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AN L^p-INEQUALITY WITH APPLICATION TO ERGODIC THEORY[1]

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In the last few years a large number of papers have appeared in the literature dealing with the weak convergence of iterates of contractions and strong convergence of their averages. The trend started with the Blum-Hanson Theorem [3]. The latest result in this direction is due to Akcoglu and Sucheston [1], [2]. The purpose of this paper is to give a simple, straightforward proof of the Adcoglu-Sucheston theorem; the present proof avoids approximation by finite-dimensional operators for which the contraction case is reduced to the invertible isometry case, and thus avoids altogether the application of Akcoglu's "Dilation Theorem".

Let $(X,\underline{F},\mu)$ be a measure space and for $1 < p < \infty$, let $L^p = L^p(X,\underline{F},\mu)$ denote the usual Banach space. We write $L^p_+ = \{ f \in L^p | f \geqslant 0\}$. The following inequality was suggested to us by [1] (where a particular case of this inequality appears):

The L^p-Inequality. *Let* $1 < p < \infty$. *Let* $f \in L^p_+$, $g \in L^p_+$. *Then for any* $0 < \epsilon < 1$ *we have, with* $\alpha = (p - 1) + \frac{1}{p-1}$:

(1) $\int f^{p-1} g \, d\mu \leqslant \epsilon \|f\|_p^p + \epsilon \|g\|_p^p + \frac{1}{\epsilon^\alpha} \int f \cdot g^{p-1} \, d\mu.$

PROOF: We may assume without loss of generality that $g > 0$. Let $0 < \eta < K$ and define

$$A = \{ f < \eta g \}, \quad B = \{ f > Kg \}, \quad C = \{ \eta g \leqslant f \leqslant Kg \} .$$

Then

$$\int_A f^{p-1} g \, d\mu \leqslant \int_A \eta^{p-1} g^{p-1} g \, d\mu \leqslant \eta^{p-1} \|g\|_p^p$$

$$\int_B f^{p-1} g \, d\mu \leqslant \int_B f^{p-1} \frac{f}{K} \leqslant \frac{1}{K} \|f\|_p^p$$

$$\int_C f^{p-1} g \, d\mu \leqslant \int_C (Kg)^{p-1} g \, d\mu = K^{p-1} \int_C g^{p-1} g \, d\mu$$

$$\leqslant K^{p-1} \int_C g^{p-1} \frac{f}{\eta} d\mu \leqslant \frac{K^{p-1}}{\eta} \int g^{p-1} f \, d\mu$$

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whence $\int f^{p-1} g \, d\mu \leqslant \eta^{p-1} \|g\|_p^p + \frac{1}{K} \|f\|_p^p + \frac{K^{p-1}}{\eta} \int g^{p-1} f \, d\mu.$

Setting $\eta^{p-1} = \epsilon, \frac{1}{K} = \epsilon$ in the preceding inequality, we obtain inequality (1).

Application to Ergodic Theory. We first need some preliminary results. In Lemmas 1 and 2 below we assume that p *is fixed,* $1 < p < \infty$, and that $\Phi: L^p \to L^q$ $(\frac{1}{p} + \frac{1}{q} = 1)$ is the "canonical duality map" given by $\Phi(u) = \operatorname{sgn} u \, |u|^{p-1}$. We shall omit the subscripts p and q when writing the norm of an element in L^p or L^q. We recall that for every $u \in L^p$,

$$(u,\Phi(u)) = \|u\| \, \|\Phi(u)\| \text{ and } \|\Phi(u)\| = \|u\|^{p-1}.$$

When p = 2, Φ is simply the identity mapping, $\Phi(u) = u$ for all $u \in L^2$.

A linear mapping $T: L^p \to L^p$ is called *positive* if $T(L^p_+) \subset L^p_+$ and is called a *contraction* if $\|T\| \leqslant 1$.

LEMMA 1. *For each $\epsilon > 0$ there is $\delta = \delta(\epsilon,p) > 0$ (depending only on p and ϵ) such that:*

(1) *For any contraction $S: L^p \to L^p$, and*

(2) *For any $u \in L^p$ with $\|u\| = 1$ and $\|u\| - \|Su\| \leqslant \delta$ we have $\|S^*(\Phi(Su)) - \Phi(u)\| \leqslant \epsilon$.*

PROOF: Let $\phi(t) = t^{p-1}$, the "gauge function" corresponding to Φ.

