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RECURSIVENESS, POSITIVITY, AND TRUNCATED MOMENT PROBLEMS

RAÚL E. CURTO
LAWRENCE A. FIALKOW

Abstract. **Using elementary techniques from linear algebra, we describe a recursive model for singular positive Hankel matrices. We then use this model to obtain necessary and sufficient conditions for existence or uniqueness of positive Borel measures which solve the truncated moment problems of Hamburger, Hausdorff and Stieltjes. We also present analogous results concerning Toeplitz matrices and the truncated trigonometric moment problem.**

Dedicated to the memory of Domingo A. Herrero

1. Introduction.

Given an infinite sequence of complex numbers $\gamma = \{\gamma_0, \gamma_1, \dots\}$ and a subset $K \subseteq \mathbb{C}$, the *K-power moment problem with data* γ entails finding a positive Borel measure μ on $\mathbb{C}$ such that

$$\int t^j d\mu(t) = \gamma_j \ (j \geq 0) \tag{1.1}$$

and

$$\text{supp } \mu \subseteq K.$$

The classical theorems of Stieltjes, Toeplitz, Hamburger and Hausdorff provide necessary and sufficient conditions for the solubility of (1.1) in case $K = [0, +\infty), K = \mathbf{T} := \{t \in \mathbb{C} : |t| = 1\}, K = \mathbb{R}$ and $K = [a, b] (a, b \in \mathbb{R})$, respectively ([**AhK**], [**Akh**], [**KrN**], [**Lan**], [**Sar**], [**ShT**]). For example, when $K = \mathbb{R}$, Hamburger's Theorem implies that (1.1) is soluble if and

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only if the Hankel matrices $A(n) := (\gamma_{i+j})_{i,j=0}^{n} (n = 0, 1, \dots)$ are positive semidefinite [**ShT**, Theorem I.1.2].

The classical theory also provides uniqueness theorems and parameterizations of the sets of solutions. Parts of the theory have been extended to cover more general support sets for μ ([**BeM**], [**Cas**], [**Sch**]), and the "multidimensional moment problem" has also been studied extensively ([**Atz**], [**Ber**], [**Fug**], [**Put**]).

For $0 \le m < \infty$, let $\gamma = (\gamma_0, \dots, \gamma_m) \in \mathbb{C}^{m+1}$, and consider the *truncated K-power moment problem*

$$\int t^j d\mu(t) = \gamma_j \ (0 \le j \le m). \tag{1.2}$$

and

$$\operatorname{supp} \mu \subseteq K.$$

For example, the "even" case of the truncated Hamburger problem corresponds to $m = 2k$ for some $k \ge 0$ and $K = \mathbb{R}$. On the basis of Hamburger's Theorem, one might surmise that the criterion for solubility of (1.2) in this case is $A(k) \ge 0$. As we will see in the sequel, this positivity condition is always *necessary*, and is *sufficient* when $A(k)$ is nonsingular [**AhK**, Theorem I.3]; however, when $A(k)$ is singular, the positivity of $A(k)$ is *not* sufficient. In particular, the classical theory for the "full" moment problem does *not* include the theory of the truncated moment problem as a special case. On the other hand, it is known that the theory of the truncated moment problem *can* be used to solve the full moment problem and to recover such results as Hamburger's Theorem [**Lan**, p. 5], [**Akh**].

Observe that the truncated Stieltjes moment problem differs slightly from the *classical* truncated Stieltjes moment problem for $\gamma = (\gamma_0, \dots, \gamma_m)$, defined by:

$$\int_0^\infty t^j d\mu(t) = \gamma_j \quad (0 \le j \le m-1)$$

and

$$\int_0^\infty t^m d\mu(t) \le \gamma_m. \tag{1.3}$$

For purposes of parameterizing the solution spaces, the classical problem has proved more amenable than the problem we consider. However, for certain basic interpolation problems in operator theory (discussed below), it is *necessary* to consider the Stieltjes moment problem of (1.2), i.e., where the inequality in (1.3) is replaced by equality. Despite the importance of truncated moment problems, their treatment in the literature seems somewhat fragmentary. Most sources ignore the "odd" case entirely, although for us this case is fundamental. The usual approach (e.g. [**Ioh**, Theorem A.II.1], [**ShT**, p. 28 ff.]) is to treat the "nonsingular even" case of Hamburger's problem and then reduce the "singular even" case to the nonsingular case by means of quasi-orthogonal polynomials and Gaussian quadrature.

In the present note we present a comprehensive and unified treatment of necessary and sufficient conditions for the solubility of (1.2) in case $K = \mathbb{R}, [a,b], [0,+\infty)$ or $\mathbf{T}$. Our approach revolves around the notion of "recursiveness" for positive Hankel and Toeplitz matrices (Theorems 2.4 and 6.4). This key ingredient, apparently neglected in the literature, allows us to recover many classical results and to obtain many new ones. Our basic result, Theorem 3.1, provides simple operator-theoretic necessary and sufficient conditions for solubility of the "odd" case of the truncated Hamburger moment problem. Our solution is based on elementary techniques concerning nonnegative polynomials, Lagrange interpolation, and linear algebra (Proposition 3.3); for the "singualr odd" case, the solution rests on a description of positive *singular* Hankel matrices. Theorem 2.4 shows that each entry of such a matrix (except perhaps the lower right-hand entry!) is recursively determined by the entries in the largest nonsingular upper left-hand "corner" of the matrix. Theorem 2.4 immediately allows us to reduce singular Hamburger moment problems to equivalent nonsingular ones without recourse to quasi-orthogonal polynomials or Gaussian quadrature (Corollary 3.4). The "even" cases of Hamburger's problem are then treated as consequences of the "odd" cases (Theorem 3.9). Theorems 3.8 and 3.10 contain our uniqueness criteria for the truncated Hamburger problem. Solutions of the Hausdorff and Stieltjes moment problems are corollaries of this approach and are presented in Sections 4 and 5, respectively. In Section 6 we treat the *trigonometric* moment problem ($K = \mathbf{T}$) through a corresponding (but independent) analysis of positive *Toeplitz* matrices.

The present work supplements [**CuF**], where, as part of a detailed study of k-hyponormality and quadratic hyponormality for unilateral weighted shifts, we solved the following "subnormal completion problem" of J. G. Stampfli [**Sta**]: When may a finite sequence of positive numbers, $\gamma_0, \ldots, \gamma_m$, be "completed" to a sequence $\gamma = \{\gamma_n\}_{n=0}^{\infty}$ which is the moment sequence of a *subnormal* unilateral weighted shift W_γ ? In [**CF**] the subnormality of W_γ was established using Berger's Theorem [**Con**]: We essentially solved the truncated Stieltjes moment problem with a measure produced from the spectral measure of a normal operator. Our motivation for the present work was the desire to develop a more transparent method of solving such truncated moment problems, without recourse to such analytic tools as the spectral theorem or quasi-orthogonal polynomials: our new treatment satisfies this requirement. In particular, Section 3 concludes with a simple computational algorithm for solving the truncated Hamburger problem.

2. Recursive Models for Positive Hankel Matrices. Our goal in this section is to prove a structure theorem for positive semi-definite Hankel matrices. If A is a Hankel matrix of size $k+1$, let $\gamma = (\gamma_0, \dots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$ represent the entries of A, in the sense that $A = A_\gamma := (\gamma_{i+j})_{i,j=0}^{k}$. The $j-th$ column of A will be denoted by $\mathbf{v}_j := (\gamma_{j+\ell})_{\ell=0}^{k}$, $0 \le j \le k$, so that A can be briefly written as $(\mathbf{v}_0 \cdots \mathbf{v}_k)$. More generally, let $\mathbf{v}(i,j) := (\gamma_{i+\ell})_{\ell=0}^{j}$; observe that $\mathbf{v}_j = \mathbf{v}(j,k)$. Since we'll be mainly interested in positive semi-definite Hankel matrices, *we shall always assume that* $\gamma_0 > 0$. The (*Hankel*) *rank* of γ, denoted rank γ, is now defined as follows: If A is nonsingular, rank $\gamma := k+1$; if A is singular, rank γ is the smallest integer $i, 1 \le i \le k$, such that $\mathbf{v}_i \in$ span $(\mathbf{v}_0, \dots, \mathbf{v}_{i-1})$. Thus, if A is singular, i.e., rank $\gamma \le k$, there exists a unique $\mathbf{\Phi} := \mathbf{\Phi}(\gamma) = (\varphi_0, \dots, \varphi_{i-1}) \in \mathbb{R}^i$ such that $\mathbf{v}_i = \varphi_0 \mathbf{v}_0 + \cdots + \varphi_{i-1}\mathbf{v}_{i-1}$. When $A \ge 0$ (i.e., A is positive semidefinite), rank γ admits a useful alternate description, which depends on the following basic facts about Hankel matrices.

Lemma 2.1. *Let* $\gamma = (\gamma_0, \dots, \gamma_{2k}) \in \mathbb{R}^{2k+1}, \gamma_0 > 0$, *and let* $A = A_\gamma$. *For* $0 \le m \le k$, *let* $A(m) := (\gamma_{i+j})_{i,j=0}^{m}$ *(*$A(m)$ *is the upper left-hand corner of* A *of size* $m+1$, *and* $A = A(k)$*).*

(i) *(cf. [***CF***, Proposition 2.7]) If* A *is positive and invertible, so is* $A(j)$ *for every* $j \le k$.

(ii) *(cf. [***CF***, Proposition 2.12]) Let* $r :=$ *rank* γ. *Then* $A(r-1)$ *is invertible, and if* $r \le k$, *we have* $\varphi(\gamma) = A(r-1)^{-1}\mathbf{v}(r, r-1)$.

Proof: (ii) The fact that $A(r-1)$ is invertible is contained in [**CF**, Proposition 2.12]. To prove the identity for $\mathbf{\Phi}(\gamma)$, let $\mathbf{\Phi}' = (\varphi'_0, \dots, \varphi'_{r-1}) :=$

$A(r-1)^{-1}\mathbf{v}(r,r-1)$. We employ the minimality condition in the definition of $r = \text{rank}\,\gamma : \mathbf{\Phi}(\gamma) = (\varphi_0,\ldots,\varphi_{r-1}) \in \mathbb{R}^r$ is the unique solution of

$$\mathbf{v}_r = \varphi_0\mathbf{v}_0 + \cdots + \varphi_{r-1}\mathbf{v}_{r-1}.$$

In particular, $\varphi_0\gamma_j + \cdots + \varphi_{r-1}\gamma_{j+r-1} = \gamma_{j+r}, 0 \le j \le r-1$; also, from the definition of $\mathbf{\Phi}'$, $\varphi'_0\gamma_j + \cdots + \varphi'_{r-1}\gamma_{j+r-1} = \gamma_{j+r}, 0 \le j \le r-1$, so the invertibility of $A(r-1)$ implies that $\mathbf{\Phi}' = \mathbf{\Phi}(\gamma)$. □

We can now relate rank γ to a simple criterion concerning singularity.

Proposition 2.2. *Let $A = (\gamma_{i+j})_{i,j=0}^k$ be a positive semi-definite Hankel matrix. Assume that A is singular. Then rank $\gamma =$ min $\{j : 1 \le j \le k$ and $A(j)$ is singular $\}$.*

Proof: Let $r :=$ rank γ. By Lemma 2.1 (ii), $A(r-1)$ is invertible. Since $A \ge 0$, Lemma 2.1 (i) implies that $A(j)$ is invertible for $0 \le j \le r-1$. On the other hand, the definition of r implies that $A(r)$ is singular (the columns are dependent), so $r =$ min $\{j : 1 \le j \le k$ and $A(j)$ is singular $\}$. □

We proceed now to describe a model for positive Hankel matrices, which reveals a degree of recursiveness present in any singular positive Hankel matrix. For $\gamma = (\gamma_0,\ldots,\gamma_{2k}) \in \mathbb{R}^{2k+1}$ as above and $\gamma_{2k+1} \in \mathbb{R}$, let $\mathbf{v}(k+1,k) := (\gamma_{k+1+j})_{j=0}^k$. For $\gamma_{2k+2} \in \mathbb{R}$ and $\tilde{\gamma} = (\gamma_0,\ldots,\gamma_{2k},\gamma_{2k+1})$, let $\tilde{A} := A_{\tilde{\gamma}}$ denote the *Hankel extension* of A given by

$$\tilde{A} = (\gamma_{r+s})_{r,s=0}^{k+1} = \begin{pmatrix} A & \mathbf{v}(k+1,k) \\ \mathbf{v}(k+1,k)^* & \gamma_{2k+2} \end{pmatrix}. \tag{2.1}$$

The following elementary, but very useful, result will be needed in the proof of Theorem 2.4; an operator version of this result is implicit in the work of Smul'jan [**Smu**] (cf. [**CuF**]).

