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ENTROPY OF DYNAMICAL SYSTEMS AND PERTURBATIONS OF OPERATORS, II

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1. Introduction. This paper is a sequel to our note [4]. In [4], for a family of unitary operators normalizing a von Neumann algebra, we introduced an invariant H_P (and a variant of it, $\tilde{H}_P$) called the perturbation-theoretic entropy. In the case of L^∞ of a probability measure space and of the unitary operator induced by a measure preserving ergodic transformation T, we showed that H_P (and hence also $\tilde{H}_P$) is in an interval $[C_1 h(T),\ C_2 h(T)]$ for some constants C_1, C_2 and $h(T)$ denoting the entropy of T.

For the class of Bernoulli shifts we prove here that $\tilde{H}_P$ is actually proportional to $h(T)$. This confirms our belief that the perturbation-theoretic entropy is an invariant equivalent to entropy. This also suggests that $\tilde{H}_P$ may have some advantages over H_P.

Note also that if we replace the assumption that the measure is invariant under T by the weaker assumption that the measure be only quasiinvariant, there is still a canonical unitary operator implementing the automorphism of $L^\infty(X)$ induced by T. Hence $\tilde{H}_P$ provides an entropy-like invariant for transformations with quasiinvariant measure. It is an open problem whether this extension is nontrivial, i.e. whether there exist such transformations without an equivalent invariant measure, for which the value of $\tilde{H}_P$ is finite and non-zero. In section 5 we obtain certain results which may lead to an example of such a transformation. We also prove that a non-atomic dissipative transformation has infinite perturbation-theoretic entropy. In particular an affirmative answer to the preceding problem would involve a conservative transformation.

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More generally, it is a consequence of the theory of the standard form of von Neumann algebras, that automorphisms have canonical unitary implementations in the standard form. This implies that the perturbation-theoretic entropy is defined for automorphisms of von Neumann algebras, without assuming the existence of invariant states.

For the shift-automorphism of the II_1 factor of a free group on generators indexed by $\mathbb{Z}$ the Connes-Stormer entropy is either 0 or ∞, but it is an open question which of these two is the actual value (as I have learned from Erling Stormer). We prove here that the perturbation-theoretic entropy of the free shift automorphism is ∞.

2. Preliminaries. Throughout this paper we will use definitions and notations of [4] with some amendments. This section is devoted to these amendments.

The definitions of the perturbation-theoretic entropy and of its variant, as well as the results of section 3 of [4], are given under hyperfiniteness conditions. It is natural to expect that at a more advanced stage in the study of the perturbation-theoretic entropy hyperfiniteness assumptions should play a role, however in the definitions and results of section 3 of [4], the hyperfiniteness assumptions are not necessary. We will therefore remove these unnecessary assumptions. We will use this in the case of the free shift automorphism.

As we mentioned in the introduction here, using the theory of the standard form, the perturbation-theoretic entropy for automorphisms of W^*-algebras can be defined also in the absence of invariant states. In more detail, let M be a W^*-algebra and let Q be a set of automorphisms of M. By [1] there is a standard form $(M, \mathcal{H}, \mathcal{J}, P)$ for M and this standard form is unique up to unitary equivalence. In particular for every $\alpha \in Q$ there is a unitary operatory $u(\alpha)$ on $\mathcal{H}$ which implements α, i.e. $u(\alpha)xu(\alpha)^* = \alpha(x)$ for $x \in M$. By the uniqueness of the standard form up to unitary equivalence, it follows that $H_P(u(Q), M)$ and $\tilde{H}_P(u(Q), M)$ depend only on Q and M.

Definition 2.1. If M is a von Neumann algebra and Q a set of automorphisms the perturbation-theoretic entropy $H_P(Q, M)$ and its variant $\tilde{H}_P(Q, M)$ are defined to be $H_P(u(Q), M)$ and, respectively, $\tilde{H}_P(u(Q), M)$.

