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SINGULAR PROJECTION NESTS IN UHF C*-ALGEBRAS

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Abstract. **A projection nest in a C*-algebra is said to be singular if the normaliser of its generated C*-algebra is trivial. Uncountably many pairwise nonconjugate singular homogeneous nests are identified (in a given UHF C*-algebra) all of which have the same binary relation invariant. The constructions are based upon (homogeneous) direct systems of upper triangular matrix algebras with rapidly increasing multiplicities and irregular embeddings.**

Introduction. The non-self-adjoint operator algebras that are most amenable to analysis are those for which there is an implicit coordinatisation. This is available if the algebra contains a maximal abelian subalgebra (masa) which is *regular* in the sense that its normaliser in the generated star algebra is generating for the star algebra. Consider [**MSS**], [**MS**], [**P1**] and [**P5**] for example. There is a wide class of norm-closed limit algebras which automatically contain such masas and which, as a consequence, have been studied successfully in recent years. These can be described as the inductive limits of finite-dimensional incidence algebras (CSL algebras) with respect to isometric embeddings that are so nice that they map matrix units to sums of matrix units. See, for example, [**B**], [**D**], [**HP**], [**P3**], [**P4**], [**PPW**], [**PW**], [**Po**], [**RP**], [**Th**], [**Th**], [**V**], ...

On the other hand there are not many (any?) studies that are concerned with nonregular non-self-adjoint subalgebras of C*-algebras and their classification. In the present paper we move in this direction and consider the most natural kinds of triangular subalgebras, namely nest subalgebras, with the most unnatural kinds of masas, namely singular masas - masas for which the normaliser is trivial. The algebras are all inductive limits of

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upper triangular matrix algebras with respect to unital star extendible (and therefore isometric) nest embeddings. For the algebras in section 3 the kth embedding

$$\phi_k : T_{n_k} \to T_{n_{k+1}}$$

is induced by a single $r_{k+1} \times r_{k+1}$ unitary matrix U_k, where r_{k+1} is the multiplicity of ϕ_k, which has the form

$$U_k = \begin{pmatrix} 0 & 0 & I \\ \beta_k & \alpha_k & 0 \\ -\alpha_k & \beta_k & 0 \end{pmatrix}$$

where α_k, β_k belong to [0,1] and where I is an identity matrix. The limit algebra, denoted $A(r_k, \alpha_k)$, is therefore determined by the two sequences (r_k) and (α_k). By choosing (r_k) to be very rapidly increasing (in fact r_k is at least 2^{2^k}) we create a context in which the local size of a matrix unit can be determined in terms of partial products of the scalars α_k and β_k.

The limit algebra $A(r_k, \alpha_k)$ is in fact a nest subalgebra of a UHF C*-algebra, namely the subalgebra of $\varinjlim(M_{n_k}, \phi_k)$, determined by the nest $\mathcal{N}(r_k, \alpha_k)$ which is the union of the images of the invariant projection nests of $T_{n_1}, T_{n_2}, \ldots$. We obtain necessary conditions and sufficient conditions for the associated nest subalgebras to be isometrically isomorphic, and show, in particular, that $A(r_k, \alpha)$ and $A(r_k, \beta)$ are not isomorphic if $\alpha \neq \beta$. These algebras all have the same essential support, Gelfand support and binary relation invariant, as defined at the end of section 3. An interesting new feature of the singular context is the need to consider approximately commuting diagrams (approximate interwinings in the terminology of Elliott [E]).

In all our examples the nest algebras A are *homogeneous* in the sense that pAp and qAq are isomorphic whenever p and q are interval projections with the same trace. This term was introduced in [HP] and our construction is reminiscent of the construction there of regular homogeneous nests. It seems to be an interesting open problem, in both the regular and singular cases, to determine which homogeneous nest subalgebras of UHF C*-algebras arise as direct limits of systems where all the embeddings are homogeneous. (See Section 2 below.)

The following terminology is used. A *nest* or projection nest, is a totally ordered family of projections. A nest is *regular* (resp. *singular*), in an approximately finite C*-algebra, if it generates an abelian C*-algebra which is regular (resp. singular). In fact we only consider maximal nests in UHF C*-algebras. We write T_n for the algebra of upper triangular $n \times n$ complex matrices. When considering the limit algebra $A = \varinjlim T_{n_k}$ we often denote the image of a matrix unit $e_{ij}^k \in T_{n_k}$ in the limit algebra by the same notation, e_{ij}^k.