Lemma 2.3. *(cf. [**CuF**, Proposition 2.3]) Let $A \in M_n(\mathbb{C}), \mathbf{b} \in \mathbb{C}^n$, $c \in \mathbb{C}$, and let*

$$\tilde{A} := \begin{pmatrix} A & \mathbf{b} \\ \mathbf{b}^* & c \end{pmatrix}.$$

(i) *If $\tilde{A} \ge 0$, then $A \ge 0, \mathbf{b} = A\mathbf{w}$ for some $\mathbf{w} \in \mathbb{C}^n$, and $c \ge \mathbf{w}^*A\mathbf{w}$.*

(ii) *Let $A \geq 0$ and $\mathbf{b} = A\mathbf{w}$ for some $\mathbf{w} \in \mathbb{C}^n$. Then $\tilde{A} \geq 0$ if and only if $c \geq \mathbf{w}^* A\mathbf{w}$.*

(iii) *If $A \geq 0$ and $\mathbf{b} = A\mathbf{w}$, then rank $\tilde{A} =$ rank A if and only if $c = \mathbf{w}^* A\mathbf{w}$.*

We now present our structure theorem for positive Hankel matrices.

Theorem 2.4. *Let $\gamma = (\gamma_0, \ldots, \gamma_{2k}), \gamma_{2k+1}, \gamma_{2k+2}, \tilde{A}$ be as in (2.1), assume that $\tilde{A} \geq 0$, and let $r :=$ rank γ. Then*

(i) *$A(r-1)$ is positive and invertible;*

(ii) *$\mathbf{\Phi} = (\varphi_0, \ldots, \varphi_{r-1}) := A(r-1)^{-1}\mathbf{v}(r, r-1)$ satisfies*

$$\gamma_{r+j} = \varphi_0\gamma_j + \cdots + \varphi_{r-1}\gamma_{r+j-1} \quad (0 \leq j \leq 2k+1-r); \tag{2.2}$$

(iii) *$\gamma_{2k+2} \geq \varphi_0\gamma_{2k+2-r} + \cdots + \varphi_{r-1}\gamma_{2k+1}$.*

Conversely, if there exist r $(1 \leq r \leq k+1)$ and constants $\varphi_0, \ldots, \varphi_{r-1} \in \mathbb{R}$ such that $A(r-1) \geq 0$, and (2.2) and (iii) hold, then $\tilde{A} \geq 0$.

Proof: (i) follows from Lemma 2.1. Assuming that (ii) holds, let $\hat{\mathbf{\Phi}} = (0, \ldots, 0, \varphi_0, \ldots, \varphi_{r-1}) \in \mathbb{R}^{k+1}$; then $\mathbf{v}(k+1, k) = A\hat{\mathbf{\Phi}}$ in (2.1). Since $\tilde{A} \geq 0$, Lemma 2.3 (ii) implies that

$$\gamma_{2k+2} \geq \hat{\mathbf{\Phi}}^* A\hat{\mathbf{\Phi}} = (A\hat{\mathbf{\Phi}}, \hat{\mathbf{\Phi}}) = (\mathbf{v}(k+1, k), \hat{\mathbf{\Phi}}) = \varphi_0\gamma_{2k+2-r} + \cdots + \varphi_{r-1}\gamma_{2k+1};$$

thus (iii) holds.

(ii) Since $\tilde{A} \geq 0$, then $\mathbf{v}(k+1, k) \in$ Ran A (by Lemma 2.3), so there exist $c_0, \ldots, c_k \in \mathbb{R}$ such that $\mathbf{v}(k+1, k) = c_0\mathbf{v}_0 + \cdots + c_k\mathbf{v}_k$. If A is invertible $(r = k+1)$, then $c_j = \varphi_j (0 \leq j \leq k)$, and (ii) holds. We may thus assume $1 \leq r \leq k$; note that $\gamma_{j+r} = \varphi_0\varphi_j + \cdots + \varphi_{r-1}\gamma_{j+r-1}$, $0 \leq j \leq k$, by the definition of $\mathbf{\Phi}$. Suppose p satisfies $0 \leq p \leq k-r$ and

$$\gamma_{j+r} = \varphi_0\gamma_j + \cdots + \varphi_{r-1}\gamma_{j+r-1}, 0 \leq j \leq k+p.$$

Then

$$\begin{aligned}
&\gamma_{k+p+r+1} \\
&= c_0\gamma_{r+p} + \cdots + c_k\gamma_{k+r+p} \\
&= c_0(\varphi_0\gamma_p + \cdots + \varphi_{r-1}\gamma_{p+r-1}) + \cdots + c_k(\varphi_0\gamma_{p+k} + \cdots + \varphi_{r-1}\gamma_{p+k+r-1}) \\
&= \varphi_0(c_0\gamma_p + \cdots + c_k\gamma_{p+k}) + \cdots + \varphi_{r-1}(c_0\gamma_{p+r-1} + \cdots + c_k\gamma_{p+k+r-1}) \\
&= \varphi_0\gamma_{p+k+1} + \cdots + \varphi_{r-1}\gamma_{p+k+r};
\end{aligned}$$

thus (ii) follows by induction.

For the converse, starting with $A(r-1) \geq 0$, we use (2.2) to prove inductively that $A(j) \geq 0, r-1 \leq j \leq k$. Assume that $A(j) \geq 0$ for some $j, r-1 \leq j \leq k-1$, and note that

$$A(j+1) = \begin{pmatrix} A(j) & \mathbf{w}_{j+1} \\ \mathbf{w}_{j+1}^* & \gamma_{2j+2} \end{pmatrix}, \text{ where } \mathbf{w}_{j+1} = \begin{pmatrix} \gamma_{j+1} \\ \vdots \\ \gamma_{2j+1} \end{pmatrix}.$$

Let $\mathbf{u}_j = (0, \ldots, 0, \varphi_0, \ldots, \varphi_{r-1}) \in \mathbb{R}^{j+1}$. Then (2.2) implies that $A(j+1)\mathbf{u}_j = \mathbf{w}_{j+1}$ and that $\gamma_{2j+2} = \mathbf{u}_j^* \mathbf{w}_{j+1}$; thus Lemma 2.3 implies that $A(j+1) \geq 0$. By induction, we have $A(k) \geq 0$, and now a similar argument using (iii) shows that $\tilde{A} \geq 0$. □

In the case when $\tilde{A}$ is *singular*, Theorem 2.4 (ii) shows that all of the elements of $\tilde{A}$ (except perhaps γ_{2k+2}) are determined recursively by the elements of $A(r-1)$. The proof of the "converse" direction of Theorem 2.4 then shows that if $\tilde{A} \geq 0$, then rank $A(j+1) = \text{rank} A(j), r-1 \leq j \leq k-1$, so in particular rank $A(k) =$ rank $A(r-1) = r =$ rank γ. Moreover, if

$$\gamma_{2k+2} = \varphi_0 \gamma_{2k+2-r} + \cdots + \varphi_{r-1} \gamma_{2k+1},$$

then rank $\tilde{A} =$ rank $A (= r)$. On the other hand, if $\tilde{A} \geq 0$ and rank $\tilde{A} >$ rank A, then rank $\tilde{A} = 1 +$ rank A (by [**Ioh**, p. 34, Corollary, and Theorem I.6.1]). Thus we have the following "rank principle" for positive Hankel matrices.

Corollary 2.5. *Suppose $\tilde{A} \geq 0$ and let $r =$ rank γ.*

(i) *rank $A_\gamma = r$;*

(ii) *$r \leq$ rank $\tilde{A} \leq r+1$; moreover, rank $\tilde{A} = r+1$ if and only if*

$$\gamma_{2k+2} > \varphi_0 \gamma_{2k+2-r} + \cdots + \varphi_{r-1} \gamma_{2k+1}.$$

The following "extension principle" for positive Hankel matrices will prove useful in developing our results on moments.

Theorem 2.6. *Let $A := A(k)$ and let $r :=$ rank γ. If $A \geq 0$, then the following are equivalent:*

(i) *A has a positive Hankel extension;*

(ii) *rank* $A = r$;

(iii) *There exist* $\varphi_0, \dots, \varphi_{r-1} \in \mathbb{R}$ *such that* $\gamma_j = \varphi_0\gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1}$, $r \le j \le 2k$.

Moreover, if (i) holds and $r \le k$, *then in any positive Hankel extension of* A, γ_{2k+1} *is recursively determined by* $\Phi(\gamma)$, *i.e.,* $\gamma_{2k+1} = \varphi_0\gamma_{2k+1-r} + \cdots + \varphi_{r-1}\gamma_{2k}$.

Proof: (i)$\Rightarrow$ (iii) This follows from Theorem 2.4 (ii).

(iii) $\Rightarrow$ (ii) If A is invertible, rank $A = k+1 =$ rank γ, so we may assume $r \le k$.
(iii) implies inductively that $\mathbf{v}_j \in$ span $\{\mathbf{v}_0, \dots, \mathbf{v}_{r-1}\}, r \le j \le k$, whence rank $A \le r$; the reverse inequality follows from the independence of $\{\mathbf{v}_0, \dots, \mathbf{v}_{r-1}\}$ (Lemma 2.1(ii)).

(ii) $\Rightarrow$ (i) Since $A \ge 0$, Lemma 2.3 (ii) implies that A has a positive Hankel extension $\tilde{A} := A(k+1)$ for each $\gamma_{2k+1} \in \mathbb{R}$ such that

$$\mathbf{v}(k+1,k) := \begin{pmatrix} \gamma_{k+1} \\ \vdots \\ \gamma_{2k+1} \end{pmatrix} \in \text{Ran } A.$$

If A is invertible, $\mathbf{v}(k+1,k) \in$ Ran A for every $\gamma_{2k+1} \in \mathbb{R}$. We may thus assume $1 \le r \le k$. Since $r \le k$, Proposition 2.2 implies that $r =$ rank $\gamma =$ rank $(\gamma_0, \dots, \gamma_{2k-2})$. Theorem 2.4(ii) (applied with $A(k)$ as $\tilde{A}$ and $A(k-1)$ as A) implies that there are unique scalars $\varphi_0, \dots, \varphi_{r-1}$ such that $\gamma_{r+j} = \varphi_0\gamma_j + \cdots + \varphi_{r-1}\gamma_{r+j-1} (0 \le j \le 2k-1-r)$. Thus, $A(k-1)\hat{\Phi} = \mathbf{v}(k,k-1)$, where $\hat{\Phi} := (0, \dots, 0, \varphi_0, \dots, \varphi_{r-1}) \in \mathbb{R}^k$; also, $r =$ rank $A(k) =$ rank $A(k-1)$. Lemma 2.3 (iii) now implies that $\gamma_{2k} = \varphi_0\gamma_{2k-r} + \cdots + \varphi_{r-1}\gamma_{2k-1}$. Let

$$\gamma_{2k+1} := \psi_0\gamma_{2k+1-r} + \cdots + \psi_{r-1}\gamma_{2k};$$

then the recursive relations imply that $\mathbf{v}_{k+1} \in$ span $\{\mathbf{v}_0, \dots, \mathbf{v}_{r-1}\} =$ Ran A. Moreover, if $\beta \ne \gamma_{2k+1}$ has the property that $(\gamma_{k+1}, \dots, \gamma_{2k}, \beta) \in$ Ran A, then $(0, 0, \dots, 1) \in$ span $\{\mathbf{v}_0, \dots, \mathbf{v}_{r-1}\}$, so there exist scalars $c_0, \dots, c_{r-1}$, not all zero, such that $c_0\mathbf{v}(0, r-1) + \cdots + c_{r-1}\mathbf{v}(r-1, r-1) = 0$, a contradiction to the invertibility of $A(r-1)$. □

Remark 2.7. Assume that $A(k)$ has a positive Hankel extension (as just described). Examination of the preceding argument shows that when

A is invertible, γ_{2k+1} can be chosen arbitrarily, while if A is singular, γ_{2k+1} is uniquely determined. In either case, once γ_{2k+1} is prescribed, γ_{2k+2} can be chosen as any value satisfying Theorem 2.4 (iii).

Let $\gamma = (\gamma_0, \ldots, \gamma_{2k})$ and let $r :=$ rank γ. In the sequel we say that γ is *positively recursively generated* if $A(r-1) \geq 0$ and $\gamma_j = \varphi_0\gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1} (r \leq j \leq 2k)$; equivalently, if $A(k) \geq 0$ and the conditions of Theorem 2.6 hold. We say that $\tilde{\gamma} := (\gamma_0, \ldots, \gamma_{2k+1})$ is *positively recursively generated* if $A(r-1) \geq 0$ and $\gamma_j = \varphi_0\gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1} (r \leq j \leq 2k+1)$; equivalently, $A(k) \geq 0$ and $\mathbf{v}(k+1, k) \in$ Ran $A(k)$ (Theorem 2.4).

