Lecture 23 Section 6.4 The Centroid of a Region; Pappus' Theorem on Volumes

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Test 3

- Test 3: Dec. 4-6 in CASA
- Material - Through 6.3.

Final Exam

- Final Exam: Dec. 14-17 in CASA

You Might Be Interested to Know ...

- I will replace your lowest test score with the percentage grade from the final exam (provided it is higher).
- I will give an A to anyone who receives 95% or above on the final exam.
- I will give a passing grade to anyone who receives at least 70% on the final exam.

Quiz 1

What is today?

a. Monday

b. Wednesday

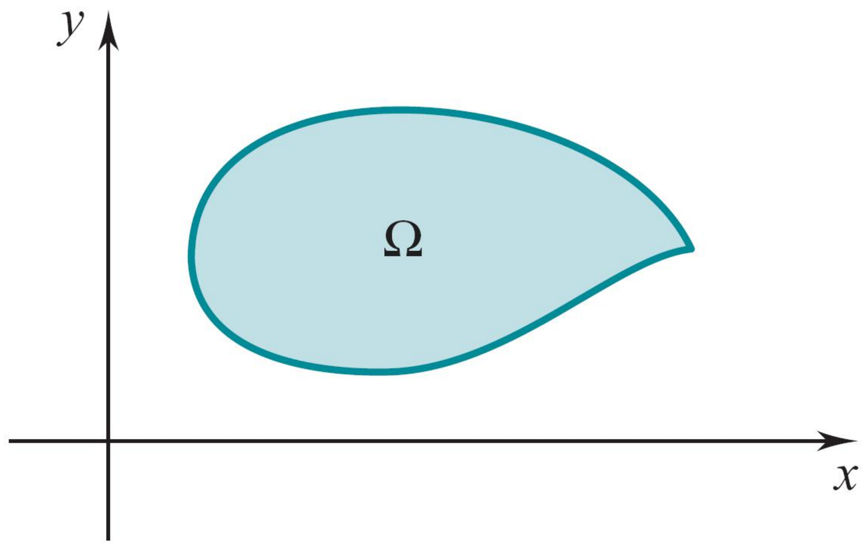
c. Friday

d. None of these

1 The Centroid of a Region; Pappus' Theorem on Volumes

1.1 The Centroid of a Region

The Centroid of a Region



The *center of mass* of a plate of constant mass density depends only on its shape Ω and falls on a point $(\bar{x}, \bar{y})$ that is called the *centroid*.

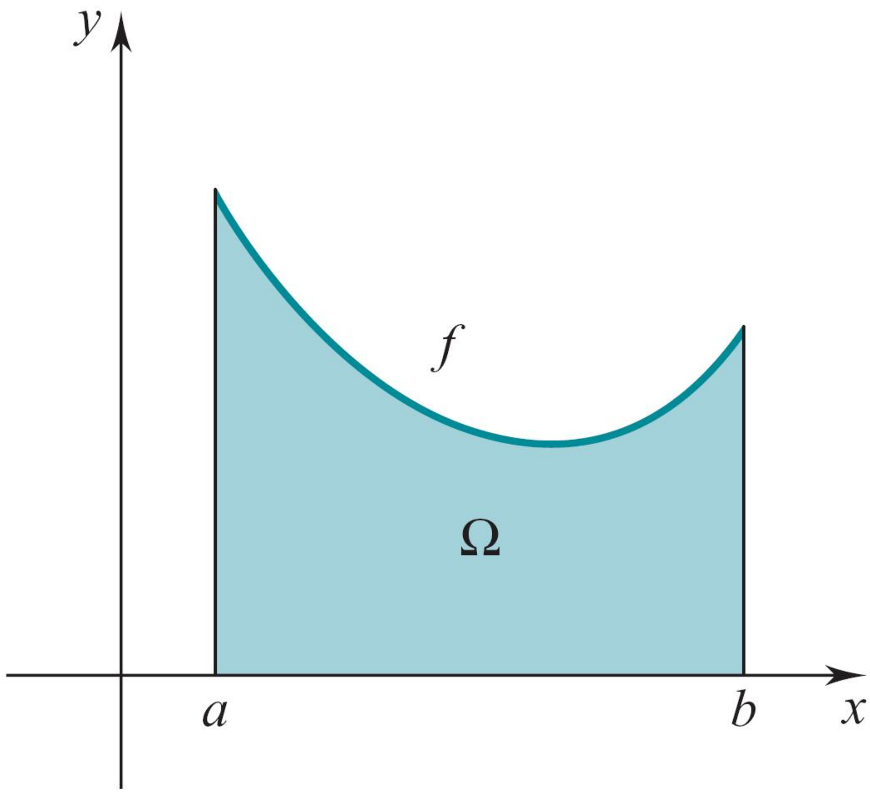
Principle 1: Symmetry
If the region has an axis of symmetry, then the centroid $(\bar{x}, \bar{y})$ lies somewhere along that axis. In particular, if the region has a center, then the center is the centroid.