We should emphasize that the singular masas constructed below are of the form $C = \overline{U_k C_k}$ where $C_1 \subset C_2 \subset \ldots$ is a chain of finite-dimensional abelian algebras with C_k a maximal abelian algebra in B_k, where $B_1 \subset B_2 \subset \ldots$ is a chain of finite-dimensional C*-algebras with dense union in the C*-algebra containing C. That such masas may fail to be regular is related to the fact that $N_{C_k}(B_k)$, the normalises of C_k in B_k, need not be contained in $N_{C_{k+1}}(B_{k+1})$. Such inclusions do prevail however in *all* the limit algebra contexts mentioned in the first paragraph above. We remark that this assumption is misleadingly omitted from the definition of the term *canonical masa* in [P2].

2. Homogeneous nest subalgebras. To construct limit algebras A which are triangular nest subalgebras of UHF C*-algebras, it is natural to consider direct limits $\varinjlim(T_{n_k}, \phi_k)$ where the embeddings ϕ_k are unital nest embeddings with C*-algebra extensions $\hat{\phi}_k : M_{n_k} \to M_{n_{k+1}}$. By a *nest embedding* $\phi : T_n \to T_{nr}$ we mean one for which

$$\phi(LatT_n) \subseteq LatT_{nr},$$

where $LatA$ denotes the invariant projection lattice. Under these circumstances we have a commuting diagram

$$\begin{array}{ccccccc}
 & \hat{\phi}_1 & & \hat{\phi}_2 & & & \\
M_{n_1} & \to & M_{n_2} & \to & \ldots & & B \\
\uparrow & & \uparrow & & & & \uparrow \\
 & \phi_1 & & \phi_2 & & & \\
T_{n_1} & \to & T_{n_2} & \to & \ldots & & A \\
\uparrow & & \uparrow & & & & \uparrow \\
 & \phi_1 & & \phi_2 & & & \\
\mathcal{N}_1 & \to & \mathcal{N}_2 & \to & \ldots & & \mathcal{N}
\end{array}$$

where $n_k | n_{k+1}$ for all k, $\mathcal{N}_k = LatT_{n_k}$, and where $\mathcal{N}$ is the countable projection nest in B which is the union of the images of the finite projection nests $\mathcal{N}_1, \mathcal{N}_2, \ldots$. Thus the Banach algebra limit algebra $A = \varinjlim(T_{n_k}, \phi_k)$ is (completely) isometrically isomorphic to a subalgebra of the UHF C*-algebra $B = \varinjlim(M_{n_k}, \hat{\phi}_k)$, and we may identify $\mathcal{N}$ and A as subsets of B.

It is easy to see that A is the nest subalgebra

$$\text{Alg}\, \mathcal{N} = \{a \in B : (1-p)ap = 0 \text{ for all } p \in \mathcal{N}\}.$$

Indeed, let $E_k(b) = \sum qbq$ where the sum extends over the atomic intervals of $\mathcal{N}_k$. Then the limit $\lim E_k(b)$ for b in B, defines a linear contractive projection E from B onto the C*-algebra $C = C^*(\mathcal{N})$. Also, let $\mathcal{U}_k$ be the upper triangular projection from B to Alg $\mathcal{N}_k$ given by

$$\mathcal{U}_k(b) = \sum_{\substack{q \neq q' \\ q \prec q'}} qbq' + E_k(b)$$

where the first sum, corresponding to the *strictly* upper triangular projection, $\mathcal{U}_k^+(b)$ say, extends over atomic intervals $q \prec q'$ in the finite nest $\mathcal{N}_k$ where $\prec$ is the usual order. Suppose now that $b \in$ Alg $\mathcal{N}$. Let $b_k \in M_{n_k}$ be a Cauchy sequence with limit b. Since $\mathcal{U}_\ell(b_k) \to b$ as $k \to \infty$, for each ℓ, we can choose a Cauchy sequence d_k with limit b where $d_k \in \mathcal{U}_k(B)$ for all k. But $\mathcal{U}_k^+(d_k) \in A$ for all k and $d_k - \mathcal{U}_k^+(d_k)$ converges to $E(b)$, which belongs to A, and so it follows that b belongs to A.

It is a simple exercise to show that the diagonal algebra C really is a masa in A.