In the case where M is $L^\infty(X, \mu)$, the standard form is on $L^2(X, \mu)$ with $\mathcal{J}f = \overline{f}$ and P the positive functions in $L^2(X, \mu)$. If α_T is the automorphism induced by a transformation T of X for which μ is quasi-invariant,

$$(u(\alpha_T)f)(\xi) = f(T^{-1}\xi)\left(\frac{d\mu T^{-1}}{d\mu}(\xi)\right)^{\frac{1}{2}}.$$

In particular, the perturbation-theoretic entropy $H_P(\alpha_T, L^\infty(X, \mu))$ is an invariant of the transformation T with quasiinvariant measure μ. Note that this invariant does not depend on the choice of μ within a given equivalence class.

Before closing this section let us point out that the assumption in [4], that the Hilbert spaces we consider are separable and that the von Neumann algebras have separable predual, will be in force throughout the present paper. The adaptation to the nonseparable case, being quite routine, is omitted here.

3. Bernoulli shifts. We prove in this section that for Bernoulli shifts the perturbation-theoretic invariant $\tilde{H}_P$ is proportional to the Kolmogorov-Sinai entropy. We will use Sinai's theorem and the following general fact:

Proposition 3.1. *If u is a unitary operator normalizing a commutative von Neumann algebra A then $\tilde{H}_P(u^n, A) = |n|\tilde{H}_P(u, A)$.*

Proof: The inequality $\leq$ and the fact that the left-hand side is not changed when n is replaced by $-n$ has been noted earlier for general A (Remark 3.5 in [4]). It will suffice to prove the inequality $\geq$ when $n > 0$. This in turn will follow if, given $B \in \mathcal{F}(A)$, we find $C \in \mathcal{F}(A)$ such that $k_\infty^-(u^n|C) \geq nk_\infty^-(u|B)$.

Since A is commutative, we may define $C = \bigvee_{0\leq k\leq n-1} u^{-k}Bu^k$. There are $X_m \in \mathcal{R}_1^+ \cap C'$ such that $X_m \uparrow I$ as $m \to \infty$ and

$$k_\infty^-(u^n|C) = \lim_{m\to\infty} |u^n X_m u^{-n} - X_m|_\infty^-.$$

Let $Y_m = n^{-1}\sum_{0\leq k\leq n-1} u^k X_m u^{-k}$. We have $Y_m \uparrow I$ and $Y_m \in \mathcal{R}_1^+ \cap \left(\bigcap_{0\leq k\leq n-1} u^k C u^{-k}\right)' \subset \mathcal{R}_1^+ \cap B'$. We have:

$$\begin{aligned} k_\infty^-(u^n|C) &= \lim_{m\to\infty} |u^n X_m u^{-n} - X_m|_\infty^- \\ &= \lim_{m\to\infty} n|uY_m u^{-1} - Y_m|_\infty^- \geq k_\infty^-(u|B) \end{aligned}$$

Q.E.D.

Corollary 3.2. *If α is an automorphism of an abelian von Neumann algebra A then $\tilde{H}_P(\alpha^n, A) = |n|\tilde{H}_P(\alpha, A)$.*

With the notations of Theorem 4.1 of [4] we have the following theorem.

Theorem 3.3. *There is a universal constant $\gamma \in [2^{-1}, 18]$ such that $\tilde{H}_P(U_T, L^\infty(X)) = \gamma h(T)$ whenever T is a Bernoulli shift.*

Proof: Combining Theorem 4.1 of [4] and the fact that $H_P \leq \tilde{H}_P \leq 3H_P$ we have:

$$2^{-1}h(T) \leq \tilde{H}_P(U_T,\ L^\infty(X)) \leq 18h(T).$$

Let T_1 and T_2 be Bernoulli shifts; by Sinai's theorem $h(T_1) \leq h(T_2)$ implies that T_1 is a factor of T_2 and hence in view of Remark 3.5 of [4], we have that

$$\tilde{H}_P(U_{T_1},\ L^\infty(X_1)) \leq \tilde{H}_P(U_{T_2},\ L^\infty(X_2)).$$

Hence there is an increasing function $\phi : [0, \infty] \to [0, \infty]$ such that $2^{-1}t \leq \phi(t) \leq 18t$ and

$$\tilde{H}_P(U_T, L^\infty(X)) = \phi(h(T))$$

if T is a Bernoulli shift.

