[g'] \subseteq R[h^{-1}] \cap W_1 \cap \dots \cap W_n$ is obvious. The converse inclusion is proved as follows: We see first that $R_S[g']$ is a regular ring hence is normal, with $S =$ (the intersection of the complements of P_i). If $b \in R[h^{-1}] \cap W_1 \cap \dots \cap W_n$, then $b \in R_S[g']$ and $b = c_0 + c_1g' + \dots + c_mg'^m$ with $c_i \in R_S$. Then by that $b \in R[h^{-1}]$, we prove that $b \in R[g']$ (cf. [1], (11.13))). Then by Theorem 2.4, p_i are prime elements in the UFD $R[g']$. If $c_i = 1$ for some i, then (g/p_i modulo p_iV_i) is transcendental over R/P_i and $W_i \subseteq V_i$, hence $W_i = V_i$. Therefore we complete the proof by induction on Σc_i. (2) follows from (1) and some well known properties of monoidal dilatations (cf. [1], §38). q.e.d.

REFERENCES

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