Note that a unital C*-extendible nest embedding $\phi : T_n \to T_{nr}$ necessarily has the form

$$\phi((a_{ij})) = (a_{ij}U_{ij})$$

where $(a_{ij}U_{ij})$ is a block matrix and the U_{ij}, for $1 \le i \le j \le n$, are unitary matrices in M_r satisfying the cocycle condition $U_{ij}U_{jk} = U_{ik}$. We say that the embedding ϕ is *homogeneous* if $U_{i,i+1} = U_{j,j+1} = U$ say, for all i and j. It is routine to check that if A is the nest subalgebra determined by a direct system of C*-extendable homogeneous nest embeddings, then A is homogeneous in the sense specified in the introduction. In [HP] it was shown that there are uncountably many regular nest subalgebras of this type. In fact if we limit attention to embeddings $\phi : T_n \to T_{nr}$ which are based on the backward shift,

$$U = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & & \ddots & & \\ & & & & 0 & 1 \\ 1 & & & & & 0 \end{pmatrix}$$

then it was shown that $A = \varinjlim T_{n_k}$ and $A' = \varinjlim T_{m_k}$ are isometrically isomorphic if and only if the direct systems for A and A' are eventually *identical* (apart, of course, from a shift). Notice that the embeddings here map standard matrix units to sums of standard matrix units. It follows from this that the image, in the limit algebra, of each standard matrix unit belongs to the partial isometry normaliser of C, i.e. to

$$N_C(A) = \{v \in A : vCv^* \subseteq C, v^*Cv \subseteq C, v \text{ a partial isometry } \}$$

So in this case A is a regular nest subalgebra.

3. Singular nests. Let $r_k \geq 2$ be positive integers and let $n_1 = r_1$, $n_{k+1} = r_{k+1}n_k$, for $k = 1, 2, \ldots$. Let $A = A(r_k, \alpha_k) = \varinjlim(T_{n_k}, \phi_k)$ be the homogeneous nest subalgebra determined by the star extendible homogeneous unital embeddings $\phi_k : T_{n_k} \to T_{n_{k+1}}$ which are based on the unitaries U_k of section 1. Write $\mathcal{N}(r_k, \alpha_k)$, or $\mathcal{N}$, for the associated projection nest.

The nest $\mathcal{N}$ need not be singular. This might be due (loosely speaking) to either of the following possibilities. Firstly, the product $\alpha_k\beta_k = \alpha_k(1 - \alpha_k^2)^{\frac{1}{2}}$ may converge to zero so rapidly that asympotically the U_k behave like permutation matrices. In this case the resulting nest is regular. Secondly there may be algebraic reasons for the presence of a regular subsystem, as in the following example.

Let $\alpha_k = \beta_k = 1/\sqrt{2}$ and let $r_k = 4$ for all k. Then $n_k = 4^k$, for all k. Note that $U_k^{16} = I$. Let $S_k \subset T_{n_k}$ be the subalgebra spanned by the matrix units e_{ij} for which $j - i = 0 \pmod{16}$. The restriction of ϕ_k to S_k is simply the refinement embedding, that is, the simplest homogeneous nest embedding associated with the unitary $U = I$. It follows that the image in the limit algebra of the matrix units of each S_k normalise the masa C* ($\mathcal{N}$).

Lemma 3.1. *Let (r_k) satisfy the growth condition $r_{k+1} \geq 2r_1r_2 \dots r_k$ for all k. Then the projection nest $\mathcal{N}(r_k, \alpha_k)$ is singular for all sequences (α_k) for which $\alpha_k\beta_k$ does not converge to zero as $k \to \infty$.*

Proof: Suppose that the nest is not singular and v belongs to $N_C(A)$ but not to C. Let $0 < \epsilon < 1$ and express v as $w + r$ where $\|r\| < \epsilon$ and $w \in A_k$, where A_k is the image of the kth triangular matrix algebra T_{n_k} in the limit algebra A. If p, q are minimal projections in $C_k = A_k \cap (A_k)^*$ then $pvq \in N_C(A)$. Thus we can find a nonzero partial isometry v in the normaliser with the form $v = \lambda e_{ij}^k + r$ where $\|r\| < \epsilon$, as before, λ is a complex scalar, and e_{ij}^k is a non-diagonal matrix unit in A_k. Clearly $1 + \epsilon > |\lambda| > 1 - \epsilon$. In A_{k+1} the matrix $\phi_k(e_{ij}^k)$ has the form of a block operator matrix where U_k^{j-i} appears in the i, j block and where all other entries are zero. Also, because of the hypothesis on (r_k) it follows that any given row of $\phi_k(e_{ij}^k)$ contains either a single nonzero entry (unity), or two nonzero entries (β_k and $\pm\alpha_k$). Let p be a minimal projection in C_{k+1} corresponding to the latter case. Then