3. The Truncated Hamburger Moment Problem. Our aim in this section is to present an elementary algebraic treatment of the truncated Hamburger moment problem. For $k \geq 0$, let $\gamma = (\gamma_0, \ldots, \gamma_{2k}) \in \mathbb{R}^{2k+1}$, $\gamma_0 > 0$. For $\gamma_{2k+1} \in \mathbb{R}$, let $\tilde{\gamma} = (\gamma_0, \ldots, \gamma_{2k+1})$. As in the previous section, we define $A(m) = (\gamma_{i+j})_{0 \leq i,j \leq m} (0 \leq m \leq k), \mathbf{v}_i = (\gamma_{i+j})_{0 \leq j \leq k} (0 \leq i \leq k+1)$, and set $r =$ rank γ. Thus $\{\mathbf{v}_0, \ldots, \mathbf{v}_{r-1}\}$ is a linearly independent set, and there exists a unique $\mathbf{\Phi}(\tilde{\gamma}) := \tilde{\mathbf{\Phi}} = (\tilde{\varphi}_0, \ldots, \tilde{\varphi}_{r-1}) \in \mathbb{R}^r$ such that

$$\mathbf{v}_r = \tilde{\varphi}_0\mathbf{v}_0 + \cdots + \tilde{\varphi}_{r-1}\mathbf{v}_{r-1}.$$

Note that $\tilde{\mathbf{\Phi}} = A(r-1)^{-1}\mathbf{v}(r, r-1)$ and that if $r \leq k$, then $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}(\gamma)$. We refer to the polynomial

$$g_{\tilde{\gamma}}(t) := t^r - (\tilde{\varphi}_0 + \cdots + \tilde{\varphi}_{r-1}t^{r-1}) \ (t \in \mathbb{R})$$

as the *generating function* of $\tilde{\gamma}$. We begin with an existence theorem for the "odd case" of the Hamburger power moment problem

$$\gamma_j = \int t^j d\mu(t), \quad 0 \leq j \leq 2k+1. \tag{3.1}$$

A *solution* to (3.1) is a positive Borel measure μ on $\mathbb{R}$ for which (3.1) holds; μ is then called a *representing measure* for $\tilde{\gamma}$.

Theorem 3.1. *(Existence Theorem, Odd Case) Let $\tilde{\gamma} = (\gamma_0, \ldots, \gamma_{2k+1}) \in \mathbb{R}^{2k+2}, \gamma_0 > 0$. The following are equivalent:*

(i) *There exists a positive Borel measure μ on $\mathbb{R}$ such that $\gamma_j = \int t^j d\mu(t)$, $0 \leq j \leq 2k+1$;*

(ii) *There exists a compactly supported representing measure for* $\tilde{\gamma}$,

(iii) *There exists a finitely atomic representing measure for* $\tilde{\gamma}$,

(iv) *There exists a rank* γ*-atomic representing measure for* $\tilde{\gamma}$, *whose support consists of the roots of* $g_{\tilde{\gamma}}$;

(v) $A := A(k) \geq 0$ *and* $\mathbf{v}_{k+1} \in$ *Ran* A;

(vi) $A(k+1) \geq 0$ *for some choice of* $\gamma_{2k+2} \in \mathbb{R}$ *(i.e.,* $A(k)$ *has a positive Hankel extension using* γ_{2k+1}*);*

(vii) $\tilde{\gamma}$ *is positively recursively generated.*

Theorem 3.1 ((i) $\Leftrightarrow$ (v)) reduces the question of existence of a representing measure to two standard problems in finite dimensional linear algebra and operator theory: the problem of determining whether $\mathbf{v}_{k+1} \in \text{Ran } A(k)$, and the problem of verifying that the eigenvalues of $A(k)$ are nonnegative; at the end of this section, we provide a simple algorithm which checks both conditions simultaneously.

We defer the proof of Theorem 3.1 in favor of some preliminaries. For $y \in \mathbb{R}$, δ_y denotes the probability measure on $\mathbb{R}$ such that $\delta_y(\{y\}) = 1$. Thus, a finitely atomic positive measure on $\mathbb{R}$ with atoms $y_0, \ldots, y_k$ may be expressed as $\mu = \sum_{j=0}^{k} \rho_j \delta_{y_j}$, where each $\rho_j \geq 0$. The next result relates rank γ to the number of atoms in any finitely atomic representing measure of $\tilde{\gamma}$.

Lemma 3.2. *If* $\tilde{\gamma}$ *has a finitely atomic representing measure with* m *atoms, then* $m \geq$ *rank* γ.

Proof: Let $\mu = \sum_{j=0}^{m-1} p_j \delta_{y_j}$ be a representing measure for $\tilde{\gamma}$ with distinct atoms $y_0, \ldots, y_{m-1}$. Let $g(t) := (t-y_0)\cdots(t-y_{m-1}) = c_0 + \cdots + c_{m-1}t^{m-1} + t^m$. In $L^2(\mu)$, we have $t^j g = 0 (0 \leq j \leq k)$; since μ is a representing measure for $\tilde{\gamma}$, it follows that if $m \leq k+1$, then $c_0\gamma_j + \cdots + c_{m-1}\gamma_{j+m-1} = -\gamma_{j+m}$ $(0 \leq j \leq k)$. Thus $-v_m = c_0 v_0 + \cdots + c_{m-1}v_{m-1}$, whence $m \geq$ rank γ. □

The next result establishes Theorem 3.1 ((v) $\Rightarrow$ (iv)) for the special case when $A(k)$ is invertible. This proposition is the key result that allows us to bypass use of the spectral theorem, or similar devices, in solving moment problems. It gives an elementary, constructive procedure for finding interpolating measures. A similar result (for the "nonsingular even" case) is given in [**AhK**, Theorem I.3], where it is proved using the theory of quasi-orthogonal polynomials. For $\mathbf{x} = (x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$, define

$$V_{\mathbf{x}} := \begin{pmatrix} 1 & \cdots & 1 \\ x_0 & & x_m \\ \vdots & & \vdots \\ x_0^m & \cdots & x_m^m \end{pmatrix};$$

$V_{\mathbf{x}}$ is invertible if and only if the $x_i's$ are distinct.

Proposition 3.3. *Assume that $A = A(k)$ is positive and invertible. Then $g_{\tilde{\gamma}}$ has $k+1$ distinct real roots, $x_0, \ldots, x_k$. Thus $V_{\mathbf{x}}$ is invertible, and if $\rho = (\rho_0, \ldots, \rho_k) := V_{\mathbf{x}}^{-1}\mathbf{v}_0$, then $\rho_j > 0 (0 \le j \le k)$. Moreover, if $\mu := \sum_{i=0}^{\mathbf{k}} \rho_i \delta_{x_i}$, then $\gamma_j = \int t^j d\mu$ $(0 \le j \le 2k+1)$.*

For the proof of Proposition 3.3, we require some auxiliary results on polynomials and positivity; our discussion is an adaptation to the truncated moment problem of the presentation in [**Akh**, Chapter I, Sections 1 and 2] concerning the *full* moment problem. Let $r(t) = r_0 + r_1 t^2 + \cdots + r_{2k} t^{2k} (t \in \mathbb{R})$ be a polynomial with complex coefficients and deg $r \le 2k$. Define the linear functional S by

$$S(r) := r_0\gamma_0 + \cdots + r_{2k}\gamma_{2k}.$$

For $p(t) = p_0 + \cdots + p_k t^k$ and $q(t) = q_0 + \cdots + q_k t^k$, let $\hat{\mathbf{p}} = (p_0, \ldots, p_k)$ and $\hat{\mathbf{q}} = (q_0, \ldots, q_k)$. Then

$$S(p\overline{q}) = S\left[\sum_{i,j=0}^{k} p_i \overline{q}_j t^{i+j}\right] = \sum_{i,j=0}^{k} p_i \overline{q}_j \gamma_{i+j} = (A\hat{\mathbf{p}}, \hat{\mathbf{q}}).$$

Thus if $p \not\equiv 0$ and A is positive and invertible, then $0 < (A\hat{\mathbf{p}}, \hat{\mathbf{p}}) = S(|p|^2)$. Now suppose deg $r \le 2k$, $r \not\equiv 0$, and $r(t) \ge 0 (t \in \mathbb{R})$. Then deg $r = 2m$ for some m, $0 \le m \le k$; thus by [**Akh**, p. 2], $r = p^2 + q^2$, where p and q are polynomials of degree m with *real* coefficients. Now $S(r) = S(p^2) + S(q^2) = (A\hat{\mathbf{p}}, \hat{\mathbf{p}}) + (A\hat{\mathbf{q}}, \hat{\mathbf{q}}) > 0$ (since $p \not\equiv 0$ or $q \not\equiv 0$). In summary,

$$(3.2) \qquad \begin{cases} A \ge 0 \text{ and invertible }, \\ \deg r \le 2k, r \not\equiv 0, r(t) \ge 0 (t \in \mathbb{R}) \end{cases} \Rightarrow S(r) > 0.$$

Proof of Proposition 3.3: The result is trivial for $k = 0$, so we assume $k > 0$. Since A is invertible, rank $\gamma = k + 1$, and from the definition of $\tilde{\mathbf{\Phi}}$ we have

$$\gamma_{k+j+1} = \tilde{\varphi}_0\gamma_j + \cdots + \tilde{\varphi}_k\gamma_{j+k} \ (0 \le j \le k-1).$$

Thus

$$S(g_{\tilde{\gamma}}t^j) = 0 \quad (0 \le j \le k-1); \tag{3.3}$$

indeed,

$$\begin{aligned} S(g_{\tilde{\gamma}}t^j) &= S([t^{k+1} - (\tilde{\varphi}_0 + \cdots + \tilde{\varphi}_k t^k)]t^j) \\ &= S(t^{k+j+1} - (\tilde{\varphi}_0 t^j + \cdots + \tilde{\varphi}_k t^{j+k})) \\ &= \gamma_{k+j+1} - (\tilde{\varphi}_0\gamma_j + \cdots + \tilde{\varphi}_k\gamma_{j+k}) = 0. \end{aligned}$$

We now adapt the argument of [**Akh**, Theorem 1.2.2] concerning roots of quasi-orthogonal polynomials. If $g_{\tilde{\gamma}}$ never changes sign, then $g_{\tilde{\gamma}}(t) \ge 0$ ($t \in \mathbb{R}$); since $g_{\tilde{\gamma}} \not\equiv 0$ and deg $g_{\tilde{\gamma}} = k + 1 \le 2k$, (3.2) implies $S(g_{\tilde{\gamma}}) > 0$, a contradiction to $S(g_{\tilde{\gamma}}) = 0$ in (3.3). Thus $g_{\tilde{\gamma}}$ changes sign. Let $t_1 < \cdots < t_r$ be the distinct points where $g_{\tilde{\gamma}}$ changes sign, $1 \le r \le k+1$; these are the real roots of $g_{\tilde{\gamma}}$ having odd multiplicities. Let $g(t) := (t - t_1)\cdots(t - t_r)$. If $r < k+1$, then $r < k$ (since deg $g_{\tilde{\gamma}} = k+1$), so deg $g \le k-1$ and deg $(g_{\tilde{\gamma}}g) \le 2k$. In this case, $f := g_{\tilde{\gamma}}g$ satisfies deg $f \le 2k, f \not\equiv 0$, $f(t) \ge 0 (t \in \mathbb{R})$, whence $S(f) > 0$ by (3.2). But since deg $g \le k-1$, (3.3) implies $S(f) = 0$. This contradiction shows that $r = k+1$, so $g_{\tilde{\gamma}}$ has $k+1$ distinct roots. We denote these roots by $x_0, \ldots, x_k$; since they are distinct, $V_{\mathbf{x}}$ is invertible and $\rho := V_{\mathbf{x}}^{-1}\mathbf{v}_0$ is well defined. Let $\mu = \sum_{i=0}^k \rho_i\delta_{x_i}$. Since $\mathbf{V}_{\mathbf{x}}\rho = \mathbf{v}_0$, it follows immediately that $\int t^j d\mu = \sum_{i=0}^k \rho_i x_i^j = \gamma_j (0 \le j \le k)$. Moreover, since

$$t^{k+1} = \tilde{\varphi}_0 + \cdots + \tilde{\varphi}_k t^k$$

in supp $\mu = \{x_0, \ldots, x_k\}$, and since

$$\gamma_{k+j} = \tilde{\varphi}_0\gamma_{j-1} + \cdots + \tilde{\varphi}_k\gamma_{k+j-1} \ (1 \le j \le k+1),$$

it follows inductively that $\gamma_j = \int t^j d\mu \ (k+1 \le j \le 2k+1)$. To complete the proof, it suffices to show that each ρ_j is positive. By Lagrange interpolation, there exists a polynomial $f^{(j)}$ of degree k such that $f^{(j)}(x_j) = 1$ and $f^{(j)}(x_m) = 0$ for $0 \le m \le k, m \ne j$. Write

$$f^{(j)}(t) = f_0^{(j)} + \cdots + f_k^{(j)}t^k,$$

and note that

$$0 < (A\hat{\mathbf{f}}^{(\mathrm{j})}, \hat{\mathbf{f}}^{(\mathrm{j})}) = S(|f^{(j)}|^2) = \int |f^{(j)}|^2 d\mu = \sum_{i=0}^{k} \rho_i [f^{(j)}(x_i)]^2 = \rho_j. \quad \square$$

We are now prepared to prove Theorem 3.1 (v) $\Rightarrow$ (iv).

