HOUSTON JOURNAL OF MATHEMATICS
Volume 17, No. 4, 1991

SPECTRAL ASSIGNMENT FOR HILBERT SPACE OPERATORS

LEONID GURVITS
LEIBA RODMAN*
ILYA SPITKOVSKY**

Abstract. **Given Hilbert space operators A and B, the possible spectra of operators of the form A − BF are described, under suitable hypotheses. The Fredholm properties of $\lambda I - (A + BF)$ are studied, as well as the situation when the operator $A + BF$ can be made algebraic, for a suitable choice of F.**

To the memory of Domingo Herrero

1. Introduction. Let H, G be (complex) Hilbert spaces, and denote by $L(G, H)$ the set of all linear bounded operators defined on G with range in H. Given $A \in L(H) = (L(H,H)), B \in L(G,H)$, the *spectrum assignment problem* is: what are the possible spectra of operators of the form $A+BF$, where $F \in L(H,G)$? More generally, one asks for a description of spectral structure of operators of the form $A+BF$.

In the finite dimensional case (dim $H < \infty$, dim $G < \infty$) this problem is one of the basic problems in the control theory of linear systems with finite dimensional state space. The solution of this problem is well-known as Rosenbrok's theorem [**12**] in case $\sum_{j=0}^{\infty} \operatorname{Im}(A^j B) = H$. The general finite dimensional case is treated in [**14, 15**], where not only the possible eigenvalues of $A + BF$ are described (with fixed A and B), but also the possible Jordan structures of $A + BF$.

In the infinite dimensional case, a complete description of the spectra of operators $A + BF$ is known when the pair (A, B) is *exactly controllable*, i.e. the operator

$$[B, AB, \ldots, A^{p-1}B] \in L(G^p, H)$$

is right invertible for some positive integer p.

*Partially supported by an NSF grant DMS-9000839
**Partially supported by an NSF grant DMS-9101143

Theorem 1.1. *The following statements are equivalent for a pair of operators $A \in L(H), B \in L(G, H)$:*

(i) *(A, B) is exactly controllable;*

(ii) *the operator $[\lambda I - A, B] \in L(H \oplus G, H)$ is right invertible for every $\lambda \in \mathbb{C}$.*

If, in addition, H and G are infinite dimensional then (i) and (ii) are equivalent to

(iii) *for every compact set $\Lambda \subset \mathbb{C}$ there exists $F \in L(H, G)$ such that $\sigma(A + BF) = \Lambda$.*

Moreover, if (iii) holds, then F can be chosen to depend continuously (in some appropriate sense) on A, B and Λ.

Various parts of Theorem 1.1 were proven in [**6**, 7, 13, 10]; see also [**11**]. Additional conditions equivalent to (i), (ii) and (iii) can be found in [**13**], one of them is quoted in Section 5 (Theorem 5.1).

In this paper we study the spectrum assignment problem for not exactly controllable pairs (A, B), assuming throughout the paper that H is infinite dimensional (if dim $H < \infty$, the problem is in fact finite dimensional). In the not exactly controllable case, not every compact set can be assigned as spectrum of $A+BF$ for a suitable F, as it follows from Theorem 1.1. A necessary condition for assignability is easily obtained:

Proposition 1.2. *For given $A \in L(H), B \in L(G, H)$ let*

$$\Sigma(A, B) = \{\lambda \in \mathbb{C} | [\lambda I - A, B] \text{ is not right invertible}\}$$

Then for every $F \in L(H, G)$

$$\sigma(A + BF) \supseteq \Sigma(A, B). \tag{1.1}$$

Proof: If $\lambda \in \Sigma(A, B)$, then operator $[\lambda I - A, B]$ is not right invertible, that is, $\mathrm{Im}(\lambda I - A, B)$ is a proper (not necessary closed) subspace of H. So

$$\mathrm{Im}(\lambda I - (A + BF)) = \mathrm{Im}((\lambda I - A) - BF) \subseteq$$
$$\subseteq \mathrm{Im}(\lambda I - A) + \mathrm{Im}BF \subseteq \mathrm{Im}(\lambda I - A) + \mathrm{Im}B = \mathrm{Im}(\lambda I - A, B) \neq H.$$

It follows that $\mathrm{Im}(\lambda I - (A+BF)) \neq H$, and therefore $\lambda \in \sigma(A+BF)$. ■

The question arises whether equality can be achieved in (1.1) for a suitable F, and more generally whether every compact set containing $\Sigma(A, B)$ can be achieved as the spectrum of $A + BF$. Easy examples (see Section 6) show that the answer is negative. By analogy with Theorem 1.1, one expects that some kind of "generalized exact controllability" is a natural sufficient condition for assignability of the spectrum of $A + BF$ provided the necessary condition (1.1) is satisfied. This is indeed the case.

To state one of the main results of this paper, introduce the following definition. For a pair of Hilbert space operators $A \in L(H)$, $B \in L(G, H)$ denote $\mathcal{C}_p(A, B) = \text{Im}[B, AB, \dots, A^{p-1}B]$. The pair of operators A, B is called *admissible* if for some positive integer p $\mathcal{C}_p(A, B) = \mathcal{C}_{p+1}(A, B)$ and the linear set $\mathcal{C}_p(A, B)$ is closed. If p is the minimal positive integer with these properties, we say that the pair (A, B) is p-admissible.

Theorem 1.3. *Let (A, B) be a p-admissible pair of Hilbert space operators, where p is a positive integer. If ImB is infinite dimensional, then for every compact set $\Lambda \neq \emptyset$ such that $\Lambda \supseteq \Sigma(A, B)$ there exists $F \in L(H, G)$ satisfying*

$$\sigma(A + BF) = \Lambda. \tag{1.2}$$

The proof of this theorem will be given in the next section.

The result of Theorem 1.3, as well as of the more general Theorem 4.2, depends essentially on the hypothesis of p-admissibility. An example illustrating this point is given at the end of the paper. In Section 3 we give the necessary and sufficient conditions for a compact set Λ to coincide with the spectrum of $A + BF$, for a suitable F, where the pair (A, B) is p-admissible but ImB is finite dimensional. In Section 4 we prove more detailed version of Theorem 2.3, with the Fredholm properties of $\lambda I - (A + BF)$ taken into account. The situation when $A+BF$ can be made algebraic (i.e. $p(A+BF) = 0$ for a non-zero polynomial $p(z)$) by a suitable choice of F is studied in Section 5.

We are grateful to Prof. D. A. Herrero for bringing to our attention the reference [**9**]. During the final stages of preparation of this paper, we were deeply saddened by the unexpected death of Prof. D. A. Herrero who was a close friend of one of us (L. R.).