If T is a Bernoulli shift, then $T^n (n > 0)$ is also a Bernoulli shift of entropy $nh(T)$; hence by Proposition 3.1 we have

$$\begin{aligned}\phi(nh(T)) = \phi(h(T^n)) &= \tilde{H}_P(U_{T^n},\ L^\infty(X)) \\ &= n\tilde{H}_P(U_T,\ L^\infty(X)) \\ &= n\phi(h(T)).\end{aligned}$$

Since $h(T)$ may be any number in $[0, \infty]$ we infer $\phi(nt) = n\phi(t)$ for all $t \in [0, \infty]$. It follows that $\phi(qt) = q\phi(t)$ for all positive rational numbers q. Combining this with the fact that ϕ is increasing we easily get that $\phi(t) = \gamma t$. Q.E.D.

4. The free shift. Let G be a free group on generators $g_n (n \in \mathbb{Z})$. Let $L(G)$ be the von Neumann algebra $(\lambda(G))''$ where λ is the left regular representation of G. Further, let α be the free shift automorphism, i.e., $\alpha \in \mathrm{Aut}(L(G))$ is such that $\alpha(\lambda(g_n)) = \lambda(g_{n+1})$.

Proposition 4.1. *Let α be the free shift automorphism on $L(G)$. Then $H_P(\alpha, L(G)) = +\infty$.*

Proof: $L(G)$ is in standard form on $\ell^2(G)$ and the unitary implementing α is $u \in B(\ell^2(G))$ which acts on the canonical basis of $\ell^2(G)$ (indexed by G) as the free shift automorphism of the group G. We have

$$H_P(\alpha,\ L(G)) = H_P(u,\ L(G)).$$

Further, let $v \in (\lambda(g_0))''$ be a unitary operator with $v^n = I$ and such that the trace of the spectral measure of v is Haar measure on the group of n-th roots of unity. We have

$$H_P(u,\ L(G)) \geq k_\infty^-(u, v, \dots, v^{n-1}) \geq \frac{1}{2} k_\infty^-(uv, \dots, uv^{n-1}).$$

If ξ is the canonical trace-vector in $\ell^2(G)$ for $L(G)$, then:

$$(uv^{P_1})(uv^{P_2}) \dots (uv^{P_k})\xi = \alpha^1(v^{P_1})\alpha^2(v^{P_2}) \dots \alpha^k(v^{P_k})\xi$$

and it is easily seen that $(uv, \dots, uv^{n-1})$ and ξ satisfy the assumptions of Proposition 2.3 in [4]. It follows that

$$H_P(u,\ L(G)) \geq \frac{1}{2} \log(n-1).$$

Since u is arbitrary this proves the proposition. Q.E.D.

5. Non-singular transformations. In this section we present some facts concerning H_P for general non-singular transformations which may lead to the construction of examples without invariant measure where the perturbation-theoretic entropy is finite and non-zero. We also prove that non-atomic dissipative transformations have infinite H_P.

We begin with a generalization of the upper bound part of Theorem 4.1 in [4].