$$v^*pv = |\lambda|^2\phi_k(e_{ij}^k)^*p\phi_k(e_{ij}^k) + r_1$$

where $\|r_1\| < 2\epsilon(1+\epsilon)+\epsilon^2$. But $\phi_k(e_{ij}^k)^*p\phi_k(e_{ij}^k)$ is a block diagonal matrix with a single nonzero 2×2 entry whose off-diagonal entry is either $+\alpha_k\beta_k$ or $-\alpha_k\beta_k$. Since $v^*pv \in C$ it follows that $\alpha_k\beta_k \leq \|r_1\|$. Looking at further images of e_{ij}^k in $A_{k+1}, A_{k+2}, \dots$ we can similarly obtain non-trivial elements of the normaliser of the form $\lambda e_{ij}^\ell + r$ with $\|r_\ell\| < \epsilon$, for $\ell = k+1, k+2, \dots$, from which it follows that $\alpha_\ell\beta_\ell < 2\epsilon(1+\epsilon) + \epsilon^2$ for all $\ell \geq k$.

Lemma 3.2. *Let $A = \varinjlim(T_{n_k}, \phi_k)$, $A' = \varinjlim(T_{m_k}, \psi_k)$, where ϕ_k, ψ_k are unital star extendible nest embeddings, and let $\theta : A \to A'$ be an isometric algebra isomorphism. Then θ induces a bijection $\mathcal{N} \to \mathcal{N}'$ between the associated projection nests. Furthermore this is the unique bijection which preserves the normalised traces on $C^*(A)$, $C^*(A')$.*

Proof: θ maps projections to projections and therefore $\theta(C) = C'$ where $C = A \cap A^*$, $C' = A' \cap (A')^*$. Furthermore, $\mathcal{N} = Lat A$ and $\mathcal{N}' = Lat A'$, so that $\theta(\mathcal{N}) = \mathcal{N}'$. (The inclusion $\mathcal{N} \subset Lat A$ is immediate from the definition of A, whilst if p is a projection in A which is invariant for A then $p \in A_k$, and is invariant for A_k, for some k, and so $p \in \mathcal{N}_k$).

Let $v \in A$ be the image in A of a standard matrix unit of T_{n_k}, for some k, so that the projections $p = vv^*$, $q = v^*v$ are interval projections with the same trace. We have $\|vqa\| = \|qa\|$, $\|apv\| = \|ap\|$ for all $a \in A$. Thus $\|\theta(v)q'a'\| = \|q'a'\|$ and $\|a'p'\theta(v)\| = \|a'p'\|$ for all $a' \in A'$, where $q' = \theta(q), p' = \theta(p)$ are interval projections in $\mathcal{N}'$. Choose k large so that $p', q' \in C_k$, the diagonal subalgebra of T_{n_k}, and so that there is an element $b \in T_{n_k}$ with $\|b - \theta(v)\| < \frac{1}{3}$.

Then

$$\|bq'b'\| \geq \frac{1}{2}\|q'b'\|,$$
$$\|b'p'b\| \geq \frac{1}{2}\|b'p'\|,$$

for all b' in T_{n_k}. It is elementary to check that this can only occur if p', q' have the same rank in T_{n_k}, and so p', q' have the same normalised trace. It now follows that p and $\theta(p)$ have the same normalised trace for all p in $\mathcal{N}$.