2. Proof of Theorem 1.3: Let (A, B) be a p-admissible pair. The subspace $H_0 = \mathcal{C}_p(A, B)$ is obviously A-invariant. With respect to the orthogonal decomposition $H = H_0 \oplus H_0^{\perp}$ we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}. \tag{2.1}$$

Proposition 2.1. *In the above notation,*

$$\Sigma(A, B) = \{\lambda \in \mathbb{C} | \lambda I - A_{22} \text{ is not right invertible}\}. \tag{2.2}$$

Proof: By Theorem 1.1, the operator $[\lambda I - A_{11}, B_1]$ is right invertible for all $\lambda \in \mathbb{C}$. Therefore,

$$[\lambda I - A, B] = \begin{bmatrix} \lambda I - A_{11} & -A_{12} & B_1 \\ 0 & \lambda I - A_{22} & 0 \end{bmatrix}$$

is right invertible if and only if $\lambda I - A_{22}$ is, and (2.2) follows. ■

In particular, $\Sigma(A,\ B) \subseteq \sigma(A_{22})$.

The parts (i)-(iii) of the next lemma are essentially proved in **[13]** (although it was not explicitly stated there). Because of the later needs the lemma contains more information that is needed for the proof of Theorem 1.3.

Lemma 2.2. *Let $D \in L(H)$, $B \in L(G, H)$ be an n-admissible exactly controllable pair of operators, where $n > 1$. Then there exist orthogonal decompositions $H = M_1 \oplus M_2$, $G = G_1 \oplus G_2$ ($M_2 \neq \{0\}$) with the following properties:*

(i) *write*

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} : M_1 \oplus M_2 \to M_1 \oplus M_2$$

with respect to this decomposition; then the pair (D_{22}, D_{21}) is $(n-1)$-admissible and exactly controllable.

(ii) *the operator $B : G_1 \oplus G_2 \to M_1 \oplus M_2$, when partitioned with respect to these orthogonal decompositions has the form*

$$B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

with invertible operator $B_1 \in L(G_1, M_1)$.

(iii) *for every* $E \in L(M_2 M_1)$ *and every* $K \in L(M_1)$ *there exists* $F = \begin{bmatrix} F_1 & F_2 \\ 0 & 0 \end{bmatrix} : M_1 \oplus M_2 \to G_1 \oplus G_2$ *such that*

$$D + BF = \begin{bmatrix} D_{11} + B_1 F_1 & D_{12} + B_1 F_2 \\ D_{21} & D_{22} \end{bmatrix} \text{ and } \begin{bmatrix} K & 0 \\ D_{21} & D_{22} + D_{21} E \end{bmatrix}$$

are similar:

$$\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} K & 0 \\ D_{21} & D_{22} + D_{21} E \end{bmatrix} \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} = D + BF.$$

(iv) *if* ImB *is infinite dimensional, then* M_1 *(and therefore also* G_1*) are infinite dimensional.*

Proof: Let us mention first of all that (iii) follows directly from (ii). Indeed,

$$\begin{bmatrix} I & E \\ 0 & I \end{bmatrix} \begin{bmatrix} K & 0 \\ D_{21} & D_{22} + D_{21} E \end{bmatrix} \begin{bmatrix} I & -E \\ 0 & I \end{bmatrix} = \begin{bmatrix} K + ED_{21} & ED_{22} - KE \\ D_{21} & D_{22} \end{bmatrix},$$

and

$$BF = \begin{bmatrix} B_1 F_1 & B_1 F_2 \\ 0 & 0 \end{bmatrix}.$$

So, F_1 and F_2 are defined by formulas:

$$F_1 = B_1^{-1}(K + ED_{21} - D_{11}), F_2 = B_1^{-1}(ED_{22} - KE - D_{12}).$$

Therefore it is sufficient to verify properties (i), (ii), and (iv).

Suppose at first that ImB is closed. Then take $M_1 = \text{Im}B$, $G_2 = \text{Ker } B$. In this case the operator matrix $B : G_1 \oplus G_2 \to M_1 \oplus M_2$ has the form $B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$, and (ii) is true with $B_{12} = 0$, $B_{22} = 0$. Due to this special structure of B

$$\mathcal{C}_n(D, B) = M_1 \oplus \mathcal{C}_{n-1}(D_{21}, D_{22}).$$

This equality proves (i).

Now assume ImB is not closed. As in [**13**], let $B^*B = \int_0^\infty t dE(t)$ be the spectral representation of B^*B, and let

$$G_2 = E(\epsilon)G, \quad M_1 = BG_1 \tag{2.3}$$

and

$$B_\epsilon = B(I - E(\epsilon)) \tag{2.4}$$

where $\epsilon > 0$ will be chosen later. With respect to orthogonal decompositions $G = G_1 \oplus G_2$, $M = M_1 \oplus M_2$ operators B and B_ϵ have the form

$$B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_{22} \end{bmatrix} \quad \text{and} \quad B_\epsilon = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix}$$

correspondingly. In particular, (ii) is satisfied. As $B_\epsilon \to B$ uniformly when $\epsilon \to 0$, and $[B, DB, \dots, D^{n-1}B]$ is right invertible, all operators $[B_\epsilon, DB_\epsilon, \dots, D^{n-1}B_\epsilon]$ also are right invertible for sufficiently small $\epsilon > 0$. In other words, $\mathcal{C}_n(D, B_\epsilon) = H$. We have $\mathcal{C}_{n+1}(D, B_\epsilon) \supseteq \mathcal{C}_n(D, B_\epsilon) = \mathcal{C}_n(D, B) = \mathcal{C}_{n+1}(D, B) \supseteq \mathcal{C}_{n+1}(D, B_\epsilon)$. Therefore the pair (D, B_ϵ) is also n-admissible and exactly controllable (for such ϵ). Since Im$B_\epsilon = M_1$ is closed, for the pair (D, B_ϵ) we are now in the situation that already was considered. As $G_2 = \text{Ker } B_\epsilon$, condition (i) is satisfied.

It remains to prove (iv). Indeed, if for any $\epsilon > 0$ the subspace ImB_ϵ is finite dimensional, then B is compact. Therefore, $[B, DB, \dots, D^{n-1}B]$ is also compact. Together with right invertibility of this operator this implies that $\mathcal{C}_n(D, B)$ is finite dimensional, and so ImB is finite dimensional as well. ■

Proof of Theorem 1.3: Let $H_0 = \mathcal{C}_n(A, B)$. With respect to the orthogonal decomposition $H = H_0 \oplus H_0^\perp$ we have

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Without loss of generality we assume $H_0 \neq H$ (otherwise Theorem 1.3 is reduced to Theorem 1.1). Assume first Im$B_1 = H_0$ (i.e. $p = 1$). As H_0 is infinite dimensional, we can write $H_0 = H_{01} \oplus H_{01}$, where H_{01} is an infinite dimensional Hilbert space. By Read's theorem **[9]**, there exist $K \in L(H_{01}), C \in L(H_0^\perp, H_{01}))$ such that

$$\sigma \begin{bmatrix} K & C \\ 0 & A_{22} \end{bmatrix} = \{\lambda \in \mathbb{C} | \lambda I - A_{22} \text{ is not right invertible } \}$$

which coincides with $\Sigma(A,B)$ by Proposition 2.1. So, given any compact set Λ containing $\Sigma(A,B)$, let $K(\Lambda) \in L(H_{01})$ be such that $\sigma(K(\Lambda)) = \Lambda$; then

$$\sigma\left(\begin{bmatrix} K(\Lambda) & 0 & 0 \\ 0 & K & C \\ 0 & 0 & A_{22} \end{bmatrix}\right) = \Lambda. \tag{2.5}$$

