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INVARIANT MEAN CHARACTERIZATIONS OF AMENABLE C*-ALGEBRAS

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Abstract. **It is shown that unital amenable and strongly amenable C*-algebras can be characterized by the existence of a right invariant mean on a certain subspace of $\ell_\infty(H)$, where H is the unitary group. A fixed-point theorem for amenable C*-algebras is obtained.**

1. Introduction. The following result of Haagerup ([**11**, Theorem 2.1]) is the main motivation for this paper. Let R be a von Neumann algebra with isometry semigroup S. Let $Bil^\sigma(R)$ be the space of bounded bilinear forms on R which are separately, σ-weakly continuous on R. Then *R is injective if and only if there exists a mean m on S such that for all $V \in Bil^\sigma(R)$ and all $a \in R$, we have*

$$\int_S V(av^*, v)dm(v) = \int_S V(v^*, va)dm(v). \tag{1}$$

Haagerup uses (1) in his proof that nuclear C^*-algebras are amenable. (Another proof which avoids (1) and the use of approximate finite dimensionality has been given by Effros ([**7**, **8**].)

Since injectivity and amenability are equivalent for R, it is natural to ask if (1) can be interpreted as asserting the existence of a suitably invariant mean on a subspace of $\ell_\infty(S)$ associated with $Bil^\sigma(R)$. A corresponding question, of course, can be asked for amenable unital C^*-algebras with the unitary group H in place of S. In both cases, the answer is positive, and this opens the way to interpreting operator algebra amenability in terms of a classical right invariant mean (RIM), replacing the more complex notion of *virtual diagonal* by the more accessible and better understood notion of *invariant mean.*

In this paper, we examine the C^*-case; the author plans to discuss the von Neumann case in another paper.

Let A be a unital C^*-algebra, and $Bil(A)$ be the Banach space of bounded bilinear forms on A. Let $Bil_{22}(A)$ be the subspace of completely bounded bilinear forms in $Bil(A)$. Recall that a C^*-algebra A is called *amenable* if there exists a virtual diagonal-for the definition, see (8) below-for A. This notion was introduced by Johnson ([**14**]); in his memoir [**15**], Johnson introduced the notion of a *strongly amenable* C^*-algebra, and Haagerup ([**11**]) has observed that this notion is characterized by the existence of a special kind of virtual diagonal. (See Proposition 4.) Our results can be interpreted as asserting that such virtual diagonals can be taken as arising from a RIM on spaces of functions on H.

We start by showing that amenability for A is associated with $Bil_{22}(A)$. We show in Proposition 2 that A is amenable if and only if there exists a virtual diagonal on $Bil_{22}(A)$. This is the analogue of the result of Effros ([**7, 8**]) that a von Neumann algebra R is amenable if and only if there exists a virtual diagonal on the subspace of completely bounded elements of $Bil^{\sigma}(R)$.

We then turn to the subspaces of $\ell_{\infty}(H)$ which support a RIM when A is amenable or strongly amenable. These spaces are quite simple to define. We define a map $\Delta : Bil(A) \to \ell_{\infty}(H)$ by

$$\Delta(V)(u) = V(u^*, u) \qquad (u \in H). \tag{2}$$

Let $B(A)$ be $\Delta(Bil(A)) \subset \ell_{\infty}(H)$. The subspace $B_{22}(A)$ of $B(A)$ is defined: $B_{22}(A) = \Delta(Bil_{22}(A))$. Both $B_{22}(A), B(A)$ are invariant and contain 1. The main result of this paper is the following (Theorem 1, Theorem 2):

