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DOMINGO HERRERO: HIS THEOREMS AND PROBLEMS

KENNETH R. DAVIDSON*

Dedicated to Domingo's wife Marta and his children Pablo Julián and Laura Inés

Domingo Herrero died April 13, 1991 after losing a battle against cancer. Domingo was a passionate man who was very close to his family and friends. I am honoured to have been one of those friends. He was also one of the best operator theorists of our time. He had an intense love for mathematics, and was relentless in his pursuit of new results. This paper is intended to be a modest survey of some of his work, and of some of the problems he posed. Domingo wrote well over one hundred papers, averaging seven papers per year for 20 years. So, I will not attempt to be anywhere near to comprehensive. Domingo had three dozen co-authors, so please forgive me if I don't mention the work he did with all of you. It was an ambition of Domingo's to have a co-author for each letter of the alphabet – he covered 17 at the last count. The bibliography contains the most up-to-date list I could obtain, but he is still writing papers with some of his co-authors even now.

Herrero was a native of Argentina. He and his wife Marta studied mathematics at the University of Buenos Aires, where they both received master's degrees and started working on their Ph.D.'s. However, the military fired many university professors in 1966, effectively shutting down most doctoral programs. Two years later, they came to the University of Chicago to complete their studies. Herrero received his Ph.D. in 1970 under the supervision of Richard Beals. After spending two years in Albany, the Herreros returned to South America. They taught in Brazil, Argentina, and finally Venezuela. But the unfavourable environment kept them moving on.

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It is incredible that in spite of this upheaval and the relative mathematical isolation, Herrero developed into a prolific mathematician. In 1980, Herrero returned to the United States. After a year in Georgia, he moved to Arizona State University. In this more stable situation, Herrero's work flourished and here he did his deepest work.

Herrero was profoundly influenced by Halmos's famous paper *Ten Problems in Hilbert Space* [**#6**]. Many of these problems raised serious questions about the structure and approximation properties of various classes of operators on Hilbert space. Herrero spent his career on such problems during a time of great progress in this direction. The deepest work in the seventies in this area was done by the Romanian school, notably Apostol, Foiaş, and Voiculescu. In the eighties, it was Herrero who set the standard. He also wrote the book – indeed two volumes he typed himself – systematizing these developments. In this short paper, I will try to give some idea of the work Herrero did, and the problems he left behind.

1. Notation. In this paper, $\mathcal{H}$ will always denote a separable Hilbert space. The space of all bounded operators is denoted $\mathcal{B}(\mathcal{H})$, and the ideal of compact operators is $\mathcal{K}(\mathcal{H})$. The quotient map onto the Calkin algebra will be denoted by π. We will need various parts of the spectrum $\sigma(T)$ such as the essential spectrum $\sigma_e(T)$, the point spectrum (eigenvalues) $\sigma_p(T)$, the 'normal' spectrum $\sigma_0(T)$ (isolated eigenvalues of finite multiplicity), and

$$\begin{aligned}\sigma_{\ell re}(T) &= \{\lambda | \pi(T-\lambda I) \text{ is neither left nor right invertible}\}\\ &= \{\lambda | T-\lambda I \text{ is not semi-Fredholm }\}\end{aligned}$$

An operator T is *semi-Fredholm* if it has closed range, and at least one of nul (T) = dim rank (T) and nul (T^*) = dim rank (T^*) is finite. The semi-Fredholm *index* is ind (T) = nul (T) − nul (T^*). An operator is Fredholm (semi-Fredholm of finite index) if and only if $\pi(T)$ is invertible, and semi-Fredholm if $\pi(T)$ is either left or right invertible. The domain of semi-Fredholmness is denoted $\rho_{s-F}(T) = \mathbf{C}\backslash\sigma_{\ell re}(T)$, consisting of those scalars λ such that $T-\lambda I$ is semi-Fredholm.

The set $\rho_{s-F}(T)$ is open, and index is constant on its components. Define min.ind $(T) = \min\{\text{nul }(T),\ \text{nul }(T^*)\}$. The function min.ind $(T-\lambda I)$ is lower semi-continuous on $\rho_{s-F}(T)$, and is constant on each component except for possibly a countable set accumulating at the boundary. This countable set is denoted $\rho^s_{s-F}(T)$, and its members are called *singular* points. The remaining part of $\rho_{s-F}(T)$ is denoted $\rho^r_{s-F}(T)$, and such points are called *regular.*

We assume familarity with the Riesz functional calculus and its basic properties. The Riesz spectral projection corresponding to a clopen set Σ of $\sigma(T)$ is denoted by $E_T(\Sigma)$.

2. Similarity. One of Halmos's 'ten problems' was: *Is every quasinilpotent operator the norm limit of nilpotent operators?*

Halmos pointed out that an example of Kakutani showed that the limit of nilpotents need not be quasinilpotent, so the 'real' question is the much less focussed: *Describe the closure of the set of nilpotent operators.* The set of nilpotent operators is invariant under similarity, and hence so is its closure. The solution of this and related problems lead to the conclusion that closed, similarity invariant classes should be determined by natural conditions on the spectrum, index and rank of the operator.