But since B_1 is right invertible, the operator in (2.5) can be written in the form

$$\begin{bmatrix} K(\Lambda) & 0 & 0 \\ 0 & K & C \\ 0 & 0 & A_{22} \end{bmatrix} = A + BF,$$

where

$$F = B_1^{-1}\left[-A_{11} + \begin{bmatrix} K(\Lambda) & 0 \\ 0 & K \end{bmatrix}, -A_{12} + [0 \quad C]\right] \in L(H_0 \oplus H_0^\perp, G),$$

and where B_1^{-1} is a right inverse of B_1. So Theorem 1.3 is proved in the case $\text{Im}B_1 = H_0$.

Assume now that $p > 1$. Apply Lemma 2.2 to the pair (A,B), with the subspace H replaced by $\mathcal{C}_p(A,B)$. We obtain

$$A = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_{22} \\ 0 & 0 \end{bmatrix},$$

with respect to the decomposition $H = M_1 \oplus M_2 \oplus (\mathcal{C}_p(A,B))^\perp, G = G_1 \oplus G_2$ where $M_1 \oplus M_2 = \mathcal{C}_p(A,B)$. Here the operator B_1 is invertible, and the pair (X_{22}, X_{21}) is exactly controllable. Observe also that by Lemma 2.2 (iv) the subspace M_1 is infinite dimensional. Let be given a non-empty compact set Λ containing $\Sigma(A,B)$. By Theorem 1.1, there exists $Y \in L(M_2, M_1)$ such that $\sigma(X_{22}+X_{21}Y)=\{\lambda_0\}$, where λ_0 is a fixed point in Λ (if M_2 happens to be finite dimensional, then the well-known pole assignment theorem for matrices ensures the existence of such Y). Let

$$A'=\begin{bmatrix} I & -Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} A \begin{bmatrix} I & Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

$$B'=\begin{bmatrix} I & -Y & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} B=\begin{bmatrix} B_1 & B_{12}-YB_{22} \\ 0 & B_{22} \\ 0 & 0 \end{bmatrix}.$$

Clearly, we can consider the pair (A',B') instead of (A,B). We have

$$A'=\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ X_{21} & Z_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix},$$

where $Z_{22}=X_{22}+X_{21}Y$. Using the invertibility of B_1 and Read's theorem [**9**], let $F=\left[\begin{smallmatrix} F_1 & F_2 & F_3 \\ 0 & 0 & 0 \end{smallmatrix}\right]\in L(H,G)$ be such that

$$A'+B'F=\begin{bmatrix} K & 0 & C \\ Z_{21} & Z_{22} & Z_{23} \\ 0 & 0 & X_{33} \end{bmatrix}$$

where the operators K and C are chosen to satisfy $\sigma\left[\begin{smallmatrix} K & C \\ 0 & X_{33} \end{smallmatrix}\right]=\Lambda$ (the same argument was used in the first part of the proof). As $\sigma(Z_{22})=\{\lambda_0\}\subseteq\Lambda$, it follows that $\sigma(A'+B'F)=\Lambda$. Indeed, $A'+B'F$ is similar to a block triangular operator with the diagonal blocks $\left[\begin{smallmatrix} K & C \\ 0 & X_{33} \end{smallmatrix}\right]$ and Z_{22}. So $\lambda I-(A'+B'F)$ is invertible for every $\lambda\notin\Lambda$ and is not invertible for every $\lambda\in\Lambda\backslash\{\lambda_0\}$. Finally, $\lambda_0 I-Z_{22}$ is not right invertible (otherwise, Z_{22} could not have a singleton spectrum), and therefore $\lambda_0 I-(A'+B'F)$ is not invertible. This completes the proof of Theorem 1.3. ■

3. Spectrum Assignment for Admissible Pairs: The Remaining Case. In this section we study the spectrum assignment for p-admissible pairs (A, B) in the relatively easy case when B is a finite rank operator. We use the decomposition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tag{3.1}$$

with respect to $H = H_0 \oplus H_0^\perp$, where $H_0 = \mathcal{C}_p(A, B)$.

Theorem 3.1. *Let (A, B) be a p-admissible pair, and assume that ImB is finite dimensional, and let be given a compact set $\Lambda \subset \mathbb{C}$. Define $q = \dim \mathcal{C}_p(A, B)(< \infty)$. Then there exists $F \in L(H, G)$ satisfying $\sigma(A+BF) = \Lambda$ if and only if $\Lambda \supseteq \sigma(A_{22})$ and $\Lambda \backslash \sigma(A_{22})$ consists of at most q points, where A_{22} is defined by the decomposition (3.1).*

Proof: From the representation (3.1) and the finite dimensionality of H_0, it follows easily that

$$\sigma(A + BF) \supseteq \sigma(A_{22}) \tag{3.2}$$

for every $F \in L(H, G)$. As dim $H_0 = q$, it is also clear that $\sigma(A + BF) \backslash \sigma(A_{22})$ consists of at most q points.

Conversely, let $\Lambda \supseteq \sigma(A_{22})$ be such that $\Lambda \backslash \sigma(A_{22})$ consists of at most q points. Since dim $H_0 = q < \infty$, by the well known pole assignment theorem in finite dimensional spaces there exists $F_1 \in L(H_0, G)$ such that $\sigma(A_{11} + B_1 F_1) = \Lambda \backslash \sigma(A_{22})$ (or $\sigma(A_{11} + B_1 F_1) \subseteq \sigma(A_{22})$ if $\Lambda = \sigma(A_{22})$). Letting $F = [F_1, 0] \in L(H, G)$ we obtain that

$$A + BF = \begin{bmatrix} A_{11} + B_1 F_1 & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

and therefore (in view of (3.2)) $\sigma(A + BF) = \Lambda$. ■

4. Assignability of semi-Fredholm properties. Theorem 1.3 admits a more precise formulation where not only the spectrum but also the semi-Fredholm properties of $A + BF$ are prescribed. Let us introduce the concepts necessary for this formulation. As before, H is an infinite dimensional Hilbert space. An operator $A \in L(H)$ is called *semi-Fredholm* if ImA is closed and at least one of Ker A and $H/\text{Im}A$ is finite dimensional. More precisely, we say that A is (α, β)*-semi-Fredholm*, where $\alpha =$ dim Ker $A, \beta =$

dim $(H/\text{Im}A)$. So α, β are nonnegative integers or infinity, but not both α and β can be infinity. Denote by Ω the set of all such pairs:

$$\Omega = (\mathbb{Z} \cup \{\infty\}) \times (\mathbb{Z} \cup \{\infty\}) \backslash (\infty, \infty).$$