For a nest subalgebra $A = \varinjlim(T_{n_k}, \phi_k)$ of a UHF C*-algebra B, determined by star extendible unital injections, we define the *essential supports* $es(a)$, $a \in A$, and $es(A)$ as follows. Let tr denote the normalised trace on B. The set $es(a)$ consists of the points (s,t) in $(0,1) \times (0,1)$ for which there exists $\delta > 0$ such that

$$\|(p_1 - p_2)a(p_3 - p_4)\| \geq \delta \text{ whenever } tr(p_2) < s < tr(p_1), tr(p_4) < t < tr(p_3),$$

where p_1, p_2, p_3, p_4 are nest projections, together with the points (s,t) on the boundary of the unit square satifying the analogous one-sided conditions. The set $es(A)$ is the union of the sets $es(a)$ for all $a \in A$. By the last lemma $es(A)$ is an isometric isomorphism invariant and so it can be used to distinguish algebras in certain families of regular nest subalgebras. (However, we remark that it is far from being a complete isomorphism invariant even in the case of regular nest subalgebras.) One can check that $es(A(r_k, \alpha_k))$ is the union of the sets $es(e_{ij}^k)$ over all the matrix units and so the essential support can be calculated (in principle). It is also easy to see that $es(A(r_k, \alpha_k)) = es(A(r_k, \alpha_k'))$ if, for some $\delta > 0$, $\delta \leq \alpha_k, \alpha_k' \leq 1 - \delta$ for all k.

In an exactly analogous way one can define a finer invariant, the *Gelfand support* $gs(a)$, for $a \in A$, to be the set of points $(x, y) \in M(C) \times$

$M(C)$ (where $C = A \cap A^*$) such that there exists $\delta > 0$ such that $\|paq\| \geq \delta$ whenever $p, q \in$ Proj (C), and $x(p) = y(q) = 1$. It is elementary to show that if $\theta : A \to A'$ is an isometric isomorphism then for each matrix unit e_{ij}^k in A, the set $gs(\theta(e_{ij}^k))$ is contained in $gs(A_j')$ for some j. (A similar assertion holds for the essential supports.) This connection can be used to obtain the classification results of sections 3,4 of [HP]. In a similar way it can be shown that if $A(r_k, \alpha_k)$ and $A(r_k', \alpha_k')$ are isometrically isomorphic then there exists an integer ℓ such that $n_{k+\ell} = n_k'$ for all large k, where $n_k = r_1 r_2 \ldots r_k$, $n_k' = r_1' r_2' \ldots r_k'$. For this reason in the next section we limit attention to the classes of algebras $A(r_k, \alpha_k)$ with r_k a fixed rapidly increasing sequence.

In the case of regular embeddings the fundamental relation, or semigroupoid, $R(A)$, as given in [P2], can be recovered from consideration of the Gelfand support. As a set $R(A) = gs(A)$. Let $gs(A)$ have the topology generated by the sets $E(a, \delta)$, for $a \in A$, $\delta > 0$ where

$$E(a, \delta) = \{(x, y) : \|paq\| \geq \delta \quad \forall p, q \in \text{ Proj } (C) \text{ with } x(p) = y(q) = 1\}.$$

Then it can be shown that $R(A)$ is isomorphic to $gs(A)$ as a topological binary relation.

In general, for singular and nonregular nests, the topological binary relation $gs(A)$ is still an isomorphism invariant, which, although incomplete, may nevertheless serve as a basis for more discriminating invariants.

4. Approximately commuting diagrams and the Algebras $\mathbf{A}(\mathbf{r}_\mathbf{k}, \alpha_\mathbf{k})$. Let r_k be rapidly increasing, as in Lemma 3.1, and let

$$A = A(r_k, \alpha_k) = \varinjlim (A_k, \phi_k),$$
$$A' = A(r_k, \alpha_k') = \varinjlim (A_k', \phi_k'),$$

where $A_k = A_k' = T_{n_k}$ and where ϕ_k, ϕ_k' are the homogeneous embeddings associated, as in the last section, with the unitary scalar matrices

$$U_k = \begin{pmatrix} \beta_k & \alpha_k \\ -\alpha_k & \beta_k \end{pmatrix}, \quad U_k' = \begin{pmatrix} \beta_k' & \alpha_k' \\ -\alpha_k' & \beta_k' \end{pmatrix}$$

respectively.