(a) A is strongly amenable if and only if there exist a RIM on $B(A)$

(b) A is amenable if and only if there exists a RIM on $B_{22}(A)$

In the final part of the paper, we prove a fixed-point theorem for amenable C^*-algebras. One would expect such a theorem to exist in view of the well known fact in the theory of amenable groups that such theorems are associated with invariant means on subspaces of $\ell_{\infty}(G)$. Bunce ([**2, 3**]) proved such a theorem for strongly amenable C^*-algebras, and this easily follows by amenable group techniques using the invariant mean result (a) above. We prove a fixed-point theorem associated with (b) above, using the notion of *weakly completely bounded* A-modules. Here, a locally convex space E which is a unital A-module is called *weakly completely bounded* if, for every $F \in E^*$ and every $x \in E$, the bilinear map F_x, where

$$F_x(a, b) = F(axb)$$

is a completely bounded bilinear form on A. This result emphasizes a theme of the paper that amenability for C^*-algebras is a completely bounded phenomenon. (An elegant account of the theory of completely bounded maps is given in [**21**].)

2. Amenable C*-algebras and invariant means. Let A be a unital C^*-algebra. Then $Bil(A) = (A\hat{\otimes}A)^*$ is the Banach space of bounded bilinear forms on $A \times A$. The norm on $Bil(A)$ can also be given by:

$$\|V\| = \sup\{|V(a, b)| : a, b \in A, \|a\| = \|b\| = 1\}.$$

Let $Bil_{22}(A)$ be the subspace of completely bounded elements of $Bil(A)$. So a bilinear form V on $A \in Bil_{22}(A)$ if it is completely bounded as a bilinear map $V : A \times A \to \mathbf{C}$. (See, for example, [4].) For our purposes, such forms can be conveniently specified as follows. Let $(a, \xi) \to a\xi$ be the universal representation of A on its Hilbert space $\mathcal{H}$. Then (cf. [8]) $V \in Bil(A)$ is *completely bounded* if and only if there exist ξ, η in $\mathcal{H}$ and $T \in B(\mathcal{H})$ such that for all $a, b \in A$,

$$V(a, b) = aTb\xi \cdot \eta. \tag{3}$$

We note that such a representation of V has been extended to the non-scalar case by Christensen and Sinclair ([**4**])-an elegant account of this is given in [**22**]. We also note that there are subspaces $Bil_{ij}(A)$ for $i, j \in \{1, 2\}$ which arise naturally and are discussed in [**16**]. These do not play a role in the present paper but are significant in the von Neumann case.

We will require another characterization of completely bounded bilinear forms in the proof of Proposition 5. For $u \in A \otimes A$, define $\|u\|_{22} \geq 0$ as follows:

$$\|u\|_{22} = \inf\left\{\|\sum a_j a_j^*\|^{\frac{1}{2}}\|\sum b_j^* b_j\|^{\frac{1}{2}} : u = \sum a_j \otimes b_j\right\}. \tag{4}$$

In [**10**, **8**], the map $\|.\|_{22}$ is shown to be a norm on $A \otimes A$ and is called the *Haagerup norm.* It is also shown that *a bilinear form on* $A \times A$ *is completely bounded if and only if it is bounded on* $A \otimes A$ *for the Haagerup norm.* Recent accounts of the Haagerup norm and other operator space norms are given in [**1**, **9**].

Let $R = A^{**}$ be the enveloping von Neumann algebra of A realised on $\mathcal{H}$. It follows from [**13**, Theorem 2.3] that each $V \in Bil(A)$ extends uniquely, without change of norm to an element, also denoted V, of $Bil^{\sigma}(R)$. (The latter space is defined in the Introduction.) So we can identify $Bil(A)$ with $Bil^{\sigma}(R)$ and can identify $Bil_{22}(A)$ with the appropriate subspace of $Bil^{\sigma}(R)$. This subspace is denoted by $Bil^{\sigma}_{22}(R)$. The elements V of $Bil^{\sigma}_{22}(R)$ are also given by the formula (3) with a, b allowed to lie in R.

We recall that $Bil(A)$ is a dual Banach A-module with actions

$$xV(a,b) = V(a,bx) \qquad Vx(a,b) = V(xa,b). \tag{5}$$

Direct checking in (3) shows that $Bil_{22}(A)$ is an invariant subspace of $Bil(A)$. There is another useful module action $\circ$ which we postpone till later ((21)).