The difference $\alpha - \beta$ is called the index of operator A and is denoted by ind A. It is well known (see, e.g., [1]) that the sets

$$\Lambda_\kappa(A) = \Lambda_\kappa = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is semi-Fredholm with index } \kappa\}$$

are open for all $\kappa \in \mathbb{Z}$. Let us denote

$$\Lambda_{\alpha,\beta}(A) = \Lambda_{\alpha,\beta} = \big\{\lambda \in \mathbb{C} : A - \lambda I$$
$$\text{is semi-Fredholm, dim Ker } (A - \lambda I) = \alpha, \text{ codim Im } (A - \lambda I) = \beta\big\}.$$

It follows directly from this definition that

$$\Lambda_\kappa = \cup\{\Lambda_{\alpha,\beta} : (\alpha, \beta) \in \Omega, \alpha - \beta = \kappa\}$$

and $\sigma(A) = \mathbb{C} \backslash \Lambda_{0,0}$. In particular,

(i) Λ_{00} contains a neighborhood of infinity.
The structure of other sets $\Lambda_{\alpha,\beta}$ is more refined. Namely,

(ii) $\Lambda_{\alpha,\beta} \cap \Lambda_{\alpha',\beta'} = \emptyset$ if $(\alpha, \beta) \neq (\alpha', \beta')$.

(iii) $\Lambda_{\alpha,0}$ and $\Lambda_{0,\beta}$ are open for all $\alpha, \beta \in \mathbb{Z} \cup \{\infty\}$.

(iv) for $\alpha > 0, \beta > 0$ sets $\Lambda_{\alpha,\beta}$ are not necessarily open; their interiors $M_{\alpha,\beta} =$ int $\Lambda_{\alpha,\beta}$ are separated, i.e. the closure of $M_{\alpha,\beta}$ does not intersect $M_{\alpha'\beta'}$ if $(\alpha, \beta) \neq (\alpha', \beta')$.

(v) For every point $\lambda \in \Lambda_{\alpha,\beta} \backslash M_{\alpha,\beta}$ there exists unique $(\alpha', \beta') \in \Omega$ such that $\lambda \in$ int $\left(\overline{M_{\alpha,\beta}}\right)$.

Therefore, $\Lambda_{\alpha,\beta} \backslash M_{\alpha,\beta}$ is at most countable discrete subset of $\Lambda_{\alpha,\beta}$. The above mentioned pair (α', β') has properties:

$$\alpha - \alpha' = \beta - \beta' > 0 \text{ if } \alpha, \beta < \infty$$
$$\alpha' = +\infty, \beta' < \beta \text{ if } \alpha = +\infty$$
$$\alpha' < \alpha, \beta' = +\infty \text{ if } \beta = +\infty$$

For finite dimensional H (the case we are not interested in) all $\Lambda_{\alpha,\beta}$ with $\alpha \neq \beta$ are void, $\Lambda_{\alpha,\alpha}$ are void except finitely many of them, for $\alpha \neq 0$ $\Lambda_{\alpha,\alpha}$ consist of finitely many points; and $\Lambda_0 = \mathbb{C}$. If H is infinite dimensional, then

(vi) $\Lambda_0 \neq \mathbb{C}$.

The system of properties i)-vi) is full. More precisely, the following result holds.

Proposition 4.1. *Let H be an infinite dimensional Hilbert space, and let $\{\Lambda_{\alpha,\beta} : (\alpha,\beta) \in \Omega\}$ be a given system of subsets in $\mathbb{C}$ satisfying (i)-(vi). Then there exists an operator $A_0 \in L(H)$ such that*

$$\Lambda_{\alpha,\beta}(A_0) = \Lambda_{\alpha,\beta} \text{ for all } (\alpha,\beta) \in \Omega$$

In particular, $\sigma(A_0) = \mathbb{C}\backslash\Lambda_{00}$.

Proof: Let us introduce some notation at first. For a bounded open domain $D \subset \mathbb{C}$ let $L_2(D)$ be the space of all square integrable complex valued functions, with respect to the planar measure in D. Let H_D be the (closed) subspace of $L_2(D)$ spanned by rational functions with poles off D, and let μ_D be the operator of multiplication by z in $H_D : (\mu_D f)(z) = zf(z)$. Clearly, the spectrum of μ_D coincides with $\overline{D}, \Lambda_{0,1}(\mu_D) = D$, and all other $\Lambda_{\alpha,\beta}(\mu_D)$ are void.

Let us now consider all pairs (α,β) such that $M_{\alpha,\beta}(=\text{ int } \Lambda_{\alpha,\beta}) \neq \emptyset$. Denote the set of all such pairs by Ω', and for $(\alpha,\beta) \in \Omega'$ put

$$S_{\alpha,\beta} = \mu^{(\beta)}_{D_{\alpha,\beta}} \oplus \left(\mu^*_{D^*_{\alpha,\beta}}\right)^{(\alpha)},$$

where $D_{\alpha,\beta} = \text{ int } \left(\overline{\Lambda_{\alpha,\beta}}\right), D^*_{\alpha,\beta} = \{\lambda : \overline{\lambda} \in D_{\alpha,\beta}\}$, and where for a given Hilbert space operator X and α (α is either a positive integer or ∞) we denote by $X^{(\alpha)}$ the block diagonal operator with all diagonal blocks equal X (the number of such blocks is equal α).

For $\lambda \in \Lambda_{\alpha,\beta} \cap D_{\alpha',\beta'}$ let us denote by N_λ the operator of multiplication by constant λ in a linear space with the (finite) dimension n_λ, where $n_\lambda = \alpha - \alpha'$ if $\alpha < \infty$, $n_\lambda = \beta - \beta'$ if $\beta < \infty$. Finally, let N be an arbitrary operator with the spectrum $\mathbb{C}\backslash \cup \{\Lambda_{\alpha,\beta} : (\alpha,\beta) \in \Omega\} = \sigma(N)$ such that $N - \lambda I$ is not semi-Fredholm for all $\lambda \in \sigma(N)$. Let us mention that such an operator N exists because the set $\cup\{\Lambda_{\alpha,\beta} : (\alpha,\beta) \in \Omega\}$ is open due to (v); we can construct N, for example, as direct sum of infinitely many copies of a normal operator with the point spectrum dense in $\sigma(N)$. It's easy to verify that the operator

(4.1)
$$A_0 = N \oplus (\oplus\{S_{\alpha,\beta} : (\alpha,\beta) \in \Omega'\}) \oplus (\oplus\{N_\lambda : \lambda \in \Lambda_{\alpha,\beta}\backslash M_{\alpha,\beta}, (\alpha,\beta) \in \Omega\})$$

has the desired semi-Fredholm properties. ■

A related result was obtained in [2] (see also Section XI.5 in [1]).