The spectrum of an operator is not continuous, but it is upper semi-continuous; meaning that if Ω is an open neighbourhood of $\sigma(T)$, then there is a neighbourhood $\mathcal{O}$ of T so that every operator $S \in \mathcal{O}$ also has $\sigma(S) \subset \Omega$, and each component of Ω which intersects $\sigma(T)$ also intersects $\sigma(S)$. In particular, if T is the limit of nilpotents, $\sigma(T)$ must be connected and contain 0. The same argument applies to $\sigma_e(T)$ by working in the Calkin algebra. The set of semi-Fredholm operators is open and index is locally constant. Thus, if $T - \lambda I$ is semi-Fredholm of index n, then $\operatorname{ind}(S - \lambda I) = n$ for all S sufficiently close to T. If S is nilpotent, it is clear that $\operatorname{ind}(S - \lambda I) = 0$ for all $\lambda \neq 0$. Thus if T is the limit of nilpotents, one must have $\operatorname{ind}(T - \lambda I) = 0$ for all $\lambda \in \rho_{s-F}(T)$.

Herrero **[22]** proved an important special case:

Theorem. *If N is a normal operator with $0 \in \sigma_e(N) = \sigma(N)$, and $\sigma(N)$ is connected, then N is the limit of nilpotents.*

This now has very simple proofs based on 'Berg's technique'. The basic idea is that a finite rank weighted shift with weights slowly growing from 0 up to 1 and down to 0 again is close to a normal matrix with eigenvalues 'thick' in the unit disc. Together with elementary conformal mapping techniques, one can build nilpotents close to any normal satisfying the spectral conditions. See Herrero's book **[57]** for a nice treatment.

Shortly thereafter, Apostol, Foiaş, and Voiculescu **[#3]** give the complete answer. This was based on Herrero's special case, and some other approximation arguments.

Theorem [AFV]. *An operator $T \in \mathcal{L}(\mathcal{H})$ is the norm limit of nilpotent operators if and only if both $\sigma(T)$ and $\sigma_e(T)$ are connected and contain 0, and ind $(T - \lambda I) = 0$ for all $\lambda \in \rho_{s-F}(T)$.*

A natural outgrowth of this work, and the approximation methods used to solve it, is the description of the closure of the similarity orbit $\overline{\mathcal{S}(T)}$ of a single operator T. For example, it is not too hard to show from the proof of the theorem above that if Q is quasinilpotent and Q^n is not compact for any $n > 0$, then $\overline{\mathcal{S}(Q)}$ equals the closure of the nilpotents. Herrero [**28**] and Apostol [#**1**] proved this independently.

In a similar way, Herrero [**30**] described the closure of the similarity orbit of a normal operator with perfect spectrum. And in [**46**], Barría and Herrero describe the closure of the similarity orbit in finite dimensions. These two results exhibit different phenomena. The normal case is much like the nilpotent case we just examined. If A is similar to a normal operator N, it has index 0 whenever it is semi-Fredholm. Thus the same holds for any limit T. The spectrum and essential spectrum of T must contain $\sigma(N) = \sigma_e(N)$, and each component must intersect $\sigma(N)$. These are sufficient conditions.

In the finite dimensional case, the notion of rank becomes important. This same condition applies to normal eigenvalues (isolated eigenvalues of finite multiplicity) in the Hilbert space case. The point is that the set of operators of rank at most k is closed. So if p is a polynomial and if $T = \lim A_n$ where A_n are similar to A, then

$$\text{rank } (p(T)) \leq \lim \text{rank } (p(A_n)) = \text{rank } p(A).$$

This turns out to be sufficient as well. There is a similar condition based on the fact that the set of operators with nullity at least k is closed. This yields the condition:

$$\text{nul } (p(T)) \geq \lim \text{nul } (p(A_n)) = \text{nul } (p(A)).$$

In finite dimensions, this condition is equivalent to the rank condition because rank $(T) +$ nul $(T) =$ dim $(\mathcal{H})$. However, in infinite dimensions they are different. Consider T to be a rank 1 projection and A to be a rank 2 projection. It is easy to compute rank $p(T)$ to be 0, 1 or ∞ while the corresponding ranks for A are 0, 2 and ∞. However, nul $(T - I) = 1 < 2 =$ nul $(A - I)$ and thus T cannot be in $\overline{\mathcal{S}(A)}$. Likewise, nul $(q(A))$ is always 0, 2 or ∞ which dominates the corresponding nullities 0, 1 and ∞ for T, but A is not in $\overline{\mathcal{S}(T)}$ either. Together, these two conditions characterize $\overline{\mathcal{S}(A)}$ for a finite rank operator A.

The philosophy which emerges is that closed, similarity invariant sets are determined by three types of conditions: (S) spectral conditions (such as 'connected containing 0'), (F) Fredholm index conditions, and (A) algebraic conditions (such as rank $(p(T)) \leq$ rank $p(A))$. This proves to be the case for 'nice' sets.