The next result seems to be well known, but for convenience we give the simple proof.

Proposition 1. *Let* $V \in Bil^{\sigma}_{22}(R)$. *Then the maps* $x \to Vx^*, x \to xV$ *are strong operator-norm continuous from* R *into* $Bil^{\sigma}(R)$.

Proof: If V is as in (3), then

$$\|Vx^* - Vy^*\| \le \|\xi\| \, \|T\| \, \|x\eta - y\eta\|, \tag{6}$$

$$\|xV - yV\| \le \|\eta\| \, \|T\| \, \|x\xi - y\xi\|. \tag{7}$$

The result now follows. □

We now discuss amenability for A. This involves the notion of a virtual diagonal for A. Let $\pi : A\hat{\otimes}A \to A$ be the multiplication map. An element M of $(A\hat{\otimes}A)^{**}$ is called a *virtual diagonal* if, for all $a \in A$:

$$aM = Ma \ \ (a \in A) \quad \pi^{**}(M) = 1. \tag{8}$$

The algebra A is called *amenable* if there exists a virtual diagonal for A.

The subspace $\pi^*(A^*)$ can easily be identified with A^* by associating $\pi^*(\phi) = V_\phi$ with ϕ, where

$$V_\phi(a, b) = \phi(ab).$$

It is simple to check that the natural A-module structure of A^* coincides with the submodule structure that it inherits as a subspace of $Bil(A)$, and that $A^* \subset Bil_{22}(A)$. Further, regarding $A \subset A^{**}$, the second equality of (8) becomes:

$$M|A^* = 1.$$

The first equality of (8) can be reformulated:

$$v^*Mv = M \quad (v \in H). \tag{9}$$

Indeed, (9) is equivalent to $Mv = vM$ for all $v \in H$, which in turn is equivalent to $aM = Ma$ for all $a \in A$ since H spans A.

Virtual diagonals for submodules of $Bil(A)$ containing A^* are defined in the obvious way.

There is a natural H-action on $Bil(A)$ associated with the module actions of (5) and (21). We define:

$$v.V(a, b) = V(v^*a, bv) \qquad V.v(a, b) = V(av^*, vb). \tag{10}$$

Clearly, $Bil(A)$ is a Banach H-module. Using (5), we have

$$v.V = vVv^*. \tag{11}$$

Since $Bil_{22}(A)$ is an A-submodule of $Bil(A)$, it follows that it is also an H-submodule.

Note also that for $\phi \in A^*$, we have $v.V_\phi = V_{v\phi v^*}$, and since $v^*v = 1$, we also have $V_\phi.v = V_\phi$. In particular, A^* is an H-submodule of $Bil(A)$. In the dual H-module action on A^{**}, where we regard $A \subset A^{**}$, we have $v.1 = 1 = 1.v$ for all $v \in H$.

The actions of (10) of course dualise to give an H-module action on $(Bil(A))^*$. These actions will be denoted by:

$$(v, M) \to v.M \qquad (M, v) \to M.v.$$

Note that, using (11):

$$M.v = v^*Mv. \tag{12}$$

The following proposition shows that for amenability for A, we require a virtual diagonal only on $Bil_{22}(A)$.

Proposition 2. *The C^*-algebra A is amenable if and only if there exists a virtual diagonal on $Bil_{22}(A)$.*

Proof: Suppose that there exists a virtual diagonal M on the space $Bil_{22}(A)$. Let G be the unitary group of R. Let $V \in Bil_{22}^{\sigma}(R)$. Let $v \in G$ and $\{u_\alpha\}$ be a net in H such that $u_\alpha \to v$ strongly in R ([**23**, Theorem 2.3.3]). Now since the strong and weak operator topologies coincide on G ([**26**, p. 84]), it follows that the map $u \to u^*$ is strong operator continuous, and using Proposition 1 and the triangular inequality, we have $\|u_\alpha^* V u_\alpha - v^* V v\| \to 0$. Hence

$$vMv^*(V) = \lim u_\alpha M u_\alpha^*(V) = M$$

and so identifying $Bil(A)$ with $Bil_{22}^{\sigma}(R)$, we see that M is a virtual diagonal on $Bil_{22}^{\sigma}(R)$. By a result of [**7**, **8**], R is amenable and so injective. So A is amenable (=nuclear) by the well-known result (due to Connes and Choi-Effros): *A is nuclear if and only if A^{**} is injective.*