Proposition 5.1. *Let T be a non-singular invertible transformation of the probability measure space $(X, \sum, \mu)$, such that the Radon-Nikodym derivative $\frac{d\mu o T}{d\mu}$ is a simple function (i.e. takes only finitely many values)* *and let* $A = sup\left| \log \frac{d\mu \circ T}{d\mu}\right|$. *Assume moreover there is a constant* C *such that for every finite partition* η *we have*

$$\limsup_{\substack{p>0,\, q>0\\ p+q\to\infty}} \frac{1}{p+q+1} I\left(\bigvee_{k=-q}^{p} T^k\eta\right)(x) \le C$$

for a.e. x *in* X. *Then we have*

$$\tilde{H}_P(U_T,\ L^\infty(X)) \le (2C + 6A\ log2)$$

where $U_T = u(\alpha_T)$.

Proof: Let $B \in \mathcal{F}(L^\infty(X))$ and let β be the corresponding partition. In view of the assumption on the Radon-Nikodym derivative we may assume, after enlarging B, that $\frac{d\mu\circ T}{d\mu} \in B$ and $\frac{d\mu\circ T^{-1}}{d\mu} \in B$.

If $p < q$ and $\lambda > 0$ let

$$\omega_{p,q}(\lambda) = \{x \in X | (1+q-p)^{-1} I(\bigvee_{j=p}^{q} T^j\beta)(x) < \lambda\}$$

Let further

$$\omega_N(\lambda = \cap_{\substack{p\le -N\\ q\ge N}} \omega_{p,q}(\lambda).$$

It is easily seen that

$$T^\alpha \omega_N(\lambda) \subset \omega_{N+1}\left(\lambda + \frac{A}{2N+3}\right)$$

if $\alpha \in \{-1,0,1\}$. Our assumptions imply $\lim_{N\to\infty} \mu(\omega_N(C+\epsilon)) = 1$ if $\epsilon > 0$. For each $n \in \mathbb{N}$ we define projections $P_n(\lambda)$ and R_N where $P_n(\lambda)$ is multiplication by the characteristic function of $\omega_N(\lambda)$, while R_N is the orthogonal projection onto the subspace of $L^2(X)$ consisting of function constants on the atoms of $T^{-N}\beta \vee \cdots \vee T^N\beta$.

If $\alpha \in \{-1,0,1\}$ we have

$$U_T^\alpha P_N(\lambda) U_T^{-\alpha} \le P_{N+1}\left(\lambda + \frac{A}{2N+3}\right).$$

Note that the assumptions

$$\frac{d\mu \circ T}{d\mu} \in B, \quad \frac{d\mu \circ T^{-1}}{d\mu} \in B$$

imply that if $\alpha \in \{-1, 0, 1\}$ then

$$U_T^{\alpha} R_N U_T^{-\alpha} \leq R_{N+1}.$$

We define

$$Y_N = N^{-1} \sum_{k=N+1}^{2N} P_k \left(C + A \sum_{t=N+1}^{k} (2t+3)^{-1} \right),$$

$$Z_n = N^{-1} \sum_{k=N+1}^{2N} R_k.$$

We have $P_N(C+\epsilon) \uparrow I$ and $R_N \uparrow Q(\beta)$, where $Q(\beta)$ is the projection onto the $\bigvee_{j=-\infty}^{\infty} T^j \beta$-measurable functions. This implies $Y_n \xrightarrow{s} I$ and $Z_N \uparrow Q(\beta)$ as $N \to \infty$.

We have

$$\begin{aligned} U_T Y_N U_T^* - Y_n &= N^{-1} \sum_{k=N+1}^{2N} \left(U_T P_k \left(\epsilon + C + \sum_{t=N+1}^{k} \frac{A}{2t+3} \right) U_T^{-1} \right. \\ &\qquad \left. - P_k \left(\epsilon + C + \sum_{t=N+1}^{k} \frac{A}{2t+3} \right) \right) \\ &\leq N^{-1} \sum_{k=N+1}^{2N} \left(P_{k+1} \left(\epsilon + C + \sum_{t=N+1}^{k} \frac{A}{2t+3} \right) \right. \\ &\qquad \left. - P_k \left(\epsilon + C + \sum_{t=N+1}^{k} \frac{A}{2t+3} \right) \right) \\ &\leq N^{-1} P_{2N+1} \left(\epsilon + C + \sum_{t=N+1}^{2N+1} \frac{A}{2t+3} \right), \end{aligned}$$