We now state the main result of this section.

Theorem 4.2. *Let (A, B) be as in Theorem 1.3. Let a family $\{\Lambda_{\alpha,\beta}, (\alpha, \beta) \in \Omega\}$ of subsets $\mathbb{C}$ with properties (i)-(vi) and such that $\Sigma(A, B) \subseteq \mathbb{C} \setminus \cup \Lambda_{\alpha,\beta}$ be given. Then there exists $F \in L(H, G)$ such that $\Lambda_{\alpha,\beta}(A + BF) = \Lambda_{\alpha,\beta}$.*

We need some preliminary results for the proof of Theorem 4.2.

Lemma 4.3. *Let H_1, H_2 be Hilbert spaces and $Y \in L(H_1), Z \in L(H_2)$ operators which are not semi-Fredholm. Then for every $W \in L(H_1, H_2)$ the operator $\begin{bmatrix} Y & 0 \\ W & Z \end{bmatrix}$ is also not semi-Fredholm.*

The proof follows from the observation that a Hilbert space operator $X \in L(H)$ is semi-Fredholm if and only if there exists $Y \in L(H)$ such that at least one of the operators XY or YX is a finite rank perturbation of the identity.

The next lemma describes the semi-Fredholm properties of $A + BF$ under the exact controllability hypothesis.

Lemma 4.4. *Let (A, B) be exactly controllable, and assume that dim $H = \infty$. Then for every family $\{\Lambda_{\alpha,\beta} : (\alpha, \beta) \in \Omega\}$ of subsets in $\mathbb{C}$ satisfying (i)-(vi) there exists $F \in L(H, G)$ such that $\Lambda_{\alpha,\beta}(A + BF) = \Lambda_{\alpha,\beta}$.*

Proof: Let p be such that

$$C_p(A, B) = H. \tag{4.2}$$

We proceed by induction on p. If $p = 1$, then B is right invertible, and every operator $A_0 \in L(H)$ can be written in the form $A_0 = A + BF$. So we are done in this case in view of Proposition 4.1.

Assume now that (4.2) holds, where $p > 1$, and that Lemma 4.4 is proved for $(p-1)$-admissible exactly controllable pairs. Using Lemma 2.2, write

$$H = M_1 \oplus M_2, \; G = G_1 \oplus G_2, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

with the properties described in Lemma 2.2. By Theorem 1.1 find $E \in L(M_2, M_1)$ such that $\sigma(A_{22} + A_{21}E) = \{\lambda_0\}$, where λ_0 is a point in $\mathbb{C} \setminus \cup$

$\{\Lambda_{\alpha,\beta} : (\alpha, \beta) \in \Omega\}$ (such E exists also when M_2 is finite dimensional, by the pole assignment theorem). Choose $K \in L(M_1)$ such that $\Lambda_{\alpha,\beta}(K) = \Lambda_{\alpha,\beta}$. This choice of K is possible because dim $M_1 = \infty$ by Lemma 2.2 (vi), and therefore Proposition 4.1 is applicable.

Let

$$X = \begin{bmatrix} K & 0 \\ A_{21} & A_{22} + A_{21}E \end{bmatrix} \tag{4.3}$$

If $\lambda \in \Lambda_{\alpha,\beta}$ then the right lower block of the operator

$$X - \lambda I = \begin{bmatrix} K - \lambda I & 0 \\ A_{21} & A_{22} + A_{21}E - \lambda I \end{bmatrix}$$

is invertible. Therefore,

$$\dim \operatorname{Ker}(X - \lambda I) = \dim \operatorname{Ker}(K - \lambda I) = \alpha, \ \operatorname{codim} \operatorname{Im}(X - \lambda I) = \operatorname{codim} \operatorname{Im}(K - \lambda I) = \beta,$$

and

$$\Lambda_{\alpha,\beta}(X) \supseteq \Lambda_{\alpha,\beta}. \tag{4.4}$$

If $\lambda \notin \cup\{\Lambda_{\alpha,\beta} : (\alpha, \beta) \in \Omega\}$, then the operator $K - \lambda I$ is not semi-Fredholm. According to Lemma 4.3, an operator $X - \lambda I$ also is not semi-Fredholm. It means that an opposite inclusion in (4.4) is also true. Now an application of Lemma 2.2 (iii) completes the proof of the induction step. ■

Proof of Theorem 4.2: The proof is modelled after the proof of Theorem 1.3. Let $H_0 = \mathcal{C}_p(A, B)$, and partition

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

with respect to $H = H_0 \oplus H_0^{\perp}$. We can assume $H_0 \neq H$ (otherwise Theorem 4.2 is reduced to Lemma 4.4).

Assume first $\operatorname{Im} B_1 = H_0$ (i.e. $p = 1$). Then we argue as in the proof of Theorem 1.3, choosing $K(\Lambda)$ with the requisite semi-Fredholm properties (which is possible by Proposition 4.1).

Assume now $p > 1$. Again, we repeat the procedure used in the proof of Theorem 1.3, choosing $\lambda_0 \in \Sigma(A, B)$. Then choose $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, $C = \begin{bmatrix} 0 \\ C_2 \end{bmatrix}$ so that $\Lambda_{\alpha,\beta}(K_1) = \Lambda_{\alpha,\beta}$ for all $(\alpha, \beta) \in \Omega$, and $\sigma \begin{bmatrix} K_2 & C_2 \\ 0 & X_{33} \end{bmatrix} = \Sigma(A, B)$. Clearly,

$$\Lambda_{\alpha,\beta} \begin{bmatrix} K & C \\ 0 & X_{33} \end{bmatrix} = \Lambda_{\alpha,\beta}.$$

As it was mentioned in the proof of Theorem 1.3, $A' + B'F$ is similar to a block triangular operator with the diagonal blocks $\begin{bmatrix} K & C \\ 0 & X_{33} \end{bmatrix}$ and Z_{22}. Now we can repeat the arguments used in the proof of Theorem 1.3 to prove that the operator X given by (4.3) has the desired properties. ■

5. Algebraic Operators. Let us remind that an operator $X \in L(H)$ is algebraic if there exists a nontrivial polynomial φ such that

$$\varphi(X) = 0. \tag{5.1}$$

The set of all polynomials φ satisfying (5.1) is obviously an ideal in the ring of polynomials. Therefore there exists the unique polynomial φ_0 of this kind having minimal degree K_0 and leading coefficient 1. In this case the condition (5.1) is satisfied if and only if φ is divisible by φ_0, and X is called an algebraic operator of degree K_0.

We say that a pair $A \in L(H), B \in L(G, H)$ is *algebraically controllable* (with the index K_0), there exists an operator $F \in L(H, G)$ such that $A+BF$ is algebraic (of degree K_0; and for any $F \in L(H, G)$ the operator $A + BF$ is not algebraic of degree less than K_0). Of course, in the case of finite dimensional H every operator $X : H \to H$ is algebraic (of degree $\leq$ dim H). Therefore, every pair (A, B) in this case is algebraically controllable.