Probably the deepest result in approximation of operators in the eighties is the theorem of Apostol, Herrero and Voiculescu [**58**, **73**] describing $\overline{\mathcal{S}(A)}$ for an arbitrary operator A. Actually, they don't quite get the whole picture when there are isolated points in $\sigma_e(A)$ of essentially nilpotent type. When these bad points are missing, the closure $\overline{\mathcal{S}(A)}$ is described solely in terms of the spectral picture of A. In other words, it can be described by knowing the different parts of the spectrum, the Fredholm index on the components of $\rho_{s-F}(A)$, and rank conditions arising from normal eigenvalues and other singular points in $\rho_{s-F}(A)$. I will state a special case. When T is *smooth*, meaning that min.ind $(T - \lambda I) = 0$ for all $\lambda \in \rho_{s-F}(T)$ (so in particular $\sigma_0(T)$ is empty), then the conditions (FA) and $(A1)$ can also be dropped.

Similarity Orbit Theorem. *Suppose that T has no isolated points in $\sigma_e(T)$ of finite order. Then $A \in \overline{\mathcal{S}(T)}$ if and only if*

- (S1) *$\sigma_0(A) \subset \sigma_0(T)$; and $\sigma_e(A) \supset \sigma_e(T)$.*
- (S2) *Each component of $\sigma_{\ell re}(A)$ meets $\sigma_e(T)$.*
- (F1) *ind $(A - \lambda I) =$ ind $(T - \lambda I)$ for all $\lambda \in \rho_{s-F}(A)$.*
- (FA) *min.ind $(A-\lambda I)^k \geq$ min.ind $(T-\lambda I)^k$ for all $\lambda \in \rho_{s-F}(A)$.*
- (A1) *rank $(E_A(\lambda)) =$ rank$(E_T(\lambda))$ for all $\lambda \in \sigma_0(A)$.*

This theorem is a real tour de force. The main ideas extend the notions introduced by the Romanian school in the solution of the nilpotent problem, and the related work on quasitriangularity. Although I can't go into any details, let me mention two main tools. One is the use of canonical models based on the direct sum of normal operators, finite rank operators and Bergman operators associated with nice domains. The other is the use of one-sided resolvents – i.e. an analytic function satisfying the resolvent equation that provides a left inverse over some domain. This latter notion is developed into a powerful tool. It's usefulness will pop up again in the next section.

Herrero [**117**] proved what he calls a '*metatheorem*' about closed, similarity invariant sets. A similarity invariant class is said to have *sufficient structure* provided that it contains a dense set of operators with no isolated points of finite order in the essential spectrum.

Metatheorem. *Suppose that $\mathcal{S}$ is a closed, similarity invariant subset of $\mathcal{B}(\mathcal{H})$ with sufficient structure. Then $\mathcal{S}$ is determined just by the set of possible fine spectral pictures. Furthermore, if $\mathcal{S}$ is invariant under compact perturbations, $\mathcal{S}$ is determined just by the set of possible spectral pictures.* (Basically, the fine detail about normal eigenvalues and singular points vanishes.)

Moreover, there is a distance formula to these sets which is of practical use. This is a very powerful theorem. Indeed, the characterizations of the closures of the nilpotents, triangular operators, n-multicyclic operators, operators with spectra contained in prescribed sets, etc., etc. all follow as corollaries from this theorem. Of course that is circular reasoning, but it indicates the scope of the result.

3. Multiplicity. An operator T is *n-multicyclic* if n is the least integer for which there are vectors $x_1, \dots, x_n$ so that

$$\text{span } \{T^k x_i | k \geq 0, 1 \leq i \leq n\} = \mathcal{H}.$$

We say that T has *multiplicity* $\overline{\mu}(T) = n$. The first observation to make is that if ind $(T - \lambda I) = -n < 0$, then T has multiplicity at least n. To see this, let $\mathcal{M} = \ker(T - \lambda I)^*$, and decompose $\mathcal{H} = \mathcal{M}^\perp \oplus \mathcal{M}$. Since $\mathcal{M}^\perp$ is T-invariant, we obtain the matrix form

$$T = \begin{bmatrix} T_{11} & T_{12} \\ 0 & \lambda I_{\mathcal{M}} \end{bmatrix}.$$

For any vectors $x_1, \dots, x_n$,

$$P_{\mathcal{M}} \text{ span}\{T^k x_i | k \geq 0, 1 \leq i \leq n\} = \text{ span } \{P_{\mathcal{M}} x_i | 1 \leq i \leq n\}.$$

Thus,

$$\overline{\mu}(T) \geq \dim(\mathcal{M}) \geq -\mathrm{ind}(T - \lambda I).$$

It follows that if $\overline{\mu}(T) = n$, then ind $(T - \lambda I) \geq -n$ for all $\lambda \in \rho_{s-F}(T)$. Moreover, if ind $(T - \lambda I) = n$, then nul $(T - \lambda I)^* = n$ and nul $(T - \lambda I) = 0$. Herrero [**40**] proves:

Theorem. *If T is n-multicyclic, then*

(i) *ind $(T - \lambda I) \geq -n$ for all $\lambda \in \rho_{s-F}(T)$.*
(ii) *Every component of $\rho_{s-F}(T)$ of index $-n$ is simply connected.*

The second condition is neither obvious nor easy to prove. This is a deep and useful result. Curiously, Herrero provided a second *easier* proof in [**86**] in his (successful) quest for a general Banach space proof. Let Γ be a smooth curve in a component Ω of index $-n$, and let $\Psi = \hat{\Gamma} \cap \Omega$. The key idea is the construction of a left resolvent function $R(\psi)$ defined on Ψ such that

$$R(\psi)(T - \psi I) = I \qquad \text{for all } \psi \in \Psi$$

and

$$R(\psi) - R(\phi) = (\phi - \psi)R(\psi)R(\phi) \qquad \text{for all } \psi, \phi \in \Psi.$$

In order to do this, fix a point $\gamma \in \Gamma$. Since $x_1, \dots, x_n$ is a cyclic set for T (and also $T - \gamma I$), they must span a complement to Ran $(T - \gamma I)$. Hence there is a unique left inverse R_γ of $T - \gamma I$ satisfying $R_\gamma x_i = 0$ for $1 \leq i \leq n$. It can be shown that $I + (\psi - \gamma)R_\gamma = R_\gamma(\psi - T)$ is invertible on Ψ. (This operator clearly has index 0, so it suffices to show that the adjoint is injective.) Then a routine computation shows that

$$R(\psi) = (I + (\psi - \gamma)R_\gamma)^{-1}R_\gamma$$

is the desired left resolvent.

It is clear that this is an analytic function. Now the Cauchy integral formula allows the construction of a left resolvent on the whole (simply connected) region bounded by Γ. From this, we can conclude that $\hat{\Gamma}$ belongs to $\rho_{s-F}(T)$. Hence Ω contains $\hat{\Gamma}$ for all curves Γ, and thus is simply connected.

Now consider the closure of the set of n-multicyclic operators. We have obtained two important conditions, (i) is of Fredholm type, and (ii) is a spectral condition. Fine structure conditions such as nul $(T-\lambda I) \geq -n$ are not necessary because any operator has a small compact perturbation such that min.ind $(T+K-\lambda I)=0$ for all $\lambda \in \rho_{s-F}(T)\backslash\sigma_0(T)$. Isolated eigenvalues of finite multiplicity cannot be removed, but they can be modified by another small compact so that all the normal eigenvalues are of multiplicity one. Herrero [**40**,72] proves:

Theorem. *The closure of the n-multicyclic operators is precisely the set of operators satisfying (i) and (ii). Furthermore, if T satisfies (i) and (ii) and $\epsilon > 0$, then there is a compact operator K with $\|K\| < \epsilon$ such that $T-K$ is n-multicyclic.*

The fact that the two conditions (i) and (ii) are sufficient is another example of the similarity paradigm at work. This theorem illustrates another familiar theme in this subject, the fact that small *compact* perturbations often suffice. This is frequently the case when there is no obstruction (of type (S), (F) or (A)) to preclude it.

There is an obvious analogue of n-multicyclicity called *rational n-multicyclicity* which uses rational functions of T rather than polynomials. The theory runs in parallel, except that the condition of simple connectedness for holes of index $-n$ is no longer present. The closure conditions are again as expected. However, this time compact perturbations do not suffice! Herrero goes on to find a third notion of n-multicyclicity with closure agreeing with the rational case, but for which compact perturbations do suffice [**86**].

The nature of the multiplicity function is rather mysterious. Herrero and Wogen [**110**] studied the sequence $\overline{\mu}(T^{(n)})$ where $T^{(n)}$ denotes the direct sum of n copies of T. They construct examples such that $\overline{\mu}(T^{(n)}) = \overline{\mu}(T^{*(n)}) = 1$ for all $n \geq 1$, answering a question of Apostol. Likewise they obtain the constant sequence 2. Taking T to be a power of a shift, a linear sequence $\{nk|n \geq 1\}$ can be achieved. The complete list known includes only a few more: $\{nk+1|n \geq 1\}$, $\{nk+2|n \geq 1\}$ and $\{k+1, 2k, 3k, \dots\}$. Here are a few test cases.

Problem. *Is every constant sequence the multiplicity sequence of $T^{(n)}$ for some T? If $\overline{\mu}(T) = \overline{\mu}(T^{(2)}) = \overline{\mu}(T^{(3)})$, is the sequence constant? If this sequence is not constant, must it satisfy $\overline{\mu}(T^{(n)}) \geq n$?*

4. Quasisimilarity.

4.1. Spectrum. Two operators A and B are quasisimilar if there exist one-to-one dense range operators X and Y so that $AX = XB$ and $YA = BY$. The range of X must be a dense operator range which is invariant for A. Furthermore, the restriction of A to this linear manifold is algebraically equivalent to B. Similarly, A is algebraically equivalent to the restriction of B to a dense invariant operator range. This notion was introduced by Sz. Nagy and Foiaş in their study of contractions. Quasisimilarity preserves the *existence* of non-trivial hyperinvariant subspaces. But it is a weak equivalence relation that doesn't preserve much that we operator theorists hold dear, such as the spectrum. On the other hand, the equivalence classes are large enough that it becomes possible to classify large classes of operators with a tractible set of invariants.