The rest of the proof is trivial. □

We now discuss invariant means on groups. Let G be a group. Convolution on $\ell_1(G)$ dualises to give a G-action on $\ell_\infty(G)$:

$$(fs_0)(s) = f(s_0 s) \qquad (s_0 f)(s) = f(s s_0)$$

for all $s_0, s \in G$ and all $f \in \ell_\infty(G)$. A *right invariant mean* (RIM) on $\ell_\infty(G)$ is a mean (=state) on $\ell_\infty(G)$ which is right invariant under the right dual G-action on $(\ell_\infty(G))^*$. So a mean m on G is a RIM if and only if

$$m(sf) = m(f)$$

for all $f \in \ell_\infty(G)$ and all $s \in G$. The group G is called *right amenable* if there exists a RIM on $\ell_\infty(G)$. *Left amenability* and *two-sided amenability* for G are defined in the obvious ways. Recent accounts of amenability theory are given in [**19**, **24**, **25**].

A subspace X of $\ell_\infty(G)$ is called *left invariant* if $sf \in X$ for all $f \in X$ and all $s \in G$. If X is left invariant and contains 1, then a RIM on X is an element $m \in X^*$ satisfying $m(1) = 1 = \|m\|$ and $m(sf) = m(f)$ for all $f \in X$ and all $s \in G$. Similarly we can define left invariant means (LIM's) for right invariant unital subspaces of $\ell_\infty(G)$. We will be concerned with invariant means on subspaces of $\ell_\infty(H)$. Since H is so large and (usually) highly non-commutative, it is rarely going to be amenable, and we are interested in the existence of invariant means on certain smaller, though significant, subspaces of $\ell_\infty(H)$.

The subspaces $B_{22}(A)$ and $B(A)$ that will concern us are associated with the following map $\Delta : Bil^\sigma(A) \to \ell_\infty(S)$:

$$\Delta(V)(v) = V(v^*, v). \tag{13}$$

We define the following subspaces of $\ell_\infty(G)$:

$$B(A) = \Delta(Bil(A)) \qquad B_{22}(A) = \Delta(Bil_{22}(A)). \tag{14}$$

We give H the relative $\sigma(A, A^*)$ (i.e. the weak) topology. Then ([**20**]) H is a topological group. The invariant, unital C^*-algebra $LUC(H)$ (resp $RUC(H)$) is the set of functions $f \in \ell_\infty(H)$ such that the map $s \to sf$ (resp $s \to fs$) is norm continuous. Since $1 \in H$, each $f \in LUC(H)$ is continuous.

We now collect some simple facts relating to the spaces $B(A)$ and $B_{22}(A)$.

Proposition 3. *(a) The map Δ is an H-equivariant, norm decreasing, linear map onto $B(A)$. Further, the spaces $B(A), B_{22}(A)$ are invariant subspaces of $\ell_\infty(H)$, and $\Delta(A^*) = \mathbf{C}1$.*

- (b) *$\Delta^*(m)$ is a virtual diagonal for every RIM m on $B(A)$.*
- (c) *Both subspaces $B(A)$ and $B_{22}(A)$ are closed under the complex conjugation map $f \to \overline{f}$.*
- (d) *$1 \in B_{22}(A) \subset LUC(H)$.*