Corollary 3.4. *Assume that $A = A(k) \geq 0$ and that $\mathbf{v}(k+1, k) \in$ Ran A. Then there exists a rank γ-atomic positive measure μ, with supp $\mu = \mathcal{Z}(g_{\tilde{\gamma}})$, the zero set of $g_{\tilde{\gamma}}$, such that*

$$\gamma_j = \int t^j d\mu \quad (0 \leq j \leq 2k+1).$$

Proof: If A is invertible, the result follows from Proposition 3.3. Let $r :=$ rank γ; if A is singular, then $r \leq k, A(r-1)$ is positive and invertible (Theorem 2.4 (i)), and $\tilde{\mathbf{\Phi}} = (\tilde{\varphi}_0, \ldots, \tilde{\varphi}_{r-1})$ is the unique solution of $A(r-1)\tilde{\mathbf{\Phi}} = (\gamma_r, \ldots, \gamma_{2r-1})$. We now apply Proposition 3.3 to the sequence $\gamma(r) := (\gamma_0, \ldots, \gamma_{2r-2})$, so that the role of A in Proposition 3.3 is played by $A(r-1)$ and the role of $\tilde{\gamma}$ in Proposition 3.3 is played by $\gamma(r)^{\sim} = (\gamma_0, \ldots, \gamma_{2r-1})$. Note, moreover, that $g_{\gamma(r)^{\sim}} = g_{\tilde{\gamma}}$. Thus Proposition 3.3 yields a positive measure $\mu := \sum_{i=0}^{r-1} \rho_i \delta_{x_i}$ (where $x_0, \ldots, x_{r-1}$ are the distinct real roots of $g_{\tilde{\gamma}}$) such that

$$\gamma_j = \int t^j d\mu \quad (0 \leq j \leq 2r-1). \tag{3.4}$$

The hypotheses and Lemma 2.3 (ii) imply that A admits a positive Hankel extension $\tilde{A} = A(k+1)$ with γ_{2k+1} from $\tilde{\gamma}$. Since $\tilde{A} \geq 0$, Theorem 2.4 (ii) implies that

$$\gamma_m = \tilde{\varphi}_0 \gamma_{m-r} + \cdots + \tilde{\varphi}_{r-1} \gamma_{m-1} (r \leq m \leq 2k+1). \tag{3.5}$$

Starting with (3.4), we use (3.5) to prove inductively that

$$\gamma_j = \int t^j d\mu \quad (2r \leq j \leq 2k+1).$$

Suppose that $2r-1 \le j \le 2k$ and $\gamma_i = \int t^i d\mu$ for $0 \le i \le j$. Then

$$\begin{aligned}\int t^{j+1} d\mu &= \int t^{j+1-r} t^r d\mu \\ &= \int t^{j+1-r}(\tilde{\varphi}_0 + \cdots + \tilde{\varphi}_{r-1} t^{r-1}) d\mu \quad (\text{since } g_{\tilde{\gamma}} \equiv 0 \text{ on supp } \mu) \\ &= \tilde{\varphi}_0 \gamma_{j+1-r} + \cdots + \tilde{\varphi}_{r-1}\gamma_j = \gamma_{j+1} \quad (\text{by } (3.5)). \qquad \square\end{aligned}$$

Remark 3.5. The preceding proof shows that for any $\tilde{\gamma}, g_{\tilde{\gamma}}$ has rank γ distinct real roots, which we denote in the sequel by $\mathcal{Z}(g_{\tilde{\gamma}}) := \{x_0, \ldots, x_{r-1}\}$.

Lemma 3.6. *If rank $\gamma \le k$ and μ is a solution of (3.1), then supp $\mu \subseteq \mathcal{Z}(g_{\tilde{\gamma}})$.*

Proof: Let $r :=$ rank γ; then $\gamma_j = \tilde{\varphi}_0\gamma_{j-r} + \cdots + \tilde{\varphi}_{r-1}\gamma_{j-1}$ $(r \le j \le r+k)$. Thus, by (3.1), $\int t^j g_{\tilde{\gamma}} d\mu = 0$, $0 \le j \le k$. Now,

$$\int -\tilde{\varphi}_j t^j g_{\tilde{\gamma}}(t) d\mu = 0 \quad (0 \le j \le r-1)$$

and

$$\int t^r g_{\tilde{\gamma}}(t) d\mu = 0$$

since $r \le k$. Thus

$$\int g_{\tilde{\gamma}}(t)^2 d\mu = \int (t^r - (\tilde{\varphi}_0 + \cdots + \tilde{\varphi}_{r-1} t^{r-1})) g_{\tilde{\gamma}}(t) d\mu = 0,$$

whence $g_{\tilde{\gamma}} = 0$ in $L^2(\mu)$. Since μ is a positive Borel measure, it follows that supp $\mu \subseteq \mathcal{Z}(g_{\tilde{\gamma}})$. $\square$

Proof of Theorem 3.1: (i)$\Rightarrow$(vi) Let μ be a representing measure for $\tilde{\gamma}$, let $p(t) = p_0 + \cdots + p_k t^k (t \in \mathbb{R})$ be a polynomial with complex coefficients, and let $\hat{\mathbf{p}} = (p_0, \ldots, p_k)$. Then as in the discussion preceding the proof of Proposition 3.3, $(A\hat{\mathbf{p}}, \hat{\mathbf{p}}) = S(|p|^2)$, and (i) implies $S(|p|^2) = \int |p|^2 d\mu \ge 0$; thus $A \ge 0$. If A is invertible, then rank $A =$ rank $\gamma = k+1$, so Theorem 2.6 ((ii)$\Rightarrow$ (i)) implies that A has a positive Hankel extension of the form $A(k+1)$. If A is singular, then rank $\gamma \le k$, so Lemma 3.6 implies that supp $\mu \subseteq \mathcal{Z}(g_{\tilde{\gamma}})$. Let $\gamma_{2k+2} := \int t^{2k+2} d\mu < +\infty$, and let $A(k+1) := A(k+1)(\mu)$. We may now extend the functional S to polynomials of degree up to $2k+2$ by $S(t^j) := \gamma_j (0 \leq j \leq 2k+2)$; if $p(t) = p_0 + \cdots + p_{k+1}t^{k+1}$, then, exactly as above, we have $(A(k+1)\hat{\mathbf{p}}, \hat{\mathbf{p}}) = S(|p|^2) = \int |p|^2 d\mu \geq 0$. Thus, $A(k+1)$ is a positive Hankel extension of A.

(vi)⇒(v) Apply Lemma 2.3.

(v)⇒(iv) Apply Corollary 3.4.

(iv)⇒(iii)⇒(ii)⇒(i) Trivial. □

For our uniqueness theorem in the "odd" case, we actually require the following result about the "even" case.

Lemma 3.7. *Let $\gamma = (\gamma_0, \ldots, \gamma_{2k})$ and $\check{\gamma} := (\gamma_0, \ldots, \gamma_{2k-1})$. If $r :=$ rank $\gamma \leq k$, and μ is any representing measure for γ, then supp $\mu \subseteq \mathcal{Z}(g_{\check{\gamma}})$.*

Proof: We consider two cases. If $r < k$, then the moment problem for $\check{\gamma}$ is a "singular odd" problem, for which μ is a solution. By Lemma 3.6, supp $\mu \subseteq \mathcal{Z}(g_{\check{\gamma}})$. If $r = k$, then $\mathbf{v}_k = \varphi_0 \mathbf{v}_0 + \ldots \varphi_{k-1}\mathbf{v}_{k-1}$ for a unique $(\varphi_0, \ldots, \varphi_{k-1}) \in \mathbb{R}^k$. This gives

$$\gamma_{j+k} = \varphi_0 \gamma_j + \cdots + \varphi_{k-1}\gamma_{j+k-1} \ (0 \leq j \leq k),$$

whence

$$\int t^j g_{\check{\gamma}} d\mu = 0 \quad (0 \leq j \leq k).$$

(Recall that $g_{\check{\gamma}} = t^k - (\varphi_0 + \cdots + \varphi_{k-1}t^{k-1})$.) It follows that $\int g_{\check{\gamma}}^2 d\mu = 0$, so supp $\mu \subseteq \mathcal{Z}(g_{\check{\gamma}})$. □

In the case when (3.1) is soluble, we may describe the set of solutions as follows.

Theorem 3.8. *(Uniqueness Theorem, Odd Case) Let $\tilde{\gamma} = (\gamma_0, \ldots, \gamma_{2k+1})$, $\gamma_0 > 0$, and suppose that $\tilde{\gamma}$ has a representing measure, i.e., $A(k) \geq 0$ and $\mathbf{v}_{k+1} \in$ Ran $A(k)$. Let $r :=$ rank γ.*

(i) If $r \leq k$, then (3.1) has the unique solution

$$\mu = \sum_{i=0}^{r-1} \rho_i \delta_{x_i}; \tag{3.6}$$

here supp $\mu = \{x_0, \ldots, x_{r-1}\} = \mathcal{Z}(g_{\tilde{\gamma}})$ and $\rho := (\rho_0, \ldots, \rho_k)$ is given by $\rho = V_{\mathbf{x}}^{-1}\mathbf{v}_0$, where $\mathbf{x} := (x_0, \ldots, x_{r-1})$.

(ii) If $r = k+1$, then (3.1) has infinitely many solutions: Let $\mathbf{\Phi} := A(k)^{-1}\mathbf{v}_{k+1}$ and let $c := \varphi^*\mathbf{v}_{k+1}$. If μ is a representing measure for $\tilde{\gamma}$, then $\gamma_{2k+2}(\mu) \geq c$; moreover, there is a unique representing measure $\tilde{\mu}$ with $\int t^{2k+2} d\tilde{\mu}(t) = c$, and $\tilde{\mu}$ has $k+1$ atoms. For each $\gamma_{2k+2} > c$ and any $\gamma_{2k+3} \in \mathbb{R}, \tilde{\tilde{\gamma}} := (\gamma_0, \dots, \gamma_{2k+3})$ has a representing measure, and any such measure μ satisfies card (supp μ) $\geq k+2$.

Proof: (i) The proof of Corollary 3.4 shows that (3.6) is a representing measure for $\tilde{\gamma}$. Suppose ν is also a representing measure. Since rank $\gamma \leq k$, Lemma 3.6 implies that supp $\nu \subseteq \mathcal{Z}(g_{\tilde{\gamma}})$; thus ν is of the form $\nu = \sum_{i=0}^{r-1} \epsilon_i \delta_{x_i}$. Since $\gamma_j = \int t^j d\nu$ $(0 \leq j \leq r-1)$, then

$$V_{\mathbf{x}}\epsilon = \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{r-1} \end{pmatrix} = V_{\mathbf{x}}\rho,$$

whence $\epsilon = \rho$ and $\nu = \mu$.

(ii) Suppose μ is a representing measure of $\tilde{\gamma}$. We claim that $\int t^{2k+2} d\mu(t) \geq c$. We may assume $\int t^{2k+2} d\mu(t) < +\infty$, and thus we may define $A(k+1) := A(k+1)(\mu)$ (using $\gamma_{2k+2} := \int t^{2k+2} d\mu(t)$). Just as in the proof of Theorem 3.1 ((i) $\Rightarrow$ (vi)), $A(k+1) \geq 0$, so Lemma 2.3 (ii) implies

$$\int t^{2k+2} d\mu(t) \geq \mathbf{\Phi}^* \mathbf{v}_{k+1} = c.$$

We next show that there is a unique representing measure $\tilde{\mu}$ with $\int t^{2k+2} d\tilde{\mu}(t) = c$. Let

$$\lambda_{2k+2} := \int t^{2k+2} d\mu(t) = c = \mathbf{\Phi}^* \mathbf{v}_{k+1} = \varphi_0 \gamma_{k+1} + \dots + \varphi_k \gamma_{2k+1}.$$

Let $\mathbf{w}_0, \dots, \mathbf{w}_{k+1}$ denote the successive columns of the corresponding Hankel matrix $A(k+1)$, so that

$$\mathbf{w}_{k+1} = \varphi_0 \mathbf{w}_0 + \dots + \varphi_k \mathbf{w}_k.$$

Let $\gamma_{2k+3} := \varphi_0 \gamma_{k+2} + \dots + \varphi_k \gamma_{2k+2}$ and let $\mathbf{w}_{k+2} := (\gamma_j)_{j=k+2}^{2k+3}$. Finally, let $\tilde{\tilde{\gamma}} := (\gamma_0, \dots, \gamma_{2k+3})$. Lemma 2.3 implies $A(k+1) \geq 0$, and clearly $\mathbf{w}_{k+2}$

$\in$ Ran $A(k+1)$. Since $A(k+1)$ is singular, Theorem 3.1 and part (i) above imply that $\tilde{\tilde{\gamma}}$ has a unique representing measure $\tilde{\mu}$ and that card (supp $\tilde{\mu}$) $= k+1$; in particular, $\int t^{2k+2} d\tilde{\mu}(t) = c$. Let ν be any representing measure for $(\gamma_0, \dots, \gamma_{2k+1}, c)$. Since $A(k+1)$ is singular, Lemma 3.7 implies that supp $\nu \subseteq \mathcal{Z}(g_{\tilde{\gamma}})$. Thus $\int t^{2k+3} d\nu(t) < +\infty$ and $\mathbf{w}_{k+2}(\nu) \in$ Ran $A(k+1)$ ($A(k+2)(\nu)$ is positive). Since $\mathbf{w}_{k+2}$ and $\mathbf{w}_{k+2}(\nu)$ are in Ran $A(k+1)$, if $\int t^{2k+3} d\nu(t) \neq \gamma_{2k+3}$, then Ran $A(k+1)$ contains the final basis vector $\mathbf{e}_{k+1} := (0, \dots, 0, 1)$ of $\mathbb{R}^{k+2}$. Since $\mathbf{w}_{k+1} \in$ span $(\mathbf{w}_0, \dots, \mathbf{w}_k)$, there exist scalars $c_0, \dots, c_k$, not all zero, such that $\mathbf{e}_{k+1} = c_0\mathbf{w}_0 + \cdots + c_k\mathbf{w}_k$, whence $0 = c_0\mathbf{v}_0 + \cdots + c_k\mathbf{v}_k$, contradicting rank $\gamma = k+1$. Thus $\int t^{2k+3} d\nu(t) = \gamma_{2k+3}$, and ν is a representing measure of $\tilde{\tilde{\gamma}}$, whence (by (i) above) $\nu = \tilde{\mu}$. Thus $\tilde{\mu}$ is the unique representing measure of $\tilde{\gamma}$ for which $\int t^{2k+2} d\tilde{\mu} = c$, and card (supp $\tilde{\mu}$)$= k+1$.