Sz. Nagy and Foiaş [#8] gave the first examples of quasisimilar operators with different spectra. A very easy example is given by using the $n \times n$ Jordan nilpotent blocks J_n. If S_n is the invertible $n \times n$ diagonal matrix with diagonal entries n^{1-i}, it is easy to see that $S_n^{-1} J_n S_n = n^{-1} J_n$. Let

$$A = \sum_{n \geq 1} \oplus J_n \qquad \text{and} \qquad B = \sum_{n \geq 1} \oplus n^{-1} J_n,$$

and let

$$X = \sum_{n \geq 1} \oplus S_n \qquad \text{and} \qquad Y = \sum_{n \geq 1} \oplus S_n^{-1} / \| S_n^{-1} \|.$$

Then it is an easy calculation to verify that $AX = XB$ and $YA = BY$. Clearly B is quasinilpotent, but $\sigma(A)$ is the whole unit disc.

Hoover [#7] showed that their spectra must intersect. To see this, consider the Rosenblum operator on $\mathcal{B}(\mathcal{H})$ given by $\tau(X) = AX - XB$. The Rosenblum lemma states that

$$\sigma(\tau) \subset \sigma(A) - \sigma(B).$$

In particular, if $\sigma(A)$ and $\sigma(B)$ are disjoint, then τ is invertible. But if $A \sim_{qs} B$ so that $AX = XB$, then there is a non-zero X in $\ker \tau$, and thus the spectra overlap. Moreover, as Herrero pointed out, each component of $\sigma(B)$ meets $\sigma(A)$. For otherwise, there is an open set Ω intersecting $\sigma(B)$ in a non-empty clopen set Σ, but is disjoint from $\sigma(A)$. Let $E = E_B(\Sigma)$ be the

Riesz spectral projection onto the spectral subspace for B corresponding to Σ. We obtain $A(XE) = (XE)(B|_{E\mathcal{H}})$. The Rosenblum lemma again implies that $XE = 0$, which is absurd since X is injective and E is non-zero.

One cannot do the same trick in the Calkin algebra because the intertwining operator X may be compact. Nevertheless, Herrero [99] showed recently that:

Theorem. *If $A \sim_{qs} B$, then every component of $\sigma_e(A)$ meets $\sigma_e(B)$ and vice versa.*

The key to this theorem is the result on n-multicyclic operators. In order to indicate the connection, let's review a few simple properties of quasisimilarity.

If $AX = XB$, then it follows easily that $r(A)X = Xr(B)$ for every rational function r with poles off $\sigma(A) \cup \sigma(B)$. In particular, X maps ker $(B - \lambda I)$ into ker $(A - \lambda I)$, whence nul $(B - \lambda I) \leq$ nul $(A - \lambda I)$. Reversing the role of A and B yields equality. By taking adjoints, we see that $A^* \sim_{qs} B^*$, so that nul $(B - \lambda I)^* =$ nul $(A - \lambda I)^*$. Thus, if both $A - \lambda I$ and $B - \lambda I$ are semi-Fredholm, then they have the same index.

Various other properties are preserved by quasisimilarity, such as being triangularizable or being n-multicyclic. To see the latter, note that if

$$\mathcal{H} = \text{ span } \{B^k x_i | 1 \leq i \leq n,\ k \geq 0\},$$

then $\{y_i = Xx_i | 1 \leq i \leq n\}$ is an n-cyclic set for A; and vice versa.

Now let us return to the problem of intersecting essential spectra. As before, if $\sigma_e(B)$ has a component disjoint from $\sigma_e(A)$, there is a clopen subset Σ of $\sigma_e(B)$ separated by an open set Ω from $\sigma_e(A)$. One can arrange that the boundary $\Gamma = \partial\Omega$ is a finite set of nice smooth curves disjoint from $\sigma_e(A) \cup \sigma_e(B)$. So both $A - \gamma I$ and $B - \gamma I$ are Fredholm on Γ of equal index and nullity (which can be taken to be locally constant). Now ind $(A - \lambda I)$ is constant on $\overline{\Omega}$, and nul $(A - \lambda I)$ is constant except on a countable set disjoint from Γ; so in fact they are both constant on all of Γ.