$$U_TY_nU_T^* - Y_n \geq N^{-1}\left(\sum_{k=N+1}^{2N}\left(P_{k-1}\left(\epsilon + C + \sum_{t=N+1}^{k-1}\frac{A}{2t+3}\right)\right.\right.$$
$$\left.\left. - P_k\left(\epsilon + C + \sum_{t=N+1}^{k}\frac{A}{2t+3}\right)\right)\right)$$
$$\geq -N^{-1}P_{2N+1}\left(\epsilon + C + \sum_{t=N+1}^{2N+1}\frac{A}{2t+3}\right).$$

We have:

$$\sum_{t=N+1}^{2N+1}\frac{1}{2t+3} \leq \frac{1}{2}\left(\ell n\, 2 + \ell n\frac{N+2}{N+3/2}\right) = c_N,$$

so that denoting $P'_{2N+1} = P_{2N+1}(\epsilon + C + Ac_N)$, we have

$$-N^{-1}P'_{2N+1} \leq U_TY_NU_T^* - Y_N \leq N^{-1}P'_{2N+1}.$$

It follows that

$$U_TY_NU_T^* - Y_N \leq N^{-1}P'_{2N+1}K_NP'_{2N+1}$$

with $\|K_N\| \leq 1$.

Similarly

$$U_TZ_NU_T^* - Z_N = N^{-1}R_{2N+1}L_NR_{2N+1}$$

where $\|L_N\| \leq 1$.

Let $D_N = Z_NY_NZ_N$. We have $D_N \in \mathcal{R}_1^+$, $D_N \xrightarrow{s} Q(\beta)$ and $[D_N, B] = 0$. We have:

$$\begin{aligned}
|[D_N, U_T]|_\infty^- &= |U_TD_NU_T^* - D_N|_\infty^- \\
&\leq |(U_TZ_NU_T^* - Z_N)Y_N|_\infty^- + |Z_N(U_TY_NU_T^* - Y_N)|_\infty^- \\
&\quad + |Y_N(U_TZ_NU_T^* - Z_N)|_\infty^- \\
&\leq N^{-1}\left(2|R_{2N+1}P'_{2N+1}|_\infty^- + |P'_{2N+1}R_{2N+1}|_\infty^-\right) \\
&\leq 3N^{-1}\left(\log \operatorname{rank} R_{2N+1}P'_{2N+1} + \delta\right)
\end{aligned}$$

where δ is a constant independent of N.

The rank of $R_{2N+1}P'_{2N+1}$ equals the number of atoms of $T^{-2N-1}\beta \vee \cdots \vee T^{2N+1}\beta$ which meet $\omega_{2N+1}(\epsilon + C + Ac_N)$. Since the measure of such an atom is at least $e^{-(C+Ac_N+\epsilon)(4N+3)}$, it follows their total number does not exceed $e^{(4N+3)(\epsilon+C+c_NA)}$. This in turn implies

$$\lim_{N\to\infty} |[D_N, U_T]|_\infty^- \leq 12C + 6A \log 2 + 12\epsilon.$$

Replacing D_N by $Q(\beta)D_NQ(\beta)$ gives

$$k_\infty^- \left((U_T|Q(\beta)L^2(X))|(B|Q(\beta)L^2(X))\right) \leq 12C + 6A \log 2.$$

Applying this to $B^{(k)} \in \mathcal{F}(L^\infty(X))$ such that $B^{(k)} \supset B$ and $Q(B^{(k)}) \uparrow I$ we easily get

$$k_\infty^-(U_T|B) \leq 12C + 6A \log 2$$

and hence the desired conclusion. Q.E.D.