Finally, it is clear that if $\gamma_{2k+2} > c$, then $A(k+1)$ is positive and invertible. Theorem 3.1 thus implies that for each $\gamma_{2k+3} \in \mathbb{R}, (\gamma_0, \dots, \gamma_{2k+3})$ has a representing measure; moreover, Lemma 3.2 implies that for each representing measure μ,

$$\text{card}(\text{supp } \mu) \geq \text{ rank } (\gamma_0, \dots, \gamma_{2k+2}) = k+2. \qquad \square$$

We now turn our attention to the "even case" of the Hamburger truncated moment problem:

$$(3.7) \qquad \gamma_j = \int t^j d\mu(t) \quad (0 \leq j \leq 2k).$$

Theorem 3.9. *(Existence, Even Case) For $k \geq 0$, let $\gamma = (\gamma_0, \dots, \gamma_{2k})$, $\gamma_0 > 0$. The following are equivalent:*

(i) *There exists a positive Borel measure μ on $\mathbb{R}$ satisfying (3.7);*
(ii) *There exists a compactly supported representing measure for γ;*
(iii) *There exists a finitely atomic representing measure for γ;*
(iv) *There exists a (rank γ)-atomic representing measure for γ;*
(v) *$A(k) \geq 0$ and rank $A(k) =$ rank γ;*
(vi) *$A(k)$ has a positive Hankel extension;*
(vii) *γ is positively recursively generated.*

Versions of this theorem appear in the literature. Shohat and Tamarkin establish a correspondence between solutions of the "even" moment problem and certain analytic functions with prescribed rational part [**ShT**, Theorem 2.2]. The case when $A(k)$ is nonsingular is treated in [**AhK**, Theorem I.3] and [**Ioh**, Theorem A.II.1] using methods from the theory of quasi-orthogonal polynomials. Additionally, [**Ioh**, Remark, p. 205] treats the case when rank $A(k) = k$.

Proof: (i) $\Rightarrow$ (vi) As in the proof of Theorem 3.1 (i) $\Rightarrow$ (vi), the existence of a representing measure μ implies $A(k) \geq 0$. We consider two cases, depending on the value of $r :=$ rank γ. If $r \geq k$, the existence of a positive Hankel extension follows from Theorem 2.6 (since rank $A =$ rank γ). If $r < k$, let $\check{\gamma} := (\gamma_0, \ldots, \gamma_{2k-1})$; since μ interpolates $\check{\gamma}$, by Theorem 3.8 (i), μ is the unique interpolating measure, and thus coincides with the finitely atomic measure produced in (3.6). Thus, $\gamma_{2k+1} := \int t^{2k+1} d\mu$ and $\gamma_{2k+2} := \int t^{2k+2} d\mu$ are finite and, as in the proof of Theorem 3.1 ((i) $\Rightarrow$(vi)), $A(k+1)(\mu)$ is a positive Hankel extension of $A(k)$.

(vi)$\Leftrightarrow$(v) Theorem 2.6.

(vi)$\Rightarrow$(iv) Suppose $A(k+1)$ is a positive Hankel extension of $A(k)$. Then Lemma 2.3 implies $A(k) \geq 0$ and $\mathbf{v}(k+1, k) \in$ Ran $A(k)$. Let $r :=$ rank γ. Theorem 3.1 ((v)$\Rightarrow$(iv)) implies that $\tilde{\gamma} := (\gamma_0, \ldots, \gamma_{2k+1})$ has an r-atomic representing measure, and this is clearly a representing measure for γ.

(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i) Trivial.

(v)$\Leftrightarrow$(vii) Remark 2.7. □

Theorem 3.10. *(Uniqueness, Even Case) Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k}), \gamma_0 > 0$. *Suppose* γ *has a representing measure, i.e.,* $A(k) \geq 0$ *and rank* $A(k) =$ *rank* γ.

(i) Suppose $r :=$ *rank* $\gamma \leq k$. *Let* $\mathbf{\Phi} := \mathbf{\Phi}(\gamma)$ *and let*

$$\gamma_{2k+1} := \varphi_0 \gamma_{2k+1-r} + \cdots + \varphi_{r-1} \gamma_{2k}.$$

Then $\tilde{\gamma} := (\gamma_0, \ldots, \gamma_{2k+1})$ *has the unique representing measure of (3.6), which is also the unique representing measure of* γ.

(ii) Suppose $r = k+1$. *For each* $\gamma_{2k+1} \in \mathbb{R}$, *let* $\tilde{\gamma} = (\gamma_0, \ldots, \gamma_{2k+1})$. *Then* $\tilde{\gamma}$ *has infinitely many representing measures (described by Theorem 3.8 (ii)) and each is a representing measure of* γ.

Proof: (i) Theorem 3.9 implies that $A(k)$ has a positive Hankel extension, and Theorem 2.6 implies that in any such extension, γ_{2k+1} is given recursively via $\Phi(\gamma)$. If μ is a representing measure for γ, then by Lemma 3.7, supp μ is finite, whence $\gamma_{2k+1}(\mu) := \int t^{2k+1} d\mu < \infty$. Since $A(k+1)(\mu)$ is a positive Hankel extension of $A(k)$, we must have $\gamma_{2k+1}(\mu) = \gamma_{2k+1}$; thus μ is a representing measure for the "singular odd" system $\tilde{\gamma}$. By Theorem 3.8(i), μ must coincide with the measure in (3.6).

(ii) Apply Remark 2.7 and Theorem 3.8(ii). □

We conclude this section with our elementary algorithm for solving the truncated Hamburger problem. Indeed, Theorems 3.1 and 3.9 lead to a simple iterative procedure for determining whether or not $\gamma := (\gamma_0, \dots, \gamma_m), \gamma_0 > 0$, admits an interpolating measure. Let $d_n :=$ det $A(n)$, $0 \leq n \leq k := [m/2]$. If some $d_n < 0$, then $A(k)$ is not positive, so γ cannot be interpolated. For the case when each $d_n \geq 0$, we consider first when m is odd, i.e., $m = 2k+1$. Let $r := 1+ \max \{n : d_n > 0\}$ (= rank $(\gamma_0, \dots, \gamma_{2k})$); then $1 \leq r \leq k+1$, and we denote $\Phi(\tilde{\gamma})$ by $\Phi = (\varphi_0, \dots, \varphi_{r-1})$. Then γ admits a representing measure if and only if $\gamma_j = \varphi_0 \gamma_{j-r} + \cdots + \varphi_{r-1} \gamma_{j-1}$ $(r \leq j \leq m)$. In this case, the roots of the generating function built from Φ give the atoms of a representing measure, and the Vandermonde equation of Proposition 3.3 determines the densities. If each $d_n \geq 0$ and m is even, we further distinguish two cases. If each $d_n > 0$ we let c be the unique scalar such that $(\gamma_{k+1}, \dots, \gamma_{2k}, c) \in$ span $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$. Now $\tilde{\gamma} := (\gamma_0, \dots, \gamma_{2k}, c)$ has an interpolating measure as determined by the above "odd" case. To construct more general solutions of the "nonsingular even" case, we may choose an arbitrary γ_{2k+1} instead of c. Finally, if m is even and some $d_n = 0$, let $\Phi = (\varphi_0, \dots, \varphi_{r-1}) := \Phi(\gamma)$. Then γ has an interpolating measure if and only if $\gamma_j = \varphi_0 \gamma_{j-r} + \cdots + \varphi_{r-1} \gamma_{j-1} (r \leq j \leq m)$. In this case, we let γ_{2k+1} be given by $\gamma_0 \gamma_{2k+1-r} + \cdots + \varphi_{r-1} \gamma_{2k}$; again, an interpolating measure for $\tilde{\gamma} := (\gamma_0, \dots, \gamma_{2k}, \gamma_{2k+1})$ can be built as before. The reader will note that by applying this procedure and the preceding existence and uniqueness theorems successively, we have a recipe for constructing every finitely atomic solution of a given truncated Hamburger moment problem.

4. The Truncated Hausdorff Moment Problem.

Let $a < b$; given $m \geq 0$ and $\gamma_0, \dots, \gamma_m \in \mathbb{R}, \gamma_0 > 0$, we seek necessary and sufficient conditions for the existence of a positive Borel measure μ such that

$$\gamma_j = \int t^j d\mu(t), \quad (0 \le j \le m), \tag{4.1}$$

and

$$\operatorname{supp} \mu \subseteq [a, b]. \tag{4.2}$$

We first consider the case $m = 2k + 1$ for some $k \geq 0$. Let $B(m) := (\gamma_{i+j+1})_{i,j=0}^{m}$, $0 \leq m \leq k$, and let $B := B(k)$.

Theorem 4.1. *(Existence, Odd Case) Let $\tilde{\gamma} = (\gamma_0, \dots, \gamma_{2k+1}), \gamma_0 > 0$, let $r := \operatorname{rank} \gamma$, and let $g_{\tilde{\gamma}}$ be as in Section 3. The following are equivalent:*

- (i) *There exists a positive Borel measure μ satisfying (4.1) and (4.2).*
- (ii) *There exists a finitely atomic representing measure μ for $\tilde{\gamma}$ satisfying (4.2).*
- (iii) *There exists an r-atomic representing measure μ satisfying (4.2) and $\operatorname{supp} \mu = \mathcal{Z}(g_{\tilde{\gamma}})$.*
- (iv) *$A(k) \geq 0$, $\mathbf{v}(k+1, k) \in \operatorname{Ran} A(k)$, and $bA(k) \geq B(k) \geq aA(k)$.*

Remark 4.2. In [**KrN**, Theorem III.2.4], Krein and Nudel'man prove that (i) is equivalent to

(v) $bA(k) \geq B(k) \geq aA(k)$.

In case $a = 0, b = 1$, the same result is given in [**Akh**, p. 74]. We know that (v) is equivalent to (iv) (because each condition is equivalent to (i)); it is easy to show directly that (v)$\Rightarrow A(k) \geq 0$, but we do not know a direct proof that (v)$\Rightarrow \mathbf{v}(k+1, k) \in \operatorname{Ran} A(k)$. Indeed, we have never seen "range conditions" of this kind in the literature of truncated moment problems.

Proof of Theorem 4.1: (i)$\Rightarrow$(iv) Suppose μ is a positive Borel measure satisfying (4.1) and (4.2). Theorem 3.1 implies that $A := A(k) \geq 0$ and $\mathbf{v}(k+1, k) \in \operatorname{Ran} A$. For $\hat{\mathbf{p}} = (p_0, \dots, p_k) \in \mathbb{C}^{k+1}$, let $p(t) = \sum_{j=0}^{k} p_j t^j$ $(t \in \mathbb{R})$. We have

$$(bA\hat{\mathbf{p}}, \hat{\mathbf{p}}) = \sum_{i,j=0}^{k} b\gamma_{i+j} p_i \overline{p_j} = \sum_{i,j=0}^{k} b \left(\int t^{i+j} d\mu \right) p_i \overline{p_j}$$

$$= b\int \left(\sum_{i=0}^{k} p_i t^i\right)\left(\sum_{j=0}^{k} \overline{p_j t^j}\right) d\mu = b\int |p|^2 d\mu \geq \int t|p|^2 d\mu$$

$$= \sum_{i,j=0}^{k} \gamma_{i+j+1} p_i \overline{p_j} = (B\hat{\mathbf{p}}, \hat{\mathbf{p}});$$

thus $bA \geq B$. A similar calculation shows that $B \geq aA$.