The important special case to consider is:

$$-\text{ind}(A - \lambda I) = \text{ nul } (A - \lambda I)^* = -\text{ind } (B - \lambda I)^* = \text{ nul } (B - \lambda I) = n > 0.$$

Pick μ on the boundary of $\hat{\Omega}$ (the polynomial convex hull of Ω). Choose n vectors $f_i = Xe_i$ in the (dense) range of X which span a complement to

Ran $(A - \lambda I)$. Set

$$\mathcal{M} = \text{ span } \{B^k e_i | 1 \leq i \leq n, k \geq 0\},$$

and

$$\mathcal{N} = \text{ span } \{A^k f_i | 1 \leq i \leq n, k \geq 0\}.$$

Let $A_0 = A|\mathcal{N}$ and $B_0 = B|\mathcal{M}$. Then A_0 and B_0 are quasisimilar, are both multicyclic of order at most n, and ind $A_0 - \mu I = -n$. Hence they both have multiplicity exactly n, and the component of $\sigma_e(A_0)$ containing Γ is simply connected! In particular, ind $(A_0 - \gamma I) = -n$ for all $\gamma \in \Gamma$.

Now let A_1 be the compression of A to $\mathcal{N}^\perp$. It follows that A_1 is invertible on Γ and is quasisimilar to the compression B_1 of B to $\mathcal{M}^\perp$. Retracing all the hypotheses, we find that the Riesz spectral projection $E_{A_1}(\Omega)$ is finite rank, whence the same must follow for $E_{B_1}(\Omega)$. This in turn implies that B has no essential spectrum in Ω, contrary to fact.

4.2. Jordan forms. Let us look at a case for quasisimilarity as a useful way of classifying certain operators. In [#2], Apostol, Douglas and Foiaş show that nilpotent operators can be classified up to quasisimilarity by their 'Jordan forms'. Herrero and I **[106]** extended this result to the largest possible class. The *Jordan operators* we consider are all operators which are direct sums of the basic building blocks $\lambda I_n + J_n$ where J_n is the $n \times n$ Jordan nilpotent matrix. We allow up to countably many eigenvalues provided they remain bounded; and for each eigenvalue, it is permissable to have blocks of all sizes, repeated as often as desired.

Recall that in finite dimensions, the Jordan structure of a matrix T is determined by the dimensions nul $(T - \lambda I)^k$. Indeed,

$$\text{nul}(T - \lambda I; k) := \text{ dim } (\ker(T - \lambda I)^k / \ker(T - \lambda I)^{k-1})$$

equals the number of Jordan blocks of T for the eigenvalue λ of size at least k. The number of size exactly k is

$$\text{nul } (T - \lambda I; k) - \text{ nul } (T - \lambda I; k+1).$$

In infinite dimensions, we can use these numbers to define the *Jordan form* $J(T)$ of T whenever T has a sufficient set of eigenvalues. The only problem lies in defining $\infty - \infty$, which we define to be ∞.

The first thing to notice is that Jordan operators are triangularizable, as are their adjoints. That is, they are *bitriangular*. Since triangularity is preserved by quasisimilarity, any quasisimilar operator is also bitriangular. Our theorem states:

Theorem. *Every bitriangular operator T is quasisimilar to its canonical Jordan form $J(T)$. Two bitriangular operators S and T are quasisimilar if and only if nul $(S-\lambda I;k) =$ nul $(T-\lambda I;k)$ for all $\lambda \in \mathbf{C}$ and $k \geq 1$.*

The beauty of this theorem lies in the very simple invariants that determine the quasisimilarity class. Bitriangular operators turn out to be the appropriate analogue in infinite dimensions of finite rank operators from many points of view. For example, the diagonal entries of any upper triangular form for T will be the point spectrum of T including multiplicity; and the point spectrum of T^* is the conjugate of the point spectrum of T, again including multiplicity. To see this, put T in upper triangular form with diagonal elements $d(T)$. It is an easy vector calculation to show that

$$\sigma_p(T^*)^* \subset d(T) \subset \sigma_p(T).$$

But as T^* is also triangular (with respect to a different basis), we conclude that

$$\sigma_p(T^*)^* = d(T) = d(T^*)^* = \sigma_p(T).$$

One can keep track of multiplicity, and one can with little more effort show that

$$\text{nul }((T-\lambda I)^*;k) = \text{nul}(T-\lambda I);k).$$

The key to the Jordan form theorem lies in another aspect of the very special form of bitriangular operators. They are all similar (via $I+$ small trace-class) to 'staircase' operators; i.e. operators of the form

$$T = \begin{bmatrix} A_1 & B_1 & & & & & \\ & C_1 & & & & & \\ & D_1 & A_2 & B_2 & & & \\ & & & C_2 & & & \\ & & & D_2 & A_3 & B_3 & \\ & & & & & C_3 & \\ & & & & & \ddots & \ddots \end{bmatrix},$$

where all blocks in sight are finite dimensional, and missing entries are 0. Indeed, this form makes it very clear that the operator is both upper and lower triangular with respect to two intertwined sequences of subspaces. One really nice aspect of this representation is that large segments of the matrix can be compressed into a single block without changing the staircase appearance. This makes manipulation of these operators rather tractable. It also makes a huge number of invariant subspaces stand out in a very useful way.