By the uniform convexity of L^q (see for instance [5], p. 473), there is $\eta = \eta(\epsilon,q)$ (depending only on q and $\epsilon > 0$) such that

$$(2) \quad \left.\begin{array}{l} x \in L^q, \ y \in L^q \\ \|x\| \leqslant 1, \ \|y\| \leqslant 1 \text{ and} \\ \|\tfrac{1}{2}(x+y)\| \geqslant 1 - \eta \end{array}\right\} \Rightarrow \|x - y\| \leqslant \epsilon.$$

Since $t \to t\phi(t)$ is continuous, there is $\delta > 0$ such that

(3) $t \leqslant 1$ and $1 - t \leqslant \delta \Rightarrow t\phi(t) \geqslant \phi(1) - \eta = 1 - \eta$.

Let now $S: X \to X$ be a contraction and $u \in X$ with $\|u\| = 1$ and $\|u\| - \|Su\| = 1 - \|Su\| \leqslant \delta$. We have:

$$\|\Phi(u)\| = \phi(\|u\|) = \phi(1) = 1,$$

$$\|S^*(\Phi(Su))\| \leqslant \|\Phi(Su)\| = \phi(\|Su\|) \leqslant \phi(1) = 1.$$

On the other hand, by (3),

$$\begin{aligned} \|S^*(\Phi(Su)) + \Phi(u)\| &\geqslant (u, S^*(\Phi(Su)) + \Phi(u)) \\ &= (u, S^*(\Phi(Su))) + (u,\Phi(u)) \\ &= (Su,\Phi(Su)) + (u,\Phi(u)) \end{aligned}$$

$$= \|Su\| \ \|\Phi(Su\| + \|u\| \ \|\Phi(u)\|$$
$$= \|Su\| \phi(\|Su\|) + 1\cdot\phi(1)$$
$$\geqslant \phi(1) - \eta + \phi(1) > 2(\phi(1) - \eta) = 2(1 - \eta)$$

and hence by (2)

$$\|S^*(\Phi(Su)) - \Phi(u)\| \leqslant \epsilon.$$

This completes the proof of Lemma 1.

REMARK. Let X be a Banch space, X′ its dual and let ψ: $R_+ \to R_+$ be a continuous strictly increasing mapping with $\psi(0) = 0$. We recall that Ψ: $X \to X'$ is called a "duality map with gauge function ψ" if for each $u \in X$ the following conditions hold (see for instance [4], p. 370):

$$(u,\Psi(u)) = \|u\| \ \|\Psi(u)\| \text{ and } \|\Psi(u)\| = \psi(\|u\|).$$

It is clear that Lemma 1 remains valid if we replace L^p by X and Φ by Ψ; the only property needed in the proof is the uniform convexity of X′.

We now return to the L^p-space; with the notation of Lemma 1 we have:

COROLLARY 1. *For each $\epsilon > 0$ there is $\delta = \delta(\epsilon,p) > 0$ such that: For any contraction S: $L^p \to L^p$, any $g \in L^p$ with $\|g\| - \|Sg\| \leqslant \delta \|g\|$, and any $h \in L^p$ we have:*

$$|(Sh,\Phi(Sg)) - (h,\Phi(g))| \leqslant \epsilon \|\Phi(g)\| \ \|h\|.$$

PROOF: Straightforward consequence of Lemma 1.

LEMMA 2. *Let T: $L^p \to L^p$ be a positive contraction (in the case $p = 2$, T: $L^2 \to L^2$ an arbitrary, not necessarily positive, contraction). Suppose that for some $f \in L^p_+$ (in the case $p = 2$, $f \in L^2$) the sequence $(T^n f)_{n \geqslant 1}$ converges to 0 weakly in L^p. Then*

$$\lim_{|i-j|\to\infty} (T^i f,\Phi(T^j f)) = 0$$

PROOF: We may assume without loss of generality that $\|f\| \leqslant 1$. Let now

$$C = \{ g \in L^p_+ | \ \|g\| \leqslant 1 \} \text{ if } p \neq 2,$$

respectively

$$C = \{ g \in L^2 | \ \|g\| \leqslant 1 \} \text{ when } p = 2.$$

Then it is clear that T: $C \to C$. By the L^p-inequality it is also clear that for each $\epsilon > 0$ we may find a constant $A(\epsilon) > 0$ such that:

(*) $u \in C, v \in C \Rightarrow |(u,\Phi(v))| \leqslant \epsilon + A(\epsilon)|(v,\Phi(u))|$.