(iv)$\Rightarrow$(iii) The conditions $A \geq 0$ and $\mathbf{v}(k+1,k) \in \text{Ran } A$, together with Theorem 3.1, imply that there exists a representing measure μ for $\tilde{\gamma}$ such that supp $\mu = \mathcal{Z}(g_{\tilde{\gamma}})$ and card (supp μ) $=$ rank γ. Let $p(t) = p_0 + \cdots + p_k t^k$, with $\hat{p} \in \mathbb{C}^{k+1}$. Since $bA - B \geq 0$, it follows as above that $\int (b-t)|p|^2 d\mu = ((bA-B)\hat{\mathbf{p}}, \hat{\mathbf{p}}) \geq 0$. Suppose that there exists $t_0 \in$ supp μ such that $t_0 > b$. Then card (supp $\mu \cap (-\infty, b]) \leq$ (rank $\gamma) - 1 \leq k$. Thus, by Lagrange interpolation, there is a polynomial p_0 with deg $p_0 \leq k$ such that $p_0|_{(\text{supp } \mu)\cap(-\infty,b]} \equiv 0$ and $p_0(t_0) \neq 0$. Now

$$0 \leq \int (b-t)|p_0|^2 d\mu = \int_{(b,+\infty)} (b-t)|p_0(t)|^2 d\mu$$
$$\leq (b-t_0)|p_0(t_0)|^2 \mu(\{t_0\}) < 0;$$

this contradiction implies that supp $\mu \subseteq (-\infty, b]$. Similarly, $B \geq aA$ implies

$$\int (t-a)|p(t)|^2 d\mu(t) \geq 0$$

for all p, deg $p \leq k$, whence (as above), supp $\mu \subseteq [a, +\infty)$.

(iii)$\Rightarrow$(ii)$\Rightarrow$(i) Trivial. □

Theorem 4.3. *(Existence, Even Case) Let* $\gamma = (\gamma_0, \ldots, \gamma_{2k}), \gamma_0 > 0$ *and let* $r :=$ *rank* γ. *The following are equivalent:*

(i) There exists a positive Borel measure μ, *with supp* $\mu \subseteq [a,b]$, *such that*

$$\gamma_j = \int t^j d\mu \ (0 \leq j \leq 2k); \tag{4.3}$$

(ii) There exists a finitely atomic representing measure μ *for* γ *with supp* $\mu \subseteq [a,b]$;

(iii) There exists an r-atomic representing measure μ for γ with supp $\mu \subseteq [a,b]$;

(iv) $A(k) \geq 0$ and there exists $\gamma_{2k+1} \in \mathbb{R}$ such that $\mathbf{v}(k+1,k) \in$ Ran $A(k)$ and $bA(k) \geq B(k) \geq aA(k)$.

Remark 4.4. Krein and Nudel'man proved in [**KrN**, Theorem II.2.3] that (i) is equivalent to

(v) $A(k) \geq 0$ and $(a+b)B(k-1) \geq abA(k-1) + C$,

where $C := (\gamma_{i+j})_{i,j=1}^{k}$. Of course, this condition is more concrete than our condition (iv), but the proof depends on the Markov-Lukács Representation Theorem for polynomials that are nonnegative on $[a,b]$; our proof of (iv)$\Rightarrow$(i) does not use this theorem. The case when $a=0, b=1$, is treated in [**Akh**, p. 74].

Proof of Theorem 4.3: (i) $\Rightarrow$ (iv) Since supp $\mu \subseteq [a,b], \gamma_{2k+1} := \int t^{2k+1} d\mu$ is finite, and μ is a representing measure for $\tilde{\gamma} := (\gamma_0, \ldots, \gamma_{2k+1})$; (iv) now follows from Theorem 4.1 ((i) $\Rightarrow$ (iv)).

(iv)$\Rightarrow$(iii) Theorem 4.1 implies that $\tilde{\gamma}$ has a representing measure μ which is r-atomic and satisfies supp $\mu \subseteq [a,b]$; clearly, μ is also a representing measure for γ.

(iii)$\Rightarrow$(ii)$\Rightarrow$(i) Trivial. □

Remark 4.5. (Uniqueness in the Truncated Hausdorff Problem) For the singular case, if there is a solution, it is unique. Indeed, if μ and ν interpolate $\gamma = (\gamma_0, \ldots, \gamma_m)$, have supports inside $[a,b]$, and if rank $\gamma \leq k$, then μ and ν are measures on $(-\infty, +\infty)$ which interpolate γ, so from Theorem 3.8 or Theorem 3.10, $\mu = \nu$. For the nonsingular case, we do not know when uniqueness holds. The issue seems to be whether the matrix inequalities of Theorem 4.1 (iv) and Theorem 4.3 (iv) can be extended to $A(k+1)$ and $B(k+1)$ for some choices of γ_{2k+2} and γ_{2k+3}. We have not seen uniqueness for the truncated Hausdorff problem treated in the standard references.

The results of this section (and their proofs) show that when the Hausdorff moment problem is solvable, i.e., the conditions of Theorem 4.1 (iv) or Theorem 4.3 (iv) are satisfied, the algorithm at the end of Section 3 can be used to produce a solution.

5. The Truncated Stieltjes Moment Problem. We consider the truncated Stieltjes moment problem

$$\gamma_j = \int t^j d\mu \ (0 \le j \le m) \tag{5.1}$$

and

$$\text{supp } \mu \subseteq [0, +\infty). \tag{5.2}$$

Our first result follows from Theorem 4.1 (and its proof) by letting $a = 0$ and $b \to +\infty$; we omit the details.

Theorem 5.1. *(Existence, Odd Case) Let $\tilde{\gamma} = (\gamma_0, \dots, \gamma_{2k+1}), \gamma_0 > 0$, and let $r :=$ rank γ. The following are equivalent:*

(i) There exists a positive Borel measure μ satisfying (5.1) and (5.2);

(ii) There exists a finitely atomic representing measure μ for $\tilde{\gamma}$ satisfying (5.2);

(iii) There exists an r-atomic representing measure μ for $\tilde{\gamma}$ satisfying (5.2);

(iv) $A(k) \ge 0, B(k) \ge 0$ and $\mathbf{v}(k+1, k) \in$ Ran $A(k)$.

Krein and Nudel'man treat the *classical* truncated Stieltjes moment problem (1.3) [**KrN**, p. 175]. Moreover, they show that the "odd" case of the truncated Stieltjes problem (1.2) has a solution if $A(k)$ and $B(k)$ are positive and invertible; of course this is a special case of (iv)$\Rightarrow$(iii) in Theorem 5.1.

Theorem 5.2. *(Uniqueness, Odd Case) Let $\tilde{\gamma} = (\gamma_0, \dots, \gamma_{2k+1}), \gamma_0 > 0$, and assume that $\tilde{\gamma}$ has a representing measure for the Stieltjes problem, i.e., $A(k) \ge 0, B(k) \ge 0$, and $\mathbf{v}(k+1, k) \in$ Ran $A(k)$. Then $\tilde{\gamma}$ has a unique representing measure if and only if either $A(k)$ is singular, or $A(k)$ is nonsingular and $B(k)$ is singular.*

Proof: Let μ be a representing measure for $\tilde{\gamma}$. Suppose first that $A(k)$ is singular; since any representing measure for $\tilde{\gamma}$ is a solution of the "singular odd" Hamburger problem for $\tilde{\gamma}$, then Theorem 3.8 (i) shows that μ is the unique solution of this Hamburger problem, whence μ is also unique for the Stieltjes problem. Suppose next that both $A(k)$ and $B(k)$ are both positive and nonsingular. By Proposition 2.3(ii), for suitably large choices of γ_{2k+2} and $\gamma_{2k+3}, A(k+1)$ and $B(k+1)$ are positive and nonsingular. Thus, by

Theorem 5.1, the Stieltjes problem with data $(\gamma_0, \dots, \gamma_{2k+3})$ has a solution; since γ_{2k+2} is essentially arbitrary, we obtain infinitely many solutions.

Finally, we seek to show that if $A(k)$ is nonsingular and $B(k)$ is singular, then μ is the unique solution. Since $A(k)$ is nonsingular and $B(k)$ is singular, rank $B(k) = k$, and, moreover, rank $(\gamma_1, \dots, \gamma_{2k+1}) = k$. In particular, there exists unique scalars $\varphi_1, \dots, \varphi_k$, such that $\varphi_1 \mathbf{v}(1,k) + \cdots + \varphi_k \mathbf{v}(k,k) = \mathbf{v}(k+1,k)$. Theorem 2.6 ((ii)$\Rightarrow$ (i)) shows that $B(k)$ has a positive Hankel extension, and in any such extension $B(k+1), \gamma_{2k+2}$ is uniquely determined by $\gamma_{2k+2} = \varphi_1 \gamma_{k+2} + \cdots + \varphi_k \gamma_{2k+1}$ (Remark 2.7). It follows that

$$\mathbf{v}(k+1,k+1) = \varphi_1 \mathbf{v}(1,k+1) + \cdots + \varphi_k \mathbf{v}(k,k+1),$$

whence $A(k+1)$ is singular. We claim that μ has moments of all orders. Without loss of generality, assume that $\mu \neq \delta_0$, and let $d\nu(t) := t d\mu(t)$; clearly, ν is a solution of the Stieltjes problem with data $(\gamma_1, \dots, \gamma_{2k+1})$. Since $B(k)$ is singular, Lemma 3.7 implies that ν is finitely atomic, hence μ has moments of all orders. Thus, in $B(k+1)(\mu)$, we must have $\gamma_{2k+2}(\mu) := \int t^{2k+2} d\mu = \gamma_{2k+2}$. Thus μ solves the "singular even" Hamburger problem with data $(\gamma_0, \dots, \gamma_{2k+2})$. By Theorem 3.10(i), we conclude that μ is the unique solution of the Stieltjes problem. $\square$

Theorem 5.3. *(Existence, Even Case) Let* $\gamma = (\gamma_0, \dots, \gamma_{2k}), \gamma_0 > 0$, *and let* $r :=$ *rank* γ. *The following are equivalent:*

(i) There exists a positive Borel measure μ *satisfying (5.1) and (5.2);*

(ii) There exists a finitely atomic representing measure μ *for* γ *satisfying (5.2);*

(iii) There exists an r*-atomic representing measure* μ *for* γ *satisfying (5.2);*

(iv) $A(k) \geq 0$, $B(k-1) \geq 0$, *and* $\mathbf{v}(k+1, k-1) \in$ *Ran* $B(k-1)$.

Krein and Nudel'man proved that the "even" case of the truncated Stieltjes problem has a solution when $A(k)$ and $B(k-1)$ are positive and nonsingular [**KrN**, p. 175]; this is a special case of (iv)$\Rightarrow$(iii) of Theorem 5.3.

Proof: (i)$\Rightarrow$ (iv) As in the proof of Theorem 3.1 ((i)$\Rightarrow$ (vi)), the existence of a representing measure μ implies $A(k) \geq 0$. Also, if $p(t) = p_0 + \cdots + p_{k-1} t^{k-1}$, then

$$(B(k-1))\hat{\mathbf{p}}, \hat{\mathbf{p}}) = \int t |p(t)|^2 d\mu \geq 0;$$

thus $B(k-1) \geq 0$. We next show that $B(k-1)$ has a positive Hankel extension. Without loss of generality, assume that μ is not the point mass at the origin. Define a positive Borel measure ν by $d\nu(t) := td\mu(t)$.ν solves the "odd" Hamburger problem with data $(\gamma_1, \ldots, \gamma_{2k})$. By Theorem 3.1 ((i)⇒(vi)), $B(k-1)$ has a positive Hankel extension using γ_{2k}. By Proposition 2.3(i) we conclude that $\mathbf{v}(k+1, k-1) \in \operatorname{Ran} B(k-1)$.

(iv)⇒(iii) Since $B(k-1) \geq 0$ and $\mathbf{v}(k+1, k-1) \in \operatorname{Ran} B(k-1)$, we may define γ_{2k+1} recursively so that $B(k) \geq 0$ and $\mathbf{v}(k+1, k) \in \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \operatorname{Ran} A(k)$. Theorem 5.1 now implies that $\tilde{\gamma}$ has an r-atomic representing measure μ with supp $\mu \subseteq [0, +\infty)$, and μ also represents γ.
The other implications are trivial. □

Theorem 5.4. *(Uniqueness, Even Case) Let $\gamma = (\gamma_0, \ldots, \gamma_{2k}), \gamma_0 > 0$. Assume that γ has a representing measure for the Stieltjes problem, i.e., $A(k) \geq 0$, $B(k-1) \geq 0$, $\mathbf{v}(k+1, k-1) \in \operatorname{Ran} B(k-1)$. Then γ has a unique representing measure if and only if $A(k)$ is singular.*

Proof: Let μ be a representing measure for γ. If $A(k)$ is singular, μ is a solution of a "singular even" Hamburger problem, so uniqueness follows from Theorem 3.10(i). Suppose $A(k)$ is nonsingular. The hypotheses imply that $B(k-1)$ has a positive Hankel extension $B(k)$, and since $A(k)$ is nonsingular, the Stieltjes problem with data $(\gamma_0, \ldots, \gamma_{2k+1})$ has a solution (Theorem 5.1). Now γ_{2k+1} is essentially arbitrary, so γ has infinitely many representing measures. □

The results of this section (and their proofs) show that when the Stieltjes moment problem is solvable, i.e., the conditions in Theorem 5.1 (iv) or Theorem 5.3(iv) are satisfied, the algorithm at the end of Section 3 can be used to produce a solution.