Lemma 4.1. *For integers $s > t > 0$, let $X_{s,t} = \phi_s \circ \cdots \circ \phi_t(e^t_{1,2})$ and let $X'_{s,t}$ be the corresponding matrix in $T_{n_{s+1}}$ for the maps ϕ'_k. If $\|X_{s,t} - X'_{s,t}\| \to 0$ as $s, t \to \infty$ then A and A' are isometrically isomorphic algebras.*

Proof: Consider the noncommuting diagram

$$\begin{array}{ccccccc} & & \phi_1 & & & \phi_2 & \\ & A_1 & \to & & A_2 & \to & \dots \\ \theta_1 & \downarrow & & \theta_2 & \uparrow & & \\ & & \phi'_1 & & & \phi'_2 & \\ & A'_1 & \to & & A'_2 & \to & \dots \end{array}$$

where each θ_k is the identity map. Let $\mu_{k,n} : A_k \to A'$ be the map arising from the composition

$$A_k \to A_{k+n} \to A'_{k+n} \to A' \, .$$

Then, by the hypothesis, $\mu_{k,n}(e^k_{1,2})$ is a Cauchy sequence in n, with limit equal to the image of $e^k_{1,2}$ under the map $A'_k \to A$. Since the maps ϕ_k, ϕ'_k are homogeneous for all k, it follows that $\mu_{k,n}(a)$ is a Cauchy sequence in n for all a in A_k. It follows that the pointwise limits $\mu_k = \lim_n \mu_{k,n}$, for $k = 1, 2, \dots$, gives maps $A_k \to A'$, which extend each other, and so define an isometric homomorphism $A \to A'$. Similarly we obtain a map $A' \to A$ which is the inverse to this map, and the proposition follows.

Lemma 4.2. *If $\sum_{k=1}^{\infty}(|\alpha_k - \alpha'_k| + |\beta_k - \beta'_k|)$ is finite, then $\|X_{s,t} - X'_{s,t}\| \to 0$ as $s, t \to \infty$.*

Proof: Recall that the map $\phi_k : A_k \to A_{k+1}$ has the form $\phi_k((a_{ij})) = (a_{ij} U_k^{j-i})$, and so $\phi_k(a) = \tilde{U}_k^* \rho_k(a) \tilde{U}_k$ where $\rho_k : A_k \to A_{k+1}$ is the map $(a_{ij}) \to (a_{ij} I_k)$, where I_k has multiplicity r_{k+1}, and where

$$\tilde{U}_k = \begin{pmatrix} I_k & 0 & 0 & \dots & \\ 0 & U_k & 0 & & \\ 0 & 0 & U_k^2 & & \\ \vdots & & \ddots & & \\ & & & & U_k^{n_k - 1} \end{pmatrix}.$$

Write $\rho(a)$ as $a \otimes I_k$. Then

$$\begin{aligned}\phi_{k+1} \circ \phi_k(a) &= \tilde{U}_{k+1}^*((\tilde{U}_k^*(a \otimes I_k)\tilde{U}_k) \otimes I_{k+1})\tilde{U}_{k+1} \\ &= [(\tilde{U}_k \otimes I_{k+1})\tilde{U}_{k+1}]^*(a \otimes I_k \otimes I_{k+1})[(U_k \otimes I_{k+1})\tilde{U}_{k+1}]\end{aligned}$$

Iterating this we see that

$$\|\phi_s \circ \cdots \circ \phi_t(e_{12}^t) - \phi_s' \circ \cdots \circ \phi_t'(e_{12}^t)\| = \|U^*(e_{12}^t \otimes I)U - (U')^*(e_{12}^t \otimes I)U'\|$$

where

$$U = (\tilde{U}_t \otimes I_{t+1} \otimes I_{t+2} \otimes \cdots \otimes I_s)(\tilde{U}_{t+1} \otimes I_{t+2} \otimes \cdots \otimes I_s) \dots (\tilde{U}_{s-1} \otimes I_s)\tilde{U}_s$$

and where U' is the corresponding unitary for $U_t', \dots, U_s'$. Write these products as $U = \hat{U}_t\hat{U}_{t+1} \dots \hat{U}_s$, $U' = \hat{U}_t'\hat{U}_{t+1}' \dots \hat{U}_s$. Because of the rapid increase of r_k we see that for each $j = 1, \dots, n_k - 1$ the unitary U_k^j has entries which are either $1, \alpha_k, -\alpha_k$, or β_k. It follows that $\|\hat{U}_k - \hat{U}_k'\| = \|\tilde{U}_k - \tilde{U}_k'\| \leq 2(|\alpha_k - \alpha_k'| + |\beta_k - \beta_k'|)$.