Since T is a contraction,

$$\|f\| \geqslant \|Tf\| \geqslant \cdots \geqslant \|T^n f\| \geqslant \|T^{n+1} f\|.$$

If $\lim_n \|T^n f\| = 0$, the conclusion of the Lemma is trivial. Hence assume that

$$\lim_n \|T^n f\| = a > 0.$$

By assumption,

$$\lim_n (T^n f, \Phi(f)) = 0.$$

By (*), since $f \in C$, $T^n f \in C$, we also have

$$\lim_n (f, \Phi(T^n f)) = 0.$$

Hence given $\epsilon > 0$, there is $N' = N'(\epsilon)$ such that

(4) $n \geqslant N' \Rightarrow |(f, \Phi(T^n f))| \leqslant \frac{\epsilon}{2}.$

On account of (*), it is enough to show that

$$\lim_{\substack{|i-j| \to \infty \\ j > i}} (T^i f, \Phi(T^j f)) = 0$$

Hence we consider the case $j > i$, $j = i+n$; we must evaluate

$$(T^i f, \Phi(T^j f)) = (T^i f, \Phi(T^{i+n} f)).$$

We now apply Corollary 1. Note first that there is $N'' = N''(\epsilon)$ such that

$$(5) \qquad \left.\begin{matrix} n \geqslant N'' \\ i \geqslant 1 \end{matrix}\right\} \Rightarrow \|T^n f\| - \|T^{n+i} f\| \leqslant \delta\left(\frac{\epsilon}{2}\right) a \leqslant \delta\left(\frac{\epsilon}{2}\right) \|T^n f\|.$$

For any $n \geqslant N''$ and any $i \geqslant 1$, apply Corollary 1 with the following identifications:

$$S = T^i,\ g = T^n f,\ h = f.$$

We obtain

(6) $|(T^i f, \Phi(T^{n+i} f)) - (f, \Phi(T^n f))| \leqslant \frac{\epsilon}{2} \|\Phi(T^n f)\| \, \|f\| \leqslant \frac{\epsilon}{2}.$

Combining (4) and (6) and letting $N_0 = \max(N', N'')$ we obtain

$$\left.\begin{matrix} n \geqslant N_0 \\ i \geqslant 1 \end{matrix}\right\} \Rightarrow |(T^i f, \Phi(T^{n+i} f))| \leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof of Lemma 2.

REMARK. Under the assumptions of Lemma 2, the weak convergence to 0 (in L^p) of the sequence $(T^n f)_{n \geqslant 1}$ implies the weak convergence to 0 (in L^q) of $(\Phi(T^n f))_{n \geqslant 1}$. If we remove the *positivity* assumption on f, *in the case* $p \neq 2$, this is no longer true, as the following simple example illustrates: Assume that $(X, \underline{E}, \mu)$ is a probability space and that τ: $X \to X$ is a measure-preserving transformation which is strongly mixing (i.e. for every $A \in \underline{F}$, $B \in \underline{F}$, $\mu(\tau^{-n}(A) \cap B) \to \mu(A)\mu(B)$).

Let T: $L^p \to L^p$ be the operator induced by τ: $Tf = f \circ \tau$, for $f \in L^p$. Then it is well known that the sequence $(T^n)_{n \geq 1}$ converges in the weak operator topology to the projection operator P, where $Pf = \int f\, d\mu$, for $f \in L^p$. Consider now $g \in L^p$ such that $\int g\, d\mu = 0$ but $\int \Phi(g) d\mu = c \neq 0$. Then it is obvious that

$$T^n g \to 0 \text{ weakly in } L^p$$

but that

$$\Phi(T^n g) \to c \neq 0 \text{ weakly in } L^q.$$

Let now (a_{ni}) be a *matrix of real numbers satisfying the following two conditions*:

(a) $m_n = \sup_i |a_{ni}| \to 0$ as $n \to \infty$

(b) $M_n = \sum_i |a_{ni}| \leq M < \infty$ for all $n \geq 1$.

It is clear that any such matrix (a_{ni}) may be written in the form

$$a_{ni} = a'_{ni} - a''_{ni},$$

where $a'_{ni} \geq 0$, $a''_{ni} \geq 0$ for all (n,i) and where both (a'_{ni}) and (a''_{ni}) satisfy conditions (a) and (b) above.

