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ON MOMENTS OF DISTRIBUTION FUNCTIONS ATTRACTED TO STABLE LAWS[1]

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§0. ABSTRACT. Let F be a distribution function in the domain of attraction of a stable law of characteristic exponent $\alpha \in (0,2)$. Then $\int |x|^\alpha dF(x) < \infty$ if and only if $\Sigma_{n=1}^\infty B_n^\alpha/n^2 < \infty$ for any sequence of normalizing coefficients $\{B_n\}$ for F. In addition, $\int |x|^{\alpha+\delta} dF(x) = \infty$ for all $\delta > 0$.

§1. **Introduction.** In three books ([1], [2] and [3]) the following theorem is stated: if a distribution function F is in the domain of attraction of a stable law of characteristic exponent $\alpha \in (0,2)$, then $\int |x|^\delta dF(x) < \infty$ for all $\delta < \alpha$. The question immediately comes to mind whether δ can ever equal α or exceed it. In this note necessary and sufficient conditions are given on F for $\delta = \alpha$, and examples are constructed to show there exist distribution functions in the domain of attraction of the stable distribution of characteristic exponent α for which $\int |x|^\alpha dF(x)$ is finite and is infinite.

Let $\underset{\sim}{D}(\alpha)$ denote the domain of attraction of the stable law of characteristic exponent α. By this we mean that $\underset{\sim}{D}(\alpha)$ is the set of distribution functions F such that if $\{X_n\}$ are independent identically distributed random variables with common distribution function F, then there exists a sequence of positive constants $\{B_n\}$, $B_n \to \infty$ as $n \to \infty$, called normalizing coefficients, and a sequence of numbers $\{A_n\}$ such that the limit distribution of $\{B_n^{-1} \Sigma_{j=1}^n X_j + A_n\}$ exists and has a characteristic function of the form

$$(1) \quad f(u) = \exp\left\{ \int_{-\infty}^{-0} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{c_1}{|x|^{1+\alpha}} dx + \int_{+0}^{\infty} \left(e^{iux} - 1 - \frac{iux}{1+x^2} \right) \frac{c_2}{|x|^{1+\alpha}} dx \right\},$$

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where $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_1 + c_2 > 0$. Such a limit distribution is called a stable distribution with characteristic exponent α. The result of this note is the following:

THEOREM: *Let* $F \in \underline{D}(\alpha)$, $0 < \alpha < 2$. *Then* $\int |x|^{\alpha} dF(x) < \infty$ *if and only if* $\Sigma_{n=1}^{\infty} B_n^{\alpha}/n^2 < \infty$ *for any sequence of normalizing coefficients* $\{B_n\}$ *for F. In addition,* $\int |x|^{\alpha+\delta} dF(x) = \infty$ *for all* $\delta > 0$.

PROOF. We first show that we need only prove the theorem in the case that F is continuous. Indeed, if we were to take a normal distribution function G, then by theorem 1 in [6], $F * G \in \underline{D}(\alpha)$ and $F * G$ has the same normalizing coefficients that F has. Also $\int |x|^{\alpha} dF * G(x) < \infty$ if and only if $\int |x|^{\alpha} dF(x) < \infty$; this follows by an easy application of the c_r-inequality found in Loève ([5], page 155). Thus we may assume F is continuous. Let X be a random variable whose distribution function is F. We select a particular sequence of normalizing coefficients for F, namely

(2) $B_n = \inf \{x : P[|X| > x] = (c_1 + c_2)/n\}$,

which are shown to be normalizing coefficients in [2] (on page 176), and where c_1 and c_2 are as in (1). Since F is continuous, $B_n < B_{n+1}$ for all $n > c_1 + c_2$. We define two discrete distribution functions, F_U and F_L with jumps only at $\{B_n\}$ by $1 - F_L(x) = (c_1 + c_2)/n$ if $B_n \leqslant x < B_{n+1}$, and $1 - F_U(x) = (c_1 + c_2)/(n+1)$ if $B_n \leqslant x < B_{n+1}$, for $n > c_1 + c_2$. Because of (2), $P[|X| \geqslant B_n] = (c_1 + c_2)/n$ for all $n > c_1 + c_2$, and if $B_n \leqslant x < B_{n+1}$, then $(c_1 + c_2)/n \geqslant P[|X| \geqslant x] > (c_1 + c_2)/(n+1)$. Hence we have

(3) $1 - F_U(x) < P[|X| \geqslant x] \leqslant 1 - F_L(x)$.

Since for any random variable $X \in L_1$ we have

(4) $E|X| = \int_0^{\infty} P[|X| \geqslant x]\, dx$,

we easily obtain for $Y \in L_{\alpha}$,

(5) $E|Y|^{\alpha} = \alpha \int_0^{\infty} y^{\alpha-1} P[|Y| \geqslant y]\, dy$.