By repeated use of the inequality

$$\|X^*YX - (X')^*ZX'\| \leq 2\|X - X'\| + \|Y - Z\|,$$

for contractions X, Y, Z, we obtain

$$\begin{aligned}\|U^*(e_{12}^t \otimes I)U - (U')^*(e_{12}^t \otimes I)U'\| &\leq 2\sum_{k=t}^{s} \|\hat{U}_k - \hat{U}_k'\| \\ &\leq 4\sum_{k=t}^{s}(|\alpha_k - \alpha_k'| + |\beta_k - \beta_k'|),\end{aligned}$$

and so the lemma follows.

Corollary 4.3. *If $\sum_{k=1}^{\infty}(|\alpha_k - \alpha_k'| + |\beta_k - \beta_k'|)$ is finite then $A(r_k, \alpha_k)$ and $A(r_k, \alpha_k')$ are isometrically isomorphic algebras.*

Lemma 4.4. *Let θ be an isometric isomorphism between A and A'. Then for each $\epsilon > 0$ there exists an integer ℓ such that for all $t > \ell$ there exists a matrix unit $e_{u,v}^t$ in T_{n_t} such that if a and a' are its images in A and*

A' respectively (under the natural maps $A_t \to A, A'_t \to A'$) then $-\epsilon < \|paq\| - \|\theta(p)a'\theta(p)\| < +\epsilon$ for all projections p, q in A.

Proof: Choose ℓ large so that

$$\theta(e^1_{12}) = \sum_{i,j=1}^{n_\ell} \lambda_{i,j} f^\ell_{i,j} + b$$

where $\|b\| < \epsilon$, the $\lambda_{i,j}$ are scalars, and where $\{f^\ell_{i,j} : 1 \leq i \leq j \leq n_\ell\}$ is the matrix unit system for A_ℓ. Because of the rapid increase of r_k there are minimal projections p_1, q_1 in T_{n_ℓ} such that $p_1 e^1_{12} q_1 = e^\ell_{u,v}$ for some u, v. (To see this repeatedly apply the fact that the image of any matrix unit e^k_{ij} under ϕ_k is a sum of matrix units, and some coefficients in this sum are unity.) By Lemma 3.2, θ restricts to a trace preserving bijection $\mathcal{N} \to \mathcal{N}'$ between the projection nests of A and A'. It follows that

$$\theta(e^\ell_{u,v}) = \theta(p_1 e^1_{12} q_1) = \lambda f^\ell_{u,v} + b_1$$

where $\lambda = \lambda_{u,v}$, and $b_1 = \theta(p_1) b \theta(q_1)$ has norm less than ϵ. Since θ is an isometry we have $1 - \epsilon < |\lambda| < 1 + \epsilon$. Thus if p, q are arbitrary projections of A, then

$$\theta(p e^\ell_{u,v} q) = \lambda \theta(p) f^\ell_{u,v} \theta(q) + b_2$$

where $\|b_2\| < \epsilon$, and so, with $a = e^\ell_{u,v}, a' = f^\ell_{u,v}$ it follows that

$$-2\epsilon \leq \|paq\| - \|\theta(p)a'\theta(q)\| \leq 2\epsilon.$$

The argument can be repeated for any $t > \ell$, and so the lemma follows.

Theorem 4.5. *Let α, α' belong to [0,1] and let $\alpha_k = \alpha, \alpha'_k = \alpha'$ for all k. Then $A(r_k, \alpha_k)$ and $A(r_k, \alpha'_k)$ are isometrically isomorphic if and only if $\alpha = \alpha'$. Furthermore $A(r_k, \alpha_k)$ is a singular nest algebra for all α in (0,1).*

Proof: If $e^t_{u,v}$ is a matrix unit in $T_{n_t} = A_t \subset A$, then for each $k \geq t$ there exist minimal projections p, q in $T_{n_{k+1}}$ such that $\|p(e^t_{u,v})q\| = \alpha_k$. It follows from the last lemma that if $A(r_k, \alpha_k)$ and $A(r_k, \alpha'_k)$ are isometrically isomorphic, then $\alpha'_k - \alpha_k \to 0$ as $k \to \infty$. The theorem now follows from this together with Lemma 3.1.

Problems 1. For fixed rapidly increasing r_k classify the conjugacy classes of the nests $\mathcal{N}(r_k, \alpha_k)$. We conjecture that the sufficient condition of Lemma 4.1 is also necessary. (Is this matrix condition on the α_k, α'_k equivalent to the summation condition of Corollary 4.3?)

2. Classify the singular masas $C^*(\mathcal{N}(r_k, \alpha_k))$ up to conjugacy by star automorphisms.

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