Proof: (a) For $V \in Bil(A)$, $u, v \in H$, we have

$$\Delta(V.v)(u) = V.v(u^*, u) = V(u^*v^*, vu) = \Delta(V)(vu) = \Delta(V)v(u)$$
$$\Delta(v.V)(u) = v.V(u^*, u) = V(v^*u^*, uv) = \Delta(V)(uv) = v\Delta(V)(u)$$

so that Δ is H-equivariant. Obviously, Δ is norm-decreasing and linear. Since Δ is equivariant and the spaces $Bil(A), Bil_{22}(A)$ are H-modules, it follows that $B(A)$ and $B_{22}(A)$ are invariant. Finally, if $\phi \in A^*$, then

$$\Delta(V_\phi)(v) = V_\phi(v^*, v) = \phi(v^*v) = \phi(1) \tag{15}$$

so that $\Delta(V_\phi) = \phi(1)1$.

(b) If m is a RIM on $B(A)$, then, for $v \in H$, $\phi \in A^*$, using (a), (12) and (15):

$$v^*\Delta^*(m)v = \Delta^*(m).v = \Delta^*(mv) = \Delta^*(m)$$
$$\Delta^*(m)(V_\phi) = m(\Delta(V_\phi)) = \phi(1) = 1(V_\phi).$$

So using (9), $\Delta^*(m)$ is a virtual diagonal.

(c) For $V \in Bil(A)$, define $V^* \in Bil(A)$ by:

$$V^*(a,b) = \overline{V(b^*, a^*)}.$$

Then $\overline{\Delta(V)} = \Delta(V^*)$, and $B(A)$ is closed under complex-conjugation. The same property holds for $B_{22}(A)$: we observe that the conjugate $\overline{f}$ of $f \in B_{22}(A)$ is obtained by replacing the T in (3) by its adjoint and interchanging ξ and η.

(d) Since $A^* \subset Bil_{22}(A)$, it follows from (a) that $1 \in B_{22}(A)$. If $V \in Bil_{22}(A)$, then for $u, v \in H$,

$$\|u\Delta(V) - v\Delta(V)\| \leq \|Vu^* - Vv^*\| + \|uV - vV\|. \tag{16}$$

Now $u_\delta \to u$ weakly in A if and only if $u_\delta \to u$ in the strong operator topology of $R = A^{**}$. It follows from (16) and Proposition 1 that $\Delta(V) \in LUC(H)$. □

The next result gives an invariant mean characterization of amenable C^*-algebras.

Theorem 1. *The following statements are equivalent:*

(a) *A is amenable*
(b) *there exists a RIM on $B_{22}(A)$*
(c) *there exist a RIM on $LUC(H)$*

Proof: The equivalence of (a) and (c) follows by [**20**]. Since $B_{22}(A) \subset LUC(H)$ by (d) of Proposition 3, we have that (c) implies (b). Now suppose that (b) holds and let $R = A^{**}$. Let m be a RIM on $B_{22}(A)$. By (3), each $f \in B_{22}(A)$ is of the form $f_{T\xi\eta}$, where:

$$f_{T\xi\eta}(u) = u^*Tu\xi.\eta. \tag{17}$$

For $g \in B_{22}(A)$, define $g^* \in \ell_\infty(H)$ by setting $g^*(u) = f(u^{-1})$, and let $Y = \{g^* : g \in B_{22}(A)\}$. Then m^*, where $m^*(g^*) = m(g)$, is a left invariant mean (LIM) on Y. Now H is strongly dense in the unitary group G of R, and G is a topological group in the strong operator topology ([**12**]). From (17), each $g = f_{T\xi\eta}$ extends uniquely by continuity to a continuous function g' on G-just allow u in (17) to belong to G. Let $Y' = \{g' : g \in Y\}$. Then $Y' \subset RUC(G)$: this is easily checked as in Proposition 1. (See also [**12**].) As in [**20**, Proposition 1], there exists an LIM on Y', and a result of de la Harpe (cf [**19**, p. 78]) gives that R is injective. Hence A is nuclear and so amenable. So (b) implies (a). □

We will show in Theorem 2 below that strong amenability for A is equivalent to the existence of a RIM on $B(A)$. For convenience, we write $\overline{co}S$ for the *weak** closure of the convex hull of a subset S of a Banach space dual X^*, and for any Banach space X, will regard $X \subset X^{**}$.