Because of (3) and (4), we have

(6) $\int |x|^{\alpha} dF_U(x) \leqslant \int |x|^{\alpha} dF(x) \leqslant \int |x|^{\alpha} dF_L(x)$.

Now, by (5),

$$\int |x|^{\alpha} dF_L(x) = (c_1 + c_2)\, \alpha\, \Sigma\, n^{-1} \int_{B_n}^{B_{n+1}} y^{\alpha-1}\, dy$$

$$= (c_1 + c_2)\, \Sigma\, n^{-1} \left(B_{n+1}^{\alpha} - B_n^{\alpha} \right)$$

Now, recalling the formula for summation by parts, namely,

$$\Sigma_{n=1}^{N} a_n (b_{n+1} - b_n) = a_N b_{N+1} - \Sigma_{n=1}^{N} b_n (a_n - a_{n-1}),$$

we obtain

(7) $\int |x|^\alpha dF_L(x) = (c_1 + c_2) \Sigma B_n^\alpha / n(n-1)$.

A similar argument yields

(8) $\int |x|^\alpha dF_U(x) = (c_1 + c_2) \Sigma B_n^\alpha / n(n+1)$.

From (6), (7) and (8) we may conclude that $\int |x|^\alpha dF(x) < \infty$ if and only if $\Sigma B_n^\alpha/n^2 < \infty$ *for this particular sequence of normalizing coefficients.* However, if $\{B_n'\}$ is any other sequence of normalizing coefficients for F, then by a known result (see, e.g., [5], page 205), $B_n \sim B_n'$. Hence the last series converges if and only if $\Sigma (B_n')^\alpha/n^2 < \infty$, and thus the condition holds for every sequence of normalizing coefficients for F. Last we prove that $\int |x|^{\alpha+\delta} dF(x) = \infty$, for every $\delta > 0$. We need only prove $\int |x|^{\alpha+\delta} dF_U(x) = \infty$. Again by (5) and summation by parts we get

$$(9)\quad \int |x|^{\alpha+\delta} dF_U(x) = (c_1 + c_2) \lim_{N\to\infty} \Sigma_{n=1}^{N} (1/n)(B_{n+1}^{\alpha+\delta} - B_n^{\alpha+\delta})$$

$$= (c_1 + c_2) \lim_{N\to\infty} \left\{ B_{N+1}^{\alpha+\delta}/N + \Sigma_{n=1}^{N} B_n^{\alpha+\delta}/n(n+1) \right\} .$$

By lemma 5 in [6], $B_n = n^{1/\alpha} \varphi(n)$, where φ is a measurable slowly varying function. Now it is known that $x^{\delta/\alpha}(\varphi(x))^{\alpha+\delta} \to \infty$ as $x \to \infty$ (see [4], page 59), and applying this to (9) we obtain the desired conclusion, q.e.d.

This result does not hold for $\alpha = 2$. In case $\alpha = 2$, then by theorem 4 on page 181 of [2], $\int x^2 dF(x) < \infty$ if and only if F belongs to the domain of normal attraction of the normal distribution, i.e., $B_n = K n^{1/2}$ for some $K > 0$. In this case $\Sigma B_n^\alpha/n^2 = \infty$. Indeed this latter series diverges for all $F \in \underline{D}(2)$; this follows from the fact that if $\{B_n\}$ are normalizing coefficients for F, then $B_n \sim n^{1/2}\varphi(n)$, where φ is a non-decreasing slowly varying function (see lemma 5 in [6]).

It must be mentioned here that for every $\alpha \in (0,2)$ it is possible to find an $F_1 \in \underline{D}(\alpha)$ such that $\int |x|^\alpha dF_1(x) < \infty$ and an $F_2 \in \underline{D}(\alpha)$ such that $\int |x|^\alpha dF_2(x) = \infty$. Indeed, for $n = 2, 3, \cdots$, define $B_n^{(1)} = n^{1/\alpha}/(\log n)^{1/\alpha}$ and $B_n^{(2)} = n^{1/\alpha}/(\log n)^{2/\alpha}$. Clearly, $(\log x)^{-1/\alpha}$ and $(\log x)^{-2/\alpha}$ are slowly varying functions. Hence by the converse to lemma 5 in [6] there exist $F_1 \in \underline{D}(\alpha)$ and $F_2 \in \underline{D}(\alpha)$ such that $\{B_n^{(1)}\}$ and $\{B_n^{(2)}\}$ are normalizing coefficients for F_1 and F_2 respectively. However, it is easy to verify that $\Sigma (B_n^{(1)})^\alpha/n^2 = \infty$ and $\Sigma (B_n^{(2)})^\alpha/n^2 < \infty$, and thus we have shown that *some but not all* distribution functions in $\underline{D}(\alpha)$ have infinite absolute α^{th}

moments. In particular, it may be trivially observed that if F is in the domain of *normal* attraction of a stable distribution of characteristic exponent α, then $\int |x|^{\alpha} dF(x) = \infty$.

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