The following result is proved in [**13**].

Theorem 5.1. *A pair (A, B) is exactly controllable, that is, $C_p(A, B) = H$ for some p, if and only if for any polynomial φ of degree p there exists an operator $F \in L(G, H)$ such that*

$$\varphi(A + BF) = 0. \tag{5.2}$$

So an exactly controllable pair is algebraically controllable.

We shall show in this section that in the general case the algebraic controllability is connected very closely to the property

$$\mathrm{Im}\Psi(A) \subseteq \mathcal{C}_k(A,B). \tag{5.3}$$

This property is satisfied automatically for all polynomials Ψ when (A,B) is exactly controllable and $k = p$. In the general situation all polynomials Ψ satisfying (5.3) with the fixed k form an ideal in the ring of polynomials. Therefore, as it was in the case of annihilating polynomials φ, there exists a unique polynomial Ψ_0 having minimal degree and leading coefficient 1, satisfying (5.3). Moreover, (5.3) is satisfied if and only if Ψ is divisible by Ψ_0.

Lemma 5.2. *If (5.3) is satisfied, then* $\mathcal{C}_i(A,B) = \mathcal{C}_{i+1}(A,B)$ *for* $i \geq \max\{k,\ \deg \Psi\}$.

Proof: If $\deg \Psi = m < k$, the condition (5.3) is also satisfied when polynomial Ψ is substituted by $z^{k-m}\Psi(z)$. If $m > k$, then $\mathcal{C}_k(A,B) \subseteq \mathcal{C}_m(A,B)$ and the inclusion $\mathrm{Im}\Psi(A) \subseteq \mathcal{C}_m(A,B)$ is true along with (5.3). Therefore it is sufficient to consider the case $m = k$. Then according to (5.3) for every $x \in H$ the vector $A^k x$ can be represented as

$$A^k x = \sum_{j=0}^{k-1} c_j A^j x + z_x$$

with suitable scalars c_j and $z_x \in \mathcal{C}_k(A,B)$. In particular, for $x = By$:

$$A^k By = \sum_{j=0}^{k-1} c_j A^j By + z_{By}.$$

Therefore $A^k By \in \mathcal{C}_k(A,B)$. ■

Lemma 5.3. *If (5.2) is satisfied for a polynomial φ of degree k, then (5.3) is true with $\Psi = \varphi$.*

Proof: It's easy to verify that for any operator Z, $\varphi(A+Z)$ can be represented as

$$\varphi(A) + \sum_{j=0}^{k-1} A^j Z C_j$$

with some suitable operators C_j. In particular, for $Z = BF$,

$$0 = \varphi(A + BF) = \varphi(A) + \sum_{j=0}^{k-1} A^j BFC_j.$$

Therefore,

$$\mathrm{Im}\varphi(A) = \mathrm{Im}\sum_{j=0}^{k-1} A^j BFC_j \subseteq \mathrm{Im}[BFC_0, ABFC_1, \dots A^{k-1}BFC_{k-1}]$$
$$\subseteq \mathcal{C}_k(A,B). \quad \blacksquare$$

It follows from Lemmas 5.2 and 5.3 that if the pair (A, B) is algebraically controllable, then (5.3) holds and the increasing sequence of lineals

$$\mathcal{C}_1(A,B) \subseteq \mathcal{C}_2(A,B) \subseteq \dots \mathcal{C}_k(A,B) \subseteq \dots \tag{5.4}$$

stabilizes; moreover, the *stabilization index*, i.e. the least positive integer k such that $\mathcal{C}_k(A,B) = \mathcal{C}_{k+1}(A,B)$ does not exceed the minimal degree of a polynomial φ satisfying $\varphi(A+BF) = 0$ for some $F \in L(H,G)$. The converse statement is only partially valid. Namely, if (A, B) is a pair of Hilbert space operators for which the sequence (5.4) stabilizes with the stabilization index 1, and (5.3) holds for some $k \geq 1$ and some nonconstant polynomial Ψ of degree 1, then (A, B) is algebraically controllable. Indeed, in this case (5.3) takes the form $\mathrm{Im}\Psi(A) \subseteq \mathrm{Im}B$. By Douglas' lemma [4], there exists operator G_1 such that

$$\Psi(A) = BG_1. \tag{5.5}$$

Write $\Psi(z) = \alpha z + \beta$, $\alpha \neq 0$; then (5.5) implies $\Psi(A + BF) = \alpha(A + BF) + \beta I = \alpha A + \beta I + \alpha BF = BG_1 + \alpha BF = 0$ if $F = -\alpha^{-1}G_1$.

It is easy to come up with examples showing that condition (5.3) (for some nonzero polynomial Ψ and some $k \geq 1$) and stabilization of (5.4) are not enough to ensure that the pair (A, B) is algebraically controllable. For example, consider any positive compact infinite dimensional operator A, and put $B = A^m$. Then for any k $\mathcal{C}_k(A,B) = \mathrm{Im}\ B = \mathrm{Im}\ A^m$. Therefore, the condition (5.3) can be rewritten in the form $\mathrm{Im}\varphi(A) \subseteq \mathrm{Im}A^m$, and is satisfied if and only if $\varphi(z)$ is divisible by z^m. Nevertheless, for any choice of F the operator $\varphi(A + A^m F)$ is different from zero. Indeed, if $c_k (k \geq m)$ is the last nonzero coefficient of φ, then $\varphi(A + A^m F) = c_k(I + Z)A^k$ for some compact operator Z. Being Fredholm, operator $I + Z$ has a finite dimensional kernel. Therefore, if $\varphi(A + A^m F) = 0$, then Im A^k (which is contained in Ker $(I + Z)$) is also finite-dimensional, which is a contradiction.

Let us mention that in this example the sequence of lineals (5.4) stabilizes from the very first element, but these lineals are not closed. Therefore, the pair (A, B) is not p-admissible for any p. The situation changes when we turn to p-admissible pairs.

Theorem 5.4. *A p-admissible pair (A, B) is algebraically controllable if and only if there exists a nonzero polynomial Ψ such that*

$$\text{Im}\Psi(A) \subseteq \mathcal{C}_p(A, B). \tag{5.6}$$

If this condition is satisfied, and Ψ_0 is a polynomial of minimal degree satisfying (5.6), then an operator F such that $\varphi(A + BF) = 0$ exists if and only if φ is divisible by Ψ_0 and deg $\varphi \geq p$.