4.3. The Unilateral Shift. A description of the set of operators quasisimilar to the unilateral shift S has proven to be an elusive problem. Since S is cyclic ($\overline{\mu}(S) = 1$), any operator $T \sim_{qs} S$ is also cyclic. Hence ind $(T - \lambda I) \geq -1$, and components of index -1 are simply connected. Now $S - \lambda I$ is always injective, and $(S - \lambda I)^*$ is injective for $|\lambda| > 1$. Thus the same holds for T. So any point in $\sigma(T)\backslash\overline{\mathbf{D}}$ must belong to $\sigma_{\ell re}(T)$. On the other hand, $\mathbf{D}$ must belong to $\sigma(T)$. Hence $\sigma_e(T)$ contains $\mathbf{T}$. From above, every component of $\sigma_e(T)$ must intersect $\mathbf{T}$, and therefore $\sigma_e(T)$ is connected. (Now the simple connectedness of the components of $\mathbf{D}\backslash\sigma_e(T)$ is assured.) The Fredholm index of $T - \lambda I$ must be -1 for $\lambda \in \mathbf{D}\backslash\sigma_e(T)$.

The Similarity Orbit Theorem shows that we have just described $\overline{\mathcal{S}(S)}$. That is, the closure of the similarity orbit of the shift consists of all operators T such that (i) $\sigma_{\ell re}(T)$ is connected and contains $\mathbf{T}$, (ii) $\sigma(T) = \sigma_{\ell re}(T) \cup \mathbf{D}$, and (iii) ind $(T - \lambda I) = -1$ for all $\lambda \in \sigma(T)\backslash\sigma_e(T)$. A simpler question than a '*description*' of the operators quasisimilar to S is whether all of these spectral pictures occurs for some operator quasisimilar to the shift.

This is indeed the case. Agler, Franks and Herrero [**124**] prove:

Theorem. *Let T be a biquasitriangular operator such that $\sigma(T) = \sigma_e(T)$ and $\sigma(T) \cup \mathbf{T}$ is connected. Then given $\epsilon > 0$, there is a compact operator K with $\|K\| < \epsilon$ such that*

$$S \oplus T + K \sim_{qs} S.$$

This leaves a natural open problem:

Problem. *Suppose that $T \in \overline{\mathcal{S}(S)}$ and $\epsilon > 0$. Is there a compact operator K with $\|K\| < \epsilon$ such that $T - K \sim_{qs} S$?*

5. Quasidiagonality. Not everything Herrero worked on was invariant under similarity. Many interesting classes of operators are unitarily invariant but not similarity invariant. One very important class is the set of *quasidiagonal* operators, which, according to Herrero [**51**], '*abhors similarities*'. Recall that T is quasidiagonal if it there is a sequence P_n of finite rank projections increasing to I such that

$$\lim_{n\to\infty} \|P_nT - TP_n\| = 0.$$

This is precisely the closure of the *block diagonal* operators, and every quasidiagonal operator is the sum of a block diagonal operator and a small compact operator.

The main problem that stimulated a lot of work on approximation of quasidiagonal operators was posed by L. Williams [**#12**]: *If T is quasidiagonal and is also a limit of nilpotent operators, is T the limit of quasidiagonal nilpotents?*

Williams speculated that the answer was no. However, Herrero wrote several papers [**51**,53,68] proving special cases in an attempt to prove it true. It turned out to be false, and Herrero's counterexample [**80**] is very interesting because it uses the trace to obtain a norm estimate for operators on infinite dimensional space. Let

$$T = \begin{bmatrix} I & I \\ 0 & H \end{bmatrix},$$

where H is a diagonal operator with $\sigma(H) = [0,1]$. If we write $H = \operatorname{diag}(\{h_n\})$, then it is easy to see that

$$T \cong \sum_{n\geq 1} \oplus \begin{bmatrix} 1 & 1 \\ 0 & h_n \end{bmatrix}$$

and hence is block diagonal. Since $\sigma(T) = [0,1]$, T satisfies the hypotheses of Theorem [**AFV**] and hence is the limit of nilpotents.

Now suppose that N is a quasidiagonal nilpotent within ϵ of T, and let P_k be a sequence of finite rank projections assymptotically commuting with N. For large k, $\|[T,P_k]\| < \epsilon$. Writing P_k as a 2×2 matrix

$$P_k = \begin{bmatrix} W & X \\ X^* & Y \end{bmatrix},$$

one obtains

$$\|[P_k,T]\| = \left\| \begin{bmatrix} -X^* & (W-Y) + X(H-I) \\ X^*(I-H) & X^* + [Y,H] \end{bmatrix} \right\| < \epsilon.$$

It follows that X is small, and $W-Y$ is small. So P_k is close to a projection of the form

$$Q_k = \begin{bmatrix} R_k & 0 \\ 0 & R_k \end{bmatrix}.$$

¿From the upper-semicontinuity of the spectrum, spr (P_kNP_k) tends to 0. Thus

$$\lim_{k\to\infty} \frac{|tr(P_kNP_k)|}{tr(P_k)} \leq \lim_{k\to\infty} spr(P_kNP_k) = 0.$$

On the other hand,

$$\frac{|tr(Q_kNQ_k)|}{tr(Q_k)} \geq \frac{|tr(R_kIR_k)|}{2tr(R_k)} = \frac{1}{2}.$$

These various estimates are incompatible.