6. Positive Toeplitz Matrices. In this section we give a simple description of positive Toeplitz matrices, along the lines of the description given in Section 2 for positive Hankel matrices. Let $\gamma = (\gamma_{-k}, \ldots, \gamma_{-1}, \gamma_0, \gamma_1, \ldots, \gamma_k)$ be given, assume that $\gamma_{-j} = \overline{\gamma}_j (j = 1, \ldots, k)$ and that $\gamma_0 > 0$, and consider the Toeplitz matrix

$$T(k) := \begin{pmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_{r-1} & \gamma_r & \cdots & \gamma_k \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{r-2} & \gamma_{r-1} & \cdots & \gamma_{k-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{1-r} & \gamma_{2-r} & \cdots & \gamma_0 & \gamma_1 & \cdots & \gamma_{k+1-r} \\ \gamma_{-r} & \gamma_{1-r} & \cdots & \gamma_{-1} & \gamma_0 & \cdots & \gamma_{k-r} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \gamma_{-k} & \gamma_{1-k} & \cdots & \gamma_{-k+r-1} & \gamma_{-k+r} & \cdots & \gamma_0 \end{pmatrix}.$$

By analogy with the definition of rank for Hankel matrices, we define the (*Toeplitz*) *rank* of γ as follows: If $T(k)$ is invertible, then rank $\gamma = k+1$. If $T(k)$ singular, then rank γ is the smallest integer $r, 1 \leq r$, such that $\mathbf{w}_r = \mathbf{w}(r,k) := (\gamma_r, \dots, \gamma_{r-k}) \in$ span $\{\mathbf{w}_0, \dots, \mathbf{w}_{r-1}\}$. Then $\{\mathbf{w}_0, \dots, \mathbf{w}_{r-1}\}$ is linearly independent, so there exists a unique $\mathbf{\Phi} := \mathbf{\Phi}(\gamma) = (\varphi_0, \dots, \varphi_{r-1}) \in \mathbb{C}^r$ such that $\mathbf{w}_r = \varphi_0 \mathbf{w}_0 + \cdots + \varphi_{r-1}\mathbf{w}_{r-1}$, i.e.,

$$\gamma_j = \varphi_0 \gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1} \quad (r-k \leq j \leq r).$$

The results of Section 2 show that the *Hankel rank* of a finite collection of real numbers is intimately related to the rank of its associated Hankel matrix. In particular, for

$$\gamma = (\gamma_0, \dots, \gamma_{2k}) \in \mathbb{R}^{2k+1}, \gamma_0 \neq 0, r := \text{ rank } \gamma \text{ and}$$
$$A = A_\gamma = A(k) := (\gamma_{i+j})_{i,j=0}^k,$$

Lemma 2.1(ii) implies that $A(r-1)$ is always invertible. By way of contrast, the next example shows that the analogous result does not hold for Toeplitz matrices.

Example 6.1. Let $\gamma = (x,2,1,2,1,2,1,2,x), x \neq 1$, so that

$$T := T(4) = \begin{pmatrix} 1 & 2 & 1 & 2 & x \\ 2 & 1 & 2 & 1 & 2 \\ 1 & 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 & 2 \\ x & 2 & 1 & 2 & 1 \end{pmatrix}$$

Since $x \neq 1$, $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ x & 2 & 1 \end{vmatrix} \neq 0$, so $\{\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2\}$ are linearly independent. Since $\mathbf{w}_3 = \mathbf{w}_1$, it follows that rank $\gamma = 3$. On the other hand, $T(2) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{pmatrix}$, which is singular.

Note that $T(1)$ has negative determinant, so T is not positive semidefinite. We will show in the sequel that if $T \geq 0$, then $T(r-1)$ is positive and invertible. First we establish the recursive structure of a positive singular Toeplitz matrix. Our next proposition also shows that in the context of Toeplitz matrices there is a "rank principle" similar to the one studied in Section 2 for Hankel matrices, but this time we require positivity.

Proposition 6.2. *Let $\gamma := (\gamma_{-k}, \dots, \gamma_0, \dots, \gamma_k)$ and $T(k)$ be as above, assume that $T(k)$ is singular and positive, let $r :=$ rank γ, and let $\mathbf{\Phi} := \mathbf{\Phi}(\gamma)$. Then for $r - k \leq j \leq k$,*

$$\gamma_j = \varphi_0 \gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1}; \tag{6.1}$$

moreover, rank $T(k) = r$.

Proof: Write

$$T(k) = \begin{pmatrix} T(k-1) & \mathbf{w}(k, k-1) \\ \mathbf{w}(k, k-1)^* & \gamma_0 \end{pmatrix},$$

where $\mathbf{w}(i, m) := (\gamma_{i-j})_{j=0}^m$. Since $T(k) \geq 0$, there exist scalars $c_0, \dots, c_{k-1}$ such that

$$\gamma_m = c_0 \gamma_{m-k} + \cdots + c_{k-1}\gamma_{m-1}, 1 \leq m \leq k$$

(by Lemma 2.3 (i)). By definition of r, (6.1) holds for $r - k \leq j \leq r$. Suppose that (6.1) holds for $r - k \leq j \leq p$ where $r \leq p \leq k-1$. Then

$$\begin{aligned}
&\gamma_{p+1} \\
&= c_0\gamma_{p+1-k} + \cdots + c_{k-1}\gamma_p \\
&= c_0[\varphi_0\gamma_{p+1-k-r} + \cdots + \varphi_{r-1}\gamma_{p-k}] + \cdots + c_{k-1}[\varphi_0\gamma_{p-r} + \cdots + \varphi_{r-1}\gamma_{p-1}] \\
&= \varphi_0[c_0\gamma_{p+1-k-r} + \cdots + c_{k-1}\gamma_{p-r}] + \cdots + \varphi_{r-1}[c_0\gamma_{p-k} + \cdots + c_{k-1}\gamma_{p-1}] \\
&= \varphi_0\gamma_{p-r+1} + \cdots + \varphi_{r-1}\gamma_p.
\end{aligned}$$

Thus, using an inductive argument, (6.1) holds for $r - k \leq j \leq k$. It now follows by another inductive argument that $\mathbf{w}_j \in$ span $\{\mathbf{w}_0, \dots, \mathbf{w}_{r-1}\}$ for $r \leq j \leq k$, whence rank $T(k) \leq r$. But since $\{\mathbf{w}_0, \dots, \mathbf{w}_{r-1}\}$ is linearly independent, we obtain rank $T(k) = r$. □

The next result is an analogue of Lemma 2.1 (ii).

Proposition 6.3. *Let $\gamma, T(k)$, and r be as before, and assume that $T(k) \geq 0$. Then $T(r-1)$ is nonsingular.*

Proof: We may assume $r \leq k$. We seek to prove that $\{\mathbf{w}(0, r-1), \ldots, \mathbf{w}(r-1, r-1)\}$ is linearly independent. Suppose that $c_0\mathbf{w}(0, r-1) + \cdots + c_{r-1}\mathbf{w}(r-1, r-1) = 0$. We claim that

$$c_0\mathbf{w}(0, k) + \cdots + c_{r-1}\mathbf{w}(r-1, k) = 0.$$

We prove this by induction. Suppose that

$$c_0\mathbf{w}(0, j) + \cdots + c_{r-1}\mathbf{w}(r-1, j) = 0 \tag{6.2}$$

for some $j, r-1 \leq j < k$. We'll show that

$$c_0\mathbf{w}(0, j+1) + \cdots + c_{r-1}\mathbf{w}(r-1, j+1) = 0;$$

for this, it suffices to prove that

$$c_0\gamma_{-j-1} + \cdots + c_{r-1}\gamma_{-j-1+r-1} = 0.$$

By Proposition 6.2, we know that $\gamma_m = \varphi_0\gamma_{m-r} + \cdots + \varphi_{r-1}\gamma_{m-1}$, $r-k \leq m \leq k$, and we also have $\gamma_{-m} = \overline{\gamma}_m$, $0 \leq m \leq k$. Thus

$$\begin{aligned}
&c_0\gamma_{-j-1} + \cdots + c_{r-1}\gamma_{-j-1+r-1} \\
&= c_0\overline{\gamma}_{j+1} + \cdots + c_{r-1}\overline{\gamma}_{j+1-r+1} \\
&= c_0[\overline{\varphi}_0\overline{\gamma}_{j+1-r} + \cdots + \overline{\varphi}_{r-1}\overline{\gamma}_j] + \cdots \\
&\quad + c_{r-1}[\overline{\varphi}_0\overline{\gamma}_{j+2-2r} + \cdots + \overline{\varphi}_{r-1}\overline{\gamma}_{j-r+1}] \\
&= \overline{\varphi}_0[c_0\overline{\gamma}_{j+1-r} + \cdots + c_{r-1}\overline{\gamma}_{j+2-2r}] + \cdots \\
&\quad + \overline{\varphi}_{r-1}[c_0\overline{\gamma}_j + \cdots + c_{r-1}\overline{\gamma}_{j-r+1}] \\
&= \overline{\varphi}_0[c_0\gamma_{-j-1+r} + \cdots + c_{r-1}\gamma_{-j-1+r+(r-1)}] \\
&\quad + \cdots + \overline{\varphi}_{r-1}[c_0\gamma_{-j} + \cdots + c_{r-1}\gamma_{-j+r-1}] = 0
\end{aligned}$$

by (6.2). Thus $c_0\mathbf{w}(0, j+1) + \cdots + c_{r-1}\mathbf{w}(r-1, j+1) = 0$. By induction,

$$c_0\mathbf{w}(0, k) + \cdots + c_{r-1}\mathbf{w}(r-1, k) = 0,$$

whence $c_0 = \cdots = c_{r-1} = 0$ by the definition of rank γ. Thus $\{\mathbf{w}(0, r-1), \ldots, \mathbf{w}(r-1, r-1)\}$ is linearly independent, and therefore $T(r-1)$ is nonsingular. □

We next present our structure theorem for positive Toeplitz matrices.

Theorem 6.4. *Let γ, $T(k)$, r be as before, and assume that $T(k)$ is singular and positive. Then*

(i) *$T(r-1)$ is positive and invertible, and rank $T(k) = r$;*
(ii) *$\mathbf{\Phi} := \mathbf{\Phi}(\gamma)$ satisfies*

$$\gamma_j = \varphi_0\gamma_{j-r} + \cdots + \varphi_{r-1}\gamma_{j-1}, \quad (r-k \le j \le k).$$

Conversely, if there exist r, $1 \le r \le k$, and scalars $\varphi_0, \dots, \varphi_{r-1}$, such that $T(r-1) \ge 0$ and (ii) holds, then $T(k)$ is positive.

Proof: Apply Propositions 6.2 and 6.3. For the converse, apply Lemma 2.3. □

Remark 6.5. Notice that in the above model, *every* γ_j is uniquely determined; in the Hankel case, the entry γ_{2k+2} was free (cf. Theorem 2.4). Because of this, the (linear algebraic) rank of a Toeplitz matrix equals our (recursive) rank. In the Hankel case, we only had $r \le$ rank $A(k) \le r+1$.

We now focus on extensions of positive Toeplitz matrices. In the sequel, to emphasize the dependence on the data γ, we denote $T(k)$ by T_γ.

Corollary 6.6. *Let γ and T_γ be as before, assume that $T_\gamma \ge 0$ and that T_γ is singular. Then T_γ admits a unique positive Toeplitz extension.*

Proof: First we establish uniqueness. For $\tilde{\gamma}_{k+1} \in \mathbb{C}$, let $\tilde{\gamma} = (\overline{\tilde{\gamma}}_{k+1}, \gamma_{-k}, \dots, \gamma_0, \dots, \gamma_k, \tilde{\gamma}_{k+1})$, and assume that $T_{\tilde{\gamma}}$ is positive. By Proposition 6.3 and Lemma 2.1 (i), rank $\tilde{\gamma} =$ min $\{p : T_{\tilde{\gamma}}(p)$ is singular $\}$ and, since T_γ is singular and positive it follows that rank $\tilde{\gamma} =$ rank γ. Let $r =$ rank $\gamma, 1 \le r \le k$. Thus, $T_{\tilde{\gamma}}(r-1) = T_\gamma(r-1)$ is invertible, and so $\mathbf{\Phi}(\tilde{\gamma}) = \mathbf{\Phi}(\gamma)$. Since $T_{\tilde{\gamma}} \ge 0$, Proposition 6.2 implies that

$$\tilde{\gamma}_{k+1} = \varphi_0\gamma_{k-r+1} + \cdots + \varphi_{r-1}\gamma_k. \tag{6.3}$$

Thus $\tilde{\gamma}_{k+1}$ is uniquely determined.