Proof: Necessity. According to Lemma 5.3, if $\varphi(A + BF) = 0$ and deg $\varphi = k$, then

$$\text{Im}\varphi(A) \subseteq \mathcal{C}_k(A, B). \tag{5.7}$$

As for any integer $\ell \geq 1\, \mathcal{C}_\ell(A, B) \subseteq \mathcal{C}_p(A, B)$, the condition (5.6) is satisfied for $\Psi = \varphi$. As it was mentioned above, this means that φ is divisible by Ψ_0. From (5.7) and Lemma 5.2 it follows that $\mathcal{C}_k(A, B) = \mathcal{C}_{k+1}(A, B)$. According to the definition of a p-admissible pair, it means that $k \geq p$.

Sufficiency. We have to prove that if deg $\varphi \geq p$ and

$$\text{Im } \varphi(A) \subseteq \mathcal{C}_p(A, B), \tag{5.8}$$

then there exists an operator $F \in L(G, H)$ such that $\varphi(A + BF) = 0$. We shall use the block representations of operators A and B, obtained in the proof of the Theorem 1.3.

For $p = 1$, $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ with invertible B_1. According to (5.8), Im$\varphi(A) \subseteq$ ImB. It means that $\varphi(A_{22}) = 0$. Choose now $F = [-B_1^{-1}A_{11}, -B_1^{-1}A_{12}]$. Then $A + BF = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}$, and $\varphi(A + BF) = 0$.

Suppose now that $p > 1$. As in the proof of the Theorem 1.3, instead of the original pair (A, B) we can consider

$$A' = \begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} + X_{21}Y & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix}, \quad B' = \begin{bmatrix} B_1 & B_{12} - YB_{22} \\ 0 & B_{22} \\ 0 & 0 \end{bmatrix},$$

where (X_{22}, X_{21}) is exactly controllable and $(p-1)$-admissible pair, operator B_1 is invertible, and Y is an arbitrary (linear bounded) operator from M_2 to M_1. For

$$F = \begin{bmatrix} F_1 & F_2 & F_3 \\ 0 & 0 & 0 \end{bmatrix}$$

we have

$$A' + B'F = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ X_{21} & X_{22} + X_{21}Y & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix},$$

where we have denoted $Z_{1i} = X_{1i} + B_1F_i (i = 1, 2, 3)$. The choice of F_1, F_2, F_3 will be made later. Due to exact controllability of the pair (X_{22}, X_{21}) the operator Y can be chosen in such a way that operators X_{33} and $Z_{22} = X_{22} + X_{21}Y$ have disjoint spectra. In this case there exists an operator U such that $Z_{22}U - UX_{33} = X_{23}$ (see [**3**]). For this U the operator $A' + B'F$ is similar to

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & U \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ X_{21} & Z_{22} & X_{23} \\ 0 & 0 & X_{33} \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & I & -U \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} - Z_{12}U \\ X_{21} & Z_{22} & 0 \\ 0 & 0 & X_{33} \end{bmatrix}. \tag{5.9}$$

Let's now put

$$F_3 = B_1^{-1}(Z_{12}U - X_{13}) \tag{5.10}$$

Then $Z_{13} - Z_{12}U = 0$, and the matrix (5.9) can be re-written in a block diagonal form $\begin{bmatrix} Z_{11} & Z_{12} \\ X_{21} & Z_{22} \end{bmatrix} \oplus X_{33}$.

Now choose any polynomial φ_1 such that deg $\varphi_1 = p-1$ and φ is divisible by φ_1. This is possible due to the equality deg $\varphi = p$; moreover, the polynomial $h = \varphi/\varphi_1$ is nonconstant, and so there exists $\lambda_0 \in \mathbb{C}$ such that $h(\lambda_0) = 0$.

A pair (Z_{22}, X_{21}) is exactly controllable and $(p-1)$-admissible simultaneously with (X_{22}, X_{21}). According to the Theorem 5.1 there exists an operator V such that

$$\varphi_1(Z_{22} - X_{21}V) = 0. \tag{5.11}$$

Put

$$F_2 = B_1^{-1}(Z_{11}V + VX_{21}V - VZ_{22} - X_{12}). \tag{5.12}$$

Then

$$-Z_{11}V - VX_{21}V + Z_{12} + VZ_{22} = 0,$$

and

$$\begin{bmatrix} I & V \\ 0 & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ X_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I & -V \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} + VX_{21} & 0 \\ X_{21} & Z_{22} - X_{21}V \end{bmatrix}.$$

Finally, put

$$F_1 = B_1^{-1}(\lambda_0 I - VX_{21} - X_{11}). \tag{5.13}$$

Then the operator $A' + B'F$ is similar to $\left[\begin{smallmatrix} \lambda_0 I & 0 \\ X_{21} & Z_{22} - X_{21}V \end{smallmatrix}\right] \oplus X_{33}$, and, correspondingly, $\varphi(A' + B'F)$ is similar to

$$\varphi\left(\begin{bmatrix} \lambda_0 I & 0 \\ X_{21} & Z_{22} - X_{21}V \end{bmatrix}\right) \oplus \varphi(X_{33}).$$

The property (5.8) is equivalent to $\varphi(X_{33}) = 0$. It's also easy to prove using the functional calculus that for any function φ analytic on $\{\lambda_0\} \cup \sigma(W)$ (in particular, for any polynomial φ)

$$\varphi\begin{bmatrix} \lambda_0 I & 0 \\ X & W \end{bmatrix} = \begin{bmatrix} \varphi(\lambda_0)I & 0 \\ \tilde{\varphi}(W)X & \varphi(W) \end{bmatrix},$$

where $\tilde{\varphi}(z) = (\varphi(z) - \varphi(\lambda_0))(z - \lambda_0)^{-1}$. As $\tilde{\varphi}$ is divisible by φ_1, it follows from (5.11) that $\tilde{\varphi}(Z_{22} - X_{21}V) = \varphi(Z_{22} - X_{21}V) = 0$. Therefore

$$\varphi \begin{bmatrix} \lambda_0 I & 0 \\ X_{21} & Z_{22} - X_{21}V \end{bmatrix} = 0.$$

We have obtained that if F_1, F_2 and F_3 are chosen according to formulas (5.13), (5.12) and (5.10), then the operator $\varphi(A' + B'F)$ is similar to 0, and so it equals 0. ■

A special case of Theorem 5.4 deserves to be stated separately.