Recall that ([**15**]) the algebra A is called *strongly amenable* if, whenever X is a unital Banach A-module and $D : A \to X^*$ is a derivation, then there exists α_0 in $\overline{co}\{u^*D(u) : u \in H\}$ such that $D(a) = a\alpha_0 - \alpha_0 a$ for all $a \in A$. Haagerup ([**11**, Lemma 3.4 seq]) remarks that the following characterization of strong amenability holds.

Proposition 4. *The C^*-algebra A is strongly amenable if and only if there exists a virtual diagonal M in $\overline{co}\{u^* \otimes u : u \in H\}$.*

Theorem 2. *The C^*-algebra A is strongly amenable if and only if there exists a RIM on $B(A)$.*

Proof: Suppose that m is a RIM on $B(A)$. From (b) of Proposition 3, $\Delta^*(m)$ is a virtual diagonal for A. For $u \in G$, let $\hat{u} \in \ell_\infty(G)^*$ be given by: $\hat{u}(\phi) = \phi(u)$. It is easily checked that $\Delta^*(\hat{u}) = u^* \otimes u$. Since m is in $\overline{co}\{\hat{u} : u \in G\}$, it follows that $\Delta^*(m)$ is in $\overline{co}\{u^* \otimes u : u \in G\}$ in $(A\hat{\otimes}A)^*$. By Proposition 4, A is strongly amenable.

Conversely, suppose that A is strongly amenable, and let M be as in Proposition 4. Then there exists a net $\{f_\delta\}$ in $P(G)$ such that in the *weak** topology

$$\left(\sum_{u \in G} f_\delta(u)(u^* \otimes u)\right) \to M.$$

In particular, if $V \in Bil(A)$, then

$$(\sum f_\delta(u)\hat{u})(\Delta(V)) = (\sum f_\delta(u)(u^* \otimes u))(V) \tag{18}$$

$$\to M(V). \tag{19}$$

Define $m(\Delta(V)) = M(V)$. Then m is well-defined and is a mean on $B(A)$. Let $v \in G$. By (9) and (11), $M(v.V) = M(V)$. Further, by (a) of Proposition 3, $\Delta(v.V) = v\Delta(V)$.

It follows that m is a RIM. □

We conclude by discussing how some characterizations of amenable and strongly amenable C^*-algebras can be interpreted as fixed-point or extension theorems of classical amenability type. In particular, using modules with a certain completely bounded property, we will prove a fixed-point theorem for amenable C^*-algebras which fills a gap in the literature.

We begin with strongly amenable C^*-algebras for which the literature is more complete. In [**2**, **3**], Bunce gives six characterizations of strongly amenable C^*-algebras. An account of the results of Bunce is given in [**25**, Chapter 2]. Three of these can be interpreted as fixed-point theorems for the unitary group H analogous to the classic fixed-point theorem of Day. A fourth can be interpreted as a stronger version of a result in [**17**] which is valid for amenable Banach algebras. (See [**6**] for an elegant proof.) The remaining two give invariant extension characterizations. All of these characterizations can be readily proved using Theorem 2 and the approach of the fixed-point theorems for amenable groups ([**19**, (2.16) ff.]). We are particularly interested in the following fixed-point theorem of Bunce.

Theorem 3. *The C^*-algebra A is strongly amenable if and only if whenever X is a unital Banach A-module and S is weak*-closed convex subset of X^* such that $v^*Sv = S$ for all $v \in H$, then there exists $g \in S$ such that $v^*gv = g$ for all $v \in H$.*

Bunce gives two characterizations of amenable C^*-algebras. As in the strongly amenable case, one of these is of the Khelemskii-type and the other is an invariant extension result. Both have Banach algebra versions, the extension version appearing in [**18**]. We now discuss a fixed-point theorem for amenable C^*-algebras corresponding to Theorem 3.