We may now state the following:

THEOREM 1. *Let T: $L^p \to L^p$ be a positive contraction (in the case $p = 2$, T: $L^2 \to L^2$ an arbitrary contraction). Then for an element $f \in L^p_+$ (in the case $p = 2$ for an element $f \in L^2$) the following assertions are equivalent:*

(i) *The sequence $(T^n f)_{n \geq 1}$ converges to 0 weakly in L^p;*

(ii) $\lim_{|i-j| \to \infty} (T^i f, \Phi(T^j f)) = 0$;

(iii) *For any matrix (a_{ni}) satisfying (a) and (b), the sequence $\sum_i a_{ni} T^i f$ converges to 0 strongly in L^p.*

PROOF: As (i) $\Rightarrow$ (ii) follows from Lemma 2 and (iii) $\Rightarrow$ (i) is well known (and in any case easy to prove directly) it remains to prove (ii) $\Rightarrow$ (iii).

(ii) $\Rightarrow$ (iii). In proving (iii) we may assume without loss of generality that $a_{ni} \geq 0$ for all (n,i) and that the constant M in condition (b) above is ≤ 1. We may also assume that $\|f\| \leq 1$.

Let now C be defined as in the beginning of the proof of Lemma 2. Then the set C satisfies condition (*) and it is obvious that T^i maps C into C for all $i \geq 1$ and that

$A_n(T) = \sum_j a_{nj}T^j$ maps C into C for all $n \geq 1$.

Let now $\epsilon > 0$. We have to evaluate

$$\|A_n(T)f\| = \|\sum_j a_{nj}T^jf\|,$$

or equivalently

$$|(A_n(T)f, \Phi(A_n(T)f))|.$$

Using condition (*) we may write

$$\begin{aligned}
|(\sum_i a_{ni}T^if, \Phi(\sum_j a_{nj}T^jf))| &\leq \sum_i a_{ni}|(T^if, \Phi(\sum_j a_{nj}T^jf))| \\
&\leq \sum_i a_{ni} \{\epsilon + A(\epsilon)|(\sum_j a_{nj}T^jf, \Phi(T^if))|\} \\
&\leq \sum_i a_{ni} \{\epsilon + [\sum_j a_{nj}|(T^jf, \Phi(T^if))|] A(\epsilon)\} \\
&\leq \epsilon + A(\epsilon)[\sum_{(i,j)} a_{ni}a_{nj}[(T^jf, \Phi(T^if))|].
\end{aligned}$$

It remains to evaluate the sum

$$I(n) = \sum_{(i,j)} a_{ni}a_{nj}|(T^jf, \Phi(T^if))|$$

and to show that $\lim_n I(n) = 0$. Let $\epsilon^* > 0$; by Lemma 2, there is $N_0 = N_0(\epsilon^*)$ such that

$$i,j \geq 1, |i - j| \geq N_0 \Rightarrow |(T^jf, \Phi(T^if))| \leq \epsilon^*.$$

We deduce

$$\begin{aligned}
I(n) &= \sum_{|i-j|<N_0} + \sum_{|i-j|\geq N_0} \\
&\leq (\sum_i a_{ni})2N_0m_n + (\sum_{|i-j|\geq N_0} a_{ni}a_{nj})\epsilon^* \\
&\leq 2N_0m_n + \epsilon^*.
\end{aligned}$$

Since $m_n \to 0$ as $n \to \infty$, the assertion about I(n) is proved and thus the proof of the Theorem is concluded.

We recall that a matrix (a_{ni}) of real numbers is called "*uniformly regular*" if it satisfies conditions (a) and (b) (preceding Theorem 1) and in addition condition (c) below:

(c) $\lim_n \sum_i a_{ni} = 1$.

From Theorem 1 one easily obtains the following result recently proved by Akcoglu and Sucheston ([1], [2]):

THEOREM 2. *(Akcoglu and Sucheston). Let $T: L^p \to L^p$ be a positive contraction (in the case $p = 2$, $T: L^2 \to L^2$ an arbitrary contraction). Then the following assertions are equivalent:*

(1) *$\lim_n T^n$ exists in the weak operator topology.*

(2) *If* (a_{ni}) *is a uniformly regular matrix, then* $\lim_n \sum_i a_{ni}T^i$ *exists in the strong operator topology.*

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