All is not lost. Two problems pop up like the heads of a hydra to replace the old one.

Problem. *Is every quasinilpotent quasidiagonal operator the limit of quasidiagonal nilpotent operators?*

Herrero [**53**] shows that any quasidiagonal quasinilpotent operator of the form $Q \oplus N^{(\infty)}$, where N is a nilpotent matrix, is indeed the limit of quasidiagonal nilpotent operators. The extra 'room' provided by the nilpotent summands seems to help a lot. Such has proven to be the case in similarity invariant problems. In that context, Voiculescu's Weyl-von Neumann Theorem [**#10**] is a powerful tool. In the operator theory context, this theorem says that if ρ is a representation of $C^*(\pi(T))$, then

$$T \simeq_a T \oplus \rho(\pi(T))^{(\infty)}.$$

In the quasidiagonal context, this is likely to be useful only if $\rho(\pi(T))$ is quasidiagonal and ρ is faithful. Herrero posed the question: is this always possible? He pointed out that if it is true, then there is an affirmative answer to this problem. Unfortunately, that is not the case. Wasserman [**#11**] has constructed a counterexample.

There is another way to settle this problem affirmatively. It requires estimating the distance to the nilpotent matrices of order k (or even $k/4$) in terms of the quantity $\|T^k\|^{1/k}$. This is a problem about matrices, but the estimates must not depend on the dimension of the space. Such an estimate was shown to be valid in infinite dimensions by Apostol and Salinas [**#4**].

The other way of modifying the problem is:

Problem. *Is every quasidiagonal operator the limit of quasidiagonal algebraic operators?*

This second question is based on another theorem of Voiculescu [#9]: *The closure of the algebraic operators consists of all operators such that* ind $(T - \lambda I) = 0$ *for all* $\lambda \in \rho_{s-F}(T)$. This class clearly contains all quasidiagonal operators. Herrero, Salinas and I [**103**] obtained the following partial result:

Theorem. *Suppose that T is quasidiagonal, and there is a unital, quasidiagonal representation ρ of $C^*(\pi(T))$ such that $\sigma(\rho(\pi(T))) = \sigma_e(T)$ has connected complement. Then T is the limit of quasidiagonal algebraic operators.*

In spite of Wasserman's counterexample, the hypothesis concerning ρ is frequently satisfied. It holds whenever $C^*(T)$ is contained in a nuclear C^*-algebra. It also holds when T has a normal summand M with $\sigma_e(M) = \sigma_e(T)$. So in particular, $T \oplus M$ satisfies the hypotheses. The proof of this theorem relies heavily on a finite dimensional approximation result that we were able to prove only by applying some very big infinite dimensional guns – namely, isometric dilations (very infinite) and the quantitative version of the Brown-Douglas-Fillmore Theorem that Berg and I proved [**#5**]. It is our feeling that a better proof of this lemma (meaning a truly finite dimensional proof) would allow one to drop the condition that the spectrum be simply connected.

6. Final Remarks. I have had to leave out a lot more than I put in. Herrero did a lot of work on quasitriangularity and related notions, and was fascinated by triangular operators. Together with Fialkow, he worked on derivations. He spent a lot of energy studying the class of *finite* operators introduced by J. Williams. Like most operator theorists, he sought invariant subspaces and found some. Recently, he did some very pretty work on hypercyclic operators, which shows that many bounded linear operators exhibit peculiar chaotic behaviour.

There are perhaps two glaring omissions. Herrero (and independently Voiculescu) developed an analogue of Rota's model for operators with multiply connected spectrum. This really belongs in the section on similarity, and indeed it is lurking in the background. The other is Herrero's emphasis on compact perturbations. This is a very important aspect of much of this approximation theory. Whenever one can obtain some nice form after a small perturbation, ask yourself if it could also be a compact perturbation. If there is no obvious obstruction, it is worth the effort to try to get the stronger result. This plays a recurring theme throughout Herrero's work. Plus there are various results outside the tight area I have concentrated on here. For example, a favourite of mine is his characterization of those operators which are unitarily equivalent to a (small) compact perturbation of some operator in a fixed nest algebra.

I offer the excuse of limited space for glossing over so much. Nevertheless, I hope that I have convinced you that Herrero has played a central role in several important developments in operator theory of the past two decades. I will miss that package containing three or four more Herrero papers that used to arrive in my mail box every six months like clockwork. I will miss his incomprehensible multi-lingual jokes that he always had to translate for me. I will miss the humour and passion of a man that I was proud to call my friend. But I will not forget him, and I know that his work will outlast us all.

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Department of Pure Mathematics
Waterloo, Ontario, Canada N2L 3G1

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