Let E be a locally convex space which is a unital A-module. The module E is called *weakly completely bounded* if, for every $F \in E^*$ and every $x \in E$, the bilinear map F_x, where

$$F_x(a,b) = F(axb), \tag{20}$$

is a completely bounded bilinear form on A.

If X is a completely bounded normed A-module in the sense of ([**5**]), then X is weakly completely bounded.

We can make the Banach space $A\hat{\otimes}A$ into a unital Banach A-module with the actions $\circ$:

$$a \circ (b \otimes c) = b \otimes ac \quad (b \otimes c) \circ a = ba \otimes c. \tag{21}$$

These actions are discussed in [**2, 3**].

Proposition 5. *Let $E = Bil_{22}(A)$ with the relative weak*-topology which it inherits as a subspace of $(A\hat{\otimes}A)^*$. Then E is a weakly completely bounded A-submodule of $(A\hat{\otimes}A)^*$ under the dual actions $\circ$ for (21).*

Proof: The fact that $a \circ V \circ a' \in E$ for $a, a' \in A$ and $V \in E$ follows by expressing V in the form of (3) and checking that

$$a \circ V \circ a'(b \otimes c) = b(aTa')c\xi.\eta \tag{22}$$

which is also of the form of (3). So E is an A-module. Now the dual of E is just $(A\hat{\otimes}A)/E^{\perp}$. If F is the restriction of $(b \otimes c)$ to E, then using (22),

$$F_V(a \otimes a') = aTa'(c\xi).b^*\eta \tag{23}$$

which is also of the form of (3). Now let F be a general element of $A\hat{\otimes}A$. We can write

$$F = \sum b_i \otimes c_i \quad (\sum \|b_i\|\ \|c_i\| < \infty).$$

Let $u = \sum a_j \otimes a'_j \in A \otimes A$. Then using (23),

$$\begin{aligned}
|F_V(u)| &= |\sum_{i,j} a_j T a'_j c_i \xi . b_i^* \eta| \\
&\leq \|T\| \sum_i \sum_j \|a'_j c_i \xi\|\ \|a_j^* b_i^* \eta\| \\
&\leq \|T\| \sum_i (\sum_j \|a'_j c_i \xi\|^2)^{\frac{1}{2}} (\sum_j \|a_j^* b_i^* \eta\|^2)^{\frac{1}{2}} \\
&= \|T\| \sum_i ((\sum_j a'^*_j a'_j) c_i \xi . c_i \xi)^{\frac{1}{2}} ((\sum_j a_j a_j^*) b_i^* \eta . b_i^* \eta)^{\frac{1}{2}} \\
&\leq \|T\| \sum_i \| \sum_j a'^*_j a'_j \|^{\frac{1}{2}} \|c_i \xi\|\ \| \sum_j a_j a_j^* \|^{\frac{1}{2}} \|b_i^* \eta\|.
\end{aligned}$$

It follows that

$$|F_V(u)| \leq \|T\|(\sum_i \|b_i\| \, \|c_i\|)\|\xi\| \, \|\eta\| \, \|u\|_{22}.$$

Hence V is bounded for the Haagerup norm (4) and so is completely bounded. □

Theorem 4. *The C^*-algebra A is amenable if and only if H has the fixed-point property: whenever X is a weakly completely bounded A-module and S is a non-empty, compact, convex subset of X such that $v^*Sv = S$ for all $v \in H$, then there exists $h \in S$ such that $v^*hv = h$ for all $v \in H$.*