As for existence, let $\tilde{\gamma}_{k+1}$ be defined by (6.3), where $(\varphi_0, \dots, \varphi_{r-1}) := \mathbf{\Phi}(\gamma)$. Then Theorem 6.4 (ii), applied to $\tilde{\gamma}, T(k+1)$ and r, shows that $\mathbf{w}(k+1,k)$ satisfies $\mathbf{w}(k+1,k) = T(k)\mathbf{\Psi}$, where $\mathbf{\Psi} = (0, \dots, 0, \varphi_0, \dots, \varphi_{r-1}) \in \mathbb{C}^{k+1}$. Moreover, Theorem 6.4 (ii) also implies that

$$\begin{aligned}\mathbf{\Psi}^*\mathbf{w}(k+1,k) &= \overline{\varphi}_0\gamma_r + \cdots + \overline{\varphi}_{r-1}\gamma_1 \\ &= (\varphi_0\gamma_{-r} + \cdots + \varphi_{r-1}\gamma_{-1})^- = \overline{\gamma}_0 = \gamma_0,\end{aligned}$$

so Lemma 2.3 implies at once that $T(k+1) \geq 0$. □

Remark 6.7. An alternative proof of Corollary 6.6 can be based on Theorem 6.4 (i) and [**Ioh**, Theorem 13.2] (observe that ρ in [**Ioh**] is the usual rank).

Suppose now that T_γ is positive and invertible. Let

$$\mathbf{\Phi} = (\varphi_0, \ldots, \varphi_{k-1}) := T(k-1)^{-1}\mathbf{w}(k, k-1),$$

let

$$\tilde{\gamma}_{k+1} = \varphi_0\gamma_1 + \cdots + \varphi_{k-1}\gamma_k$$

and set

$$\tilde{\gamma} = (\overline{\tilde{\gamma}}_{k+1}, \gamma_{-k}, \ldots, \gamma_0, \ldots, \gamma_k, \tilde{\gamma}_{k+1}).$$

Proposition 6.8. *Assume that T_γ is positive and invertible, and let $\tilde{\gamma}$ be defined as above. Then $T_{\tilde{\gamma}}$ is positive and invertible.*

Proof: Since T_γ is positive and invertible, Lemma 2.3 implies that

$$\gamma_0 > \mathbf{\Phi}^*\mathbf{w}(k, k-1) = \overline{\varphi}_0\gamma_k + \cdots + \overline{\varphi}_k\gamma_1 \, (= \varphi_0\overline{\gamma}_k + \cdots + \varphi_{k-1}\overline{\gamma}_1)^-).$$

Let $\mathbf{u} = (0, \varphi_0, \ldots, \varphi_{k-1})$, so that $T_\gamma\mathbf{u} = \tilde{\mathbf{w}}(k+1, k) = (\tilde{\gamma}_{k+1}, \gamma_k, \ldots, \gamma_1)$. Then

$$\mathbf{u}^*T_\gamma\mathbf{u} = \mathbf{\Phi}^*\mathbf{w}(k, k-1) < \gamma_0 = (T_{\tilde{\gamma}})_{k+2,k+2}.$$

Since T_γ is positive and invertible, it now follows from Lemma 2.3 that $T_{\tilde{\gamma}}$ is positive and invertible. □

Corollary 6.9. *Assume that T_γ is positive and invertible. Then T_γ admits infinitely many positive and invertible Toeplitz extensions.*

Proof: For $\lambda \in \mathbb{C}$, let $\tilde{\lambda} = (\overline{\lambda}, \gamma_{-k}, \ldots, \gamma_0, \ldots, \gamma_k, \lambda)$. Since $T_{\tilde{\gamma}}$ is positive and invertible, there exists $\delta > 0$ such that if $|\lambda - \tilde{\gamma}_{k+1}| < \delta$, then $\det T_{\tilde{\lambda}} > 0$, whence the nested determinants criterion implies that $T_{\tilde{\lambda}}$ is positive and invertible. □

The problem of producing a singular positive Toeplitz extension of a nonsingular positive Toeplitz matrix is surprisingly difficult, and the solution entails Sylvester's formula for the determinant of a bordered matrix [**Ioh**, p. 98]. We now proceed to apply the model for positive Toeplitz matrices to the problem of interpolating Toeplitz data. Let $\gamma = (\gamma_{-k}, \ldots, \gamma_0, \ldots, \gamma_k) \in \mathbb{C}^{2k+1}, \gamma_0 > 0$, be given, where $\gamma_{-j} = \overline{\gamma}_j$ for all j, and let us seek a representing measure for γ, that is, a positive Borel measure μ with supp $\mu \subseteq \mathbf{T}$ and satisfying $\gamma_j = \int t^j d\mu \;\; 0 \leq j \leq k$.

Proposition 6.10. *Assume that γ admits a representing measure. Then $T_\gamma \geq 0$.*

Proof: For $\hat{\mathbf{a}} = (a_0, \ldots, a_k)$, let $a(z) = a_0 + \cdots + a_k z^k$. Then $|a(z)|^2 = \sum_{i,j=0}^{k} a_i \overline{a_j} z^{i-j}, z \in \mathbf{T}$. If μ is a representing measure for γ, then

$$0 \leq \int |a(z)|^2 d\mu = \int \sum_{i,j=0}^{k} a_i \overline{a_j} z^{i-j} d\mu = \sum_{i,j=0}^{k} a_i \overline{a_j} \gamma_{i-j} = (T(k)\hat{\mathbf{a}}, \hat{\mathbf{a}}). \quad \square$$

For the next result, we let $r :=$ rank $\gamma \leq k$, $\mathbf{\Phi} := \mathbf{\Phi}(\gamma)$, and we define $g_\gamma(z) := z^r - (\varphi_0 + \cdots + \varphi_{r-1} z^{r-1})$.

Corollary 6.11. *Assume that γ admits a representing measure μ and that rank $\gamma \leq k$. Then supp $\mu \subseteq \mathcal{Z}(g_\gamma)$.*

Proof: We define the linear functional R as follows: For a trigonometric polynomial $h(z) = a_{-k} z^{-k} + \cdots + a_0 + \cdots + a_k z^k$, let $R(h) := a_{-k}\gamma_{-k} + \cdots + a_0\gamma_0 + \cdots + a_k\gamma_k$. If $a(z) = a_0 + \cdots + a_k z^k, b(z) = b_0 + \cdots + b_k z^k$, $\hat{\mathbf{a}} := (a_0, \ldots, a_k)$, and $\hat{\mathbf{b}} := (b_0, \ldots, b_k)$, we then have

$$(T(k)\hat{\mathbf{a}}, \hat{\mathbf{b}}) = \int a(z)\overline{b(z)} d\mu = \int \sum a_i \overline{b_j} z^{i-j} d\mu = \sum a_i \overline{b_j} \gamma_{i-j} = R(a\bar{b}).$$

Also,

$$R(g_\gamma) = \gamma_r - (\varphi_0 + \cdots + \varphi_{r-1}\gamma_{r-1}) = 0,$$

and, from the definitions of rank γ and g_γ,

$$R(g_\gamma \bar{z}^j) = 0 \quad (r - k \leq j \leq k).$$

Thus, g_γ is R-orthogonal to $z^{k-r}, \ldots, z, 1, z^{-1}, \ldots, z^{-k}$, which implies that $S(g_\gamma \overline{g_\gamma}) = 0$, and this in turn gives $(T(k)\hat{\mathbf{g}}_\gamma, \hat{\mathbf{g}}_\gamma) = 0$. Therefore, $\int |g_\gamma|^2 d\mu = 0$, or $g_\gamma = 0$ a.e. $[\mu]$, so that supp $\mu \subseteq \mathcal{Z}(g_\gamma)$. $\square$

We are now ready to solve the truncated trigonometric moment problem. The following result is very similar to [**AhK**, Theorem I.I.12] and to [**Ioh**, p. 211]. Unlike these authors, we avoid Gaussian quadrature in treating the singular case, and instead we use the recursive model of Theorem 6.4.

Theorem 6.12. *Let* $\gamma = (\gamma_{-k}, \ldots, \gamma_0, \ldots, \gamma_k) \in \mathbb{C}^{2k+1}$, $\gamma_0 > 0, \gamma_{-j} = \overline{\gamma}_j$, *be given. Then there exists a representing measure* μ *for* γ *if and only if* $T_\gamma \geq 0$. *In this case,* μ *can be chosen to have* r *atoms, where* $r :=$ *rank* γ.

Proof. Case 1: ($r \leq k$) If γ has a representing measure, the conclusion follows from Proposition 6.10 and Corollary 6.11. Conversely, if $T_\gamma \geq 0$, we apply Theorem 6.4, [**Ioh**, p. 210], and the method of the proof of Proposition 3.3 and Corollary 3.4 to produce a representing measure with r atoms.

Case 2. ($r = k + 1$) As above, if γ has a representing measure, then $T_\gamma \geq 0$. Assuming that T_γ is positive and invertible, we can apply [**Ioh**, Theorem 13.1 and Remark 1] to obtain a singular Hermitian extension $T(k+1)$ of T_γ. By Lemma 2.3, we see at once that $T(k+1) \geq 0$. Moreover, $T(k + 1)$ is singular, so we can apply Case 1 to produce a representing measure with r atoms. □

Remark 6.13. As in Section 3, it is not difficult to show that in the nonsingular case there are infinitely many solutions. For the singular case, we claim that there exists a unique solution. Let μ and ν be two interpolating measures. Observe first that μ and ν have moments of all orders, and we can then define $T_\mu(j)$ and $T_\nu(j)$ for all $j \geq 0$. However, it follows from Corollary 6.6 that $T_\mu(j) = T_\nu(j)$ for all $j \geq 0$. Thus, $\int z^j d\mu = \int z^j d\nu$ for all $j \geq 0$. The F. and M. Riesz Theorem implies that Borel measures on $\mathbf{T}$ are determined by their trigonometric power moments, so we must have $\mu = \nu$. (For the class of finitely atomic measures, uniqueness is proved in [**AhK**, Theorem I.1.12].)

REFERENCES

[**AhK**] N. I. Ahiezer and M. Krein, *Some Questions in the Theory of Moments*, Transl. Math. Monographs **2** (1962), American Math. Soc., Providence.

[**Akh**] N. I. Akhiezer, *The Classical Moment Problem*, Hafner Publ. Co. (1965), New York.

[**Atz**] A. Atzmon, *A moment problem for positive measures on the unit disc*, Pacific J. Math. **59** (1975), 317-325.

[**Ber**] C. Berg, *The multidimensional moment problem and semigroups*, Proc. Symposia Appl. Math. **37** (1987), 110-124.

[**BeM**] C. Berg and P. H. Maserick, *Polynomially positive definite sequences*, Math. Ann. **259** (1982), 487-495.

[**Con**] J. B. Conway, *Subnormal Operators*, Pitman Publ. Co. (1981), London.

[CuF1] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, preprint yr1990.

[Fug] B. Fuglede, *The multidimensional moment problem*, Expo. Math. **1** (1983), 47-65.

[Ioh] I. S. Iohvidov, *Hankel and Toeplitz Matrices and Forms: Algebraic Theory*, Birkhäuser Verlag (1982), Boston.

[KoL] I. Koltracht and P. Lancaster, *A definiteness test for Hankel matrices and their lower submatrices*, Computing **39** (1987), 19-26.

[KrN] M. Krein and A. Nudel'man, *The Markov Moment Problem and Extremal Problems*, Transl. Math. Monographs **50** (1977), American Math. Soc., Providence.

[Lan] H. Landau, ed., *Moments in Mathematics*, Proc. Symposia Appl. Math. **37** (1987), Amer. Math. Soc., Providence.

[McG] J. McGregor, *Solvability criteria for certain N-dimensional moment problems*, J. Approx. Th. **30** (1980), 315-333.

[Nar] F. J. Narcowich, *R-operators II. On the approximation of certain operator-valued analytic functions and the Hermitian moment problem*, Indiana Univ. Math. J. **26** (1977), 483-513.

[Put] M. Putinar, *A two-dimensional moment problem*, J. Funct. Anal. **80** (1988), 1-8.

[Sar] D. Sarason, *Moment problems and operators in Hilbert space*, Moments in Math., Proc. Symposia Applied Math **37** (1987), 54-70, Amer. Math. Soc., Providence.

[Sch] K. Schmüdgen, *The K-moment problem for semi-algebraic sets*, Math. Ann. **289** (1991), 203-206.

[ShT] J. A. Shohat and J. D. Tamarkin, *The Problem of Moments*, Math. Surveys I, American Math. Soc. (1943), Providence.

[Smu] J. L. Smul'jan, *An operator Hellinger integral (Russian)*, Mat. Sb. **91** (1959), 381-430.

[Sta] J. Stampfli, *Which weighted shifts are subnormal*, Pacific J. Math. **17** (1966), 367-379.

[Sto] M. H. Stone, *Linear Transformations in Hilbert Space*, Amer. Math. Soc. (1932), New York.

Raúl E. Curto
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242
curto@math.uiowa.edu

Lawrence A. Fialkow
Dept. of Math. and Comp. Science
SUNY at New Paltz
New Paltz, NY 12561
fiaℓkowℓ@snynewvm.bitnet

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