Corollary 5.5. *Let (A, B) be a p-admissible pair, and φ be a polynomial of the degree at least p such that $\mathrm{Im}\varphi(A) \subseteq \mathcal{C}_p(A, B)$. Then there exists an operator F such that $\varphi(A + BF) = 0$.*

6. An Example. Let $H^2(E)$ be the Hardy space of functions with values in a Hilbert space E which are analytic in the open unit disc $\mathcal{D} = \{\lambda : |z| < 1\}$. Further, let $A \in L(H^2(E))$ be the operator of multiplication by z, and B is the natural imbedding of E into $H^2(E)$. Then $\mathcal{C}_k(A, B)$ is a subspace of E-valued polynomials of degree strictly less than k, so the pair (A, B) is not admissible. It's easy to see that

$$\mathrm{Ker}(A - \lambda I) = 0 \text{ for all } \lambda \in \mathbf{C},$$

and

$$\mathrm{Im}(A - \lambda I) = \{f \in H^2(E) : g(z) = \frac{f(z)}{z - \lambda} \in H^2(E)\}.$$

Therefore, the operator $A - \lambda I$ is invertible for $|\lambda| > 1$,

$$\mathrm{Im}(A - \lambda I) = \{f \in H^2(E) : f(\lambda) = 0\} \text{ for } |\lambda| > 1,$$

and $\mathrm{Im}(A - \lambda I)$ is a dense non-closed subspace of $H^2(E)$ such that $E \cap \mathrm{Im}(A - \lambda I) = \{0\}$ for $|\lambda| = 1$. So,

$$\mathrm{Im}(A - \lambda I) + \mathrm{Im}\, B = \mathrm{Im}(A - \lambda I) + E = H^2(E) \text{ for } |\lambda| \neq 1,$$

and $\mathrm{Im}(A - \lambda I) + \mathrm{Im}\, B$ is not closed in $H^2(E)$ for $|\lambda| = 1$ (see, e.g., Theorem 2.4 in [8]). In other words, $\Sigma(A, B) = \{\lambda : |\lambda| = 1\}$.

Consider now an operator $A + BF - \lambda I$ with $F \in L(H^2(E), E)$ and $|\lambda| < 1$. Suppose $g \in E \cap \mathrm{Im}(A + BF - \lambda I)$. Then there exists $f \in H^2(E)$ such that

$$(z - \lambda)f(z) + Ff = g, \tag{6.1}$$

where Ff and g do not depend on z. Taking $z = \lambda$ in (6.1) we find that $Ff = g$, and, consequently, $(z - \lambda)f(z) = 0$. In other words, $f = 0$, and $g = 0$. So

$$E \cap \mathrm{Im}(A + BF - \lambda I) = \{0\},$$

and the operator $A+BF-\lambda$ is not right invertible for all $\lambda \in \mathcal{D}$. Therefore, for any choice of $F \in L(H^2(E), E)$ we have $\sigma(A + BF) \supseteq \mathcal{D}$, and so the equality $\sigma(A + BF) = \Sigma(A, B)$ never occurs.

The precise description of all possible $\sigma(A + BF)$ will be given in the case of $E = \mathbb{C}$, that is, dim Im $B = 1$.

In this case $F \in (H^2(\mathbb{C}), \mathbb{C})$ is actually a linear functional on $H^2(\mathbb{C})$, and so it can be represented as

$$(Ff)(z) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\vartheta})\overline{\varphi(e^{i\vartheta})}d\vartheta = (f, \varphi)$$

for a suitable choice of $\varphi \in H^2(\mathbb{C})$.

An operator $A + BF - \lambda I$ for $|\lambda| > 1$ is Fredholm with the index 0. Therefore, it is invertible if and only if $\mathrm{Ker}(A + BF - \lambda I) = \{0\}$. But Ker $(A + BF - \lambda I) = \{f : zf(z) - \lambda f(z) + (f, \varphi) \equiv 0\}$. So any $f \in$ Ker $(A+BF-\lambda I)$ has to be of a special form $f(z) = c(\lambda-z)^{-1}$ with $c = (f, \varphi)$. Substituting the expression for $f(z)$ we find that $c((\lambda - z)^{-1}, \varphi) = c$. That is, $c = 0$ or $((\lambda - z)^{-1}, \varphi) = 1$. A computation (using the Cauchy's theorem) gives

$$((\lambda - z)^{-1}, \varphi) = \overline{\lambda}^{-1}\varphi(\overline{\lambda}^{-1}) \quad (|\lambda| > 1).$$

So the subspace Ker $(A+BF-\lambda I)$ is nontrivial if and only if $\varphi(\overline{\lambda}^{-1}) = \overline{\lambda}$. Thus,

$$\sigma(A + BF) = \{\lambda : |\lambda| \leq 1\} \cup \{\lambda : h(\overline{\lambda}^{-1}) = 0\},$$

where $h(\zeta) = 1 - \zeta\varphi(\zeta)$. According to the well-known description of the zero sets of functions in $H^2(\mathbb{C})$ (see, e.g., [5]), the general form of $\sigma(A+BF)$ is

$$\sigma(A+BF) = \{\lambda : |\lambda| \le 1\} \cup \{z_k\},$$

where $\{z_k\}$ is void, finite, or countable set; in the latter case $\sum_k(|z_k|-1) < \infty$.

This situation is totally different from the behavior of $\sigma(A+BF)$ in the case of p-admissible pairs (Theorems 1.3 and 4.2).

References

1. J. B. Conway, *A Course in Functional Analysis*, Springer-Verlag, New York, etc. (1985).
2. J. B. Conway, *Every spectral picture is possible*, Notices of Amer. Math. Soc. **24, A431** (1977).
3. Ju. L. Daleckii, M. G. Krein, *Stability of Solutions of Differential Equations in Banach Space*, Amer. Math. Soc. Transl. Math. Monographs **43** (1974), Providence, RI.
4. R. G. Douglas, *On majorization, factorization and range inclusion of operators in Hilbert space*, Proc. Amer. Math. Soc. **17** (1966), 413-415.
5. P. L. Duren, *Theory of H^p-spaces*, Academic Press, New York, etc. (1970).
6. G. Eckstein, *Exact controllability and spectrum assignment*, Operator Theory: Advances and Applications **2** (1981), 81-94.
7. M. A. Kaashoek, C. V. M. van der Mee, L. Rodman, *Analytic operator functions with compact spectrum, III. Hilbert space case: inverse problem and applications*, J. of Operator Theory **10** (1983), 219-250.
8. S. G. Krein, *Linear Equations in Banach space*, Nauka, Moscow (1971), (in Russian).
9. C. J. Read, *Extending an operator from a Hilbert to a larger Hilbert space, so as to reduce its spectrum*, Israel J. of Math. **57** (1987), 375-380.
10. L. Rodman, *On exact controllability of operators*, Rocky Mountain J. of Math. **20** (1990), 549-560.
11. L. Rodman, *An introduction to Operator Polynomials*, OT38, Birkhäuser Verlag (1989).
12. H. H. Rosenbrok, *State Space and Multivariable Theory*, Thomas Nelson, London (1970).
13. K. Takahashi, *Exact controllability and spectrum assignment*, J. of Math. Anal. and Appl. **104** (1984), 537-545.
14. I. Zaballa, *Matrices with prescribed rows and invariant factors*, Linear Algebra and its Applications **87** (1987), 113-146.
15. I. Zaballa, *Interlacing and majorization in invariant factor assignment problems*, Linear Algebra and its Applications **121** (1989), 409-421.

Robotics Research Laboratory
Courant Institute of Mathematical Sciences
New York, New York 10012

Department of Mathematics
The College of William and Mary
Williamsburg, Virginia 23187-8795

Received May 31, 1991