Proof: Suppose that A is amenable. Let X be a weakly completely bounded A-module and S be a non-empty compact, convex, invariant subset of X. Let $\alpha \in X^*$ and $g \in S$. Then in the notation of (20), $\alpha_g \in Bil_{22}(A)$. By Theorem 1, there exists a RIM m on $B_{22}(A)$. Since $\Delta(\alpha_g) \in B_{22}(A)$, we can define $h : X^* \to \mathbf{C}$ by:

$$h(\alpha) = \int_H \Delta(\alpha_g)dm = \int_H \alpha(u^*gu)dm(u).$$

Clearly, h is linear, and by approximating m by convex combinations of point masses, we can, using the invariance of S and regarding the elements of S as functionals on X^*, find a net $\{g_\delta\}$ in S such that $g_\delta \to h$ pointwise on X^*. Since S is weakly compact, it follows that $h \in S$. Now for $v \in H$,

$$v\Delta(\alpha_g)(u) = \Delta(\alpha_g)(uv) = (v\alpha v^*)(u^*gu)$$

so that

$$h(v\alpha v^*) = m(v\Delta(\alpha_g)) = m(\Delta(\alpha_g)) = h(\alpha).$$

Hence $v^*hv = h$ for all $v \in H$.

Conversely suppose that A has the fixed-point property of the theorem. The amenability of A will follow from Theorem 1 once we have shown that $B_{22}(A)$ has a RIM. For this purpose, we will use [**19**, Theorem (2.13)]. The latter asserts the existence of a RIM provided we can show that $B_{22}(A)$ is right introverted (defined below) and that for each $\phi \in B_{22}(A)$, there exists a constant function in the pointwise closure of the set

$$C_\phi = co\{\phi v : v \in H\}.$$

We will establish these two facts in turn.

Let m be a mean on H. Let $V \in Bil_{22}(A)$. We wish to define an element $V * m \in Bil_{22}(A)$ such that for $v \in H$, we have $V * \delta_v = V.v$ (as in (10)). Indeed, for $a, b \in A$, we have $bVa \in Bil^{\sigma}_{22}(R)$, and can thus define

$$V * m(a,b) = \int_H V(au^*, ub)dm(u). \tag{24}$$

It is obvious that $V * m \in Bil(A)$ and that $V * \delta_v = V.v$. In fact, by approximating m *weak** by convex combinations of elements δ_v, we see that if V satisfies (3), then $V * m$ satisfies (3) with T replaced by some ultraweak cluster point of the set $co\{v^*Tv : v \in H\}$ in $B(\mathcal{H})$. So $V * m \in Bil^{\sigma}_{22}(R)$.

A left invariant subspace Y of $\ell_\infty(H)$ is called *right introverted* ([**19**, (2.6)]) if for each $F \in \ell_\infty(H)$ and $\phi \in Y$, we have $\phi F \in Y$, where $\phi F(v) = F(v\phi)$. We claim that $Y = B_{22}(A)$ is right introverted. Indeed, if m and V are as above, then

$$\Delta(V)m(v) = m(v\Delta(V)) = \int_H v\Delta(V)(u)dm(u)$$
$$= \int_H V(v^*u^*, uv)dm(u) = V * m(v^*, v) = \Delta(V * m)(v).$$

Since $\ell_\infty(H)^*$ is spanned by means, it follows that $B_{22}(A)$ is right introverted.

We now turn to the second fact to be established. By Proposition 5, $Bil_{22}(A)$ is a weakly completely bounded A-module with the *weak**-topology and the action dual to that in (21). Note that as in (11), $V.v = v^* \circ V \circ v$. Let $V \in Bil_{22}(A)$. Let $S = \overline{co}\{v^* \circ V \circ v : v \in H\}$ in $(A\hat{\otimes}A)^*$. As in the preceding paragraph, S is a *weak**-compact convex subset of $Bil_{22}(A)$. Of course, $v^* \circ S \circ v = S$. By hypothesis, there exists $W \in S$ such that $W.v = W$ for all $v \in H$. Further there exists a net $\{g_\delta\}$ in $P(H)$ such that $V.g_\delta \to W$. Then

$$\Delta(V)g_\delta(u) = V.g_\delta(u^* \otimes u) \to W(u^*, u)$$

so that $\Delta(V)g_\delta \to \Delta(W)$ pointwise on H. Since

$$\Delta(W)(u) = W(u^*, u) = W.u(1,1) = W(1,1)$$

it follows that $\Delta(W)$ is a constant function.

This completes the proof. □

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