Proposition 5.2. *Let T_j be transformations of probability Lebesgue measure spaces $(X_j, \sum_j, \mu_j)(j = 1, 2)$ such that T_1 is non-singular while T is measure-preserving and ergodic. Then we have*

$$\tilde{H}_P(U_{T_1\times T_2}, L^\infty(X_1 \times X_2)) \geq \frac{1}{2}h(T_2).$$

Proof: Note that it will suffice to prove that if T_2 is a Bernoulli shift with weights $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$ then

$$\tilde{H}_P(U_{T_1\times T_2}, L^\infty(X_1 \times X_2)) \geq \frac{1}{2} \log(n-1).$$

The general case will then follow using Proposition 3.1 and Sinai's theorem. Indeed, if for an integer M we have $Mh(T_2) \geq \log n$ then by Sinai's theorem T_2^M has a factor isomorphic to a Bernoulli shift with weights $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Moreover, by Proposition 3.1,

$$\begin{aligned} M\tilde{H}_P(U_{T_1\times T_2}, L^\infty(X_1 \times X_2)) &= \tilde{H}_P(U_{T_2^M\times T_2^M}, L^\infty(X_1 \times X_2)) \\ &\geq \tilde{H}_P(U_{T_1^M\times T_2^M}, L^\infty(X_1 \times X_2)) \\ &\geq \frac{1}{2} \log(n-1) \end{aligned}$$

where we used Proposition 3.3 of [4] for the last inequality. Hence for $M \to \infty$ we get the desired result.

Assume now T_2 is the Bernoulli shift. In the proof of the lower bound in Theorem 4.1 [4] using Lemma 4.2 [4], we proved the existence of a unitary element $V \in A$ where $A \in \mathcal{F}(L^\infty(X_2))$, such that there is a unit vector $\xi \in L^2(X_2)$ so that if $W_j = V^j U_{T_2}(1 \le j \le n-1)$ we have that the vectors

$$\{W_{j_1} \dots W_{j_m}\xi | m \ge 0, 1 \le j_k \le n-1, \ 1 \le k \le m\}$$

are pairwise orthogonal.

Let $\zeta \in L^2(X_1)$ be a unit vector and let $\eta = \zeta \otimes \xi$. Then the vectors

$$\{(U_{T_1} \otimes W_{j_1}) \dots (U_{T_1} \otimes W_{j_m})\eta | m \ge 0, 1 \le j \le n-1, \ 1 \le k \le m\}$$

are pairwise orthogonal. By Proposition 2.3 in [4], we have:

$$\begin{aligned} k_\infty^-(I \otimes V, \dots I \otimes V^{n-1}, U_{T_1} \otimes U_{T_2} &\ge \frac{1}{2} k_\infty^-(T_{T_1} \otimes W_1, \dots, U_{T_1} \otimes W_{n-1}) \\ &\ge \frac{1}{2} \log(n-1) \end{aligned}$$

and hence

$$H_P(U_{T_1 \times T_2}, L^\infty(X_1 \times X_2)) \ge \frac{1}{2} \log(n-1).$$

Q.E.D.

Corollary 5.3. *Let T be a non-singular transformation of a non-atomic probability measure-space $(X, \sum, \mu)$. If T is dissipative, then $H_P(U_T, L^\infty(X))$ $= \infty$.*

Proof: Indeed $T = T_1 \times T_2$ with T_1, T_2 like in the preceding proposition and with $h(T_2) = \infty$. Q.E.D.

Remark 5.4 It is an open problem whether there exists a nonsingular T without invariant measure such that

$$0 < H_P(U_T, L^\infty(X)) < \infty.$$

Propositions 5.1 and 5.2 may provide a way towards finding such a T. It is however not clear whether the conditions in Proposition 5.1 can be satisfied by a transformation for which there is no equivalent invariant measure. Also checking the assumption on the lim sup of the information function appearing in Proposition 5.1 for a given T seems to represent a serious difficulty. Perhaps the results in [3] and [2] may provide some inspiration on how to deal with this question.

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