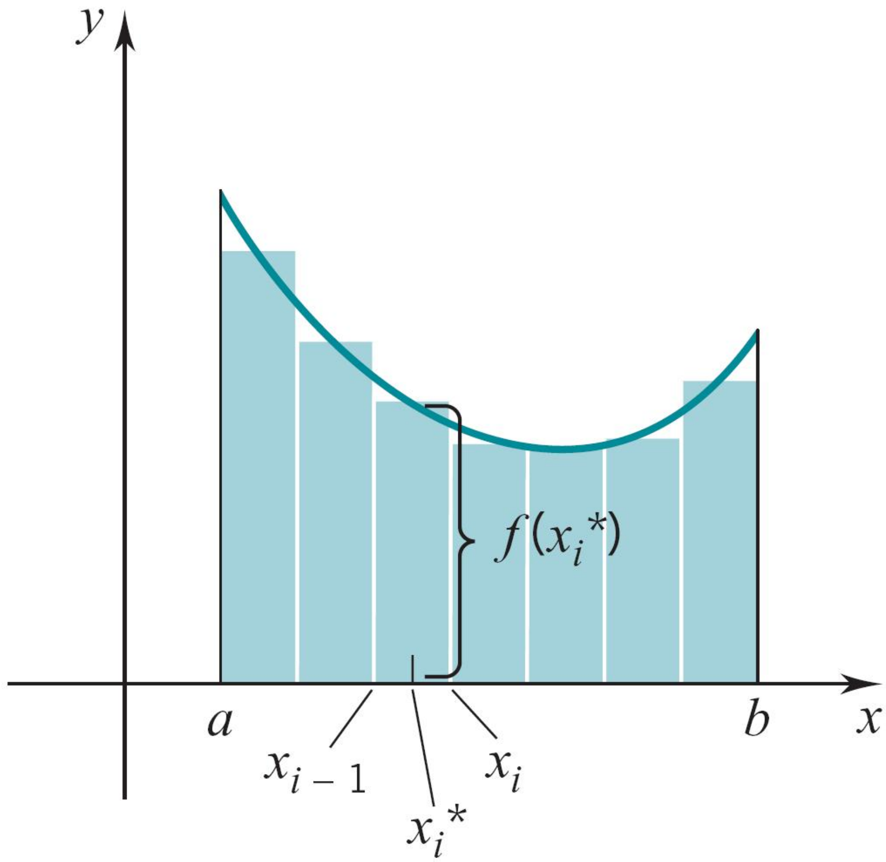
The Centroid of a Region: Principle of Additivity

Principle 2: Additivity
If the region, having area A, consists of a finite number of pieces with areas A_1, $\cdots$, A_n and centroids $(\bar{x}_1, \bar{y}_1)$, $\cdots$, $(\bar{x}_n, \bar{y}_n)$, then

$$\bar{x}A = \bar{x}_1 A_1 + \cdots + \bar{x}_n A_n,$$
$$\bar{y}A = \bar{y}_1 A_1 + \cdots + \bar{y}_n A_n.$$

Centroid of a Region below the graph of f (≥ 0)

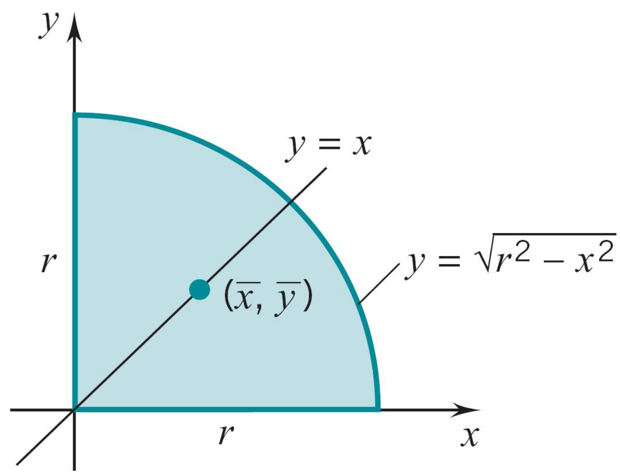




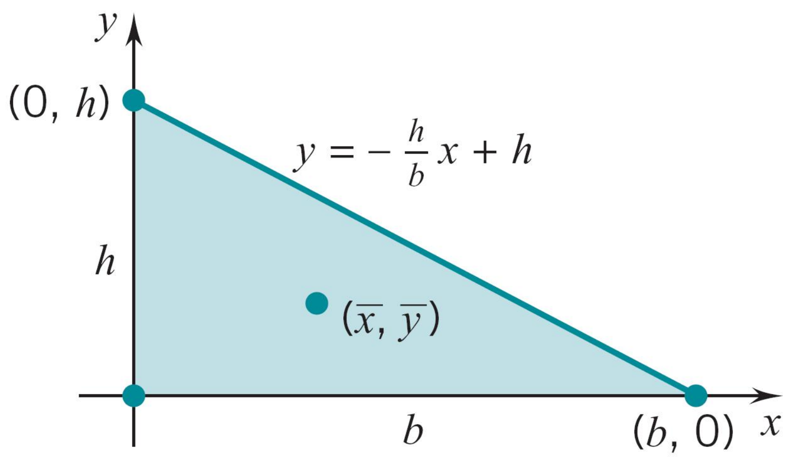
Let the region Ω under the graph of f have an area A. The centroid $(\bar{x}, \bar{y})$ of Ω is given by

$$\bar{x}A = \int_a^b x\, f(x)\, dx, \quad \bar{y}A = \int_a^b \frac{1}{2}\left[f(x)\right]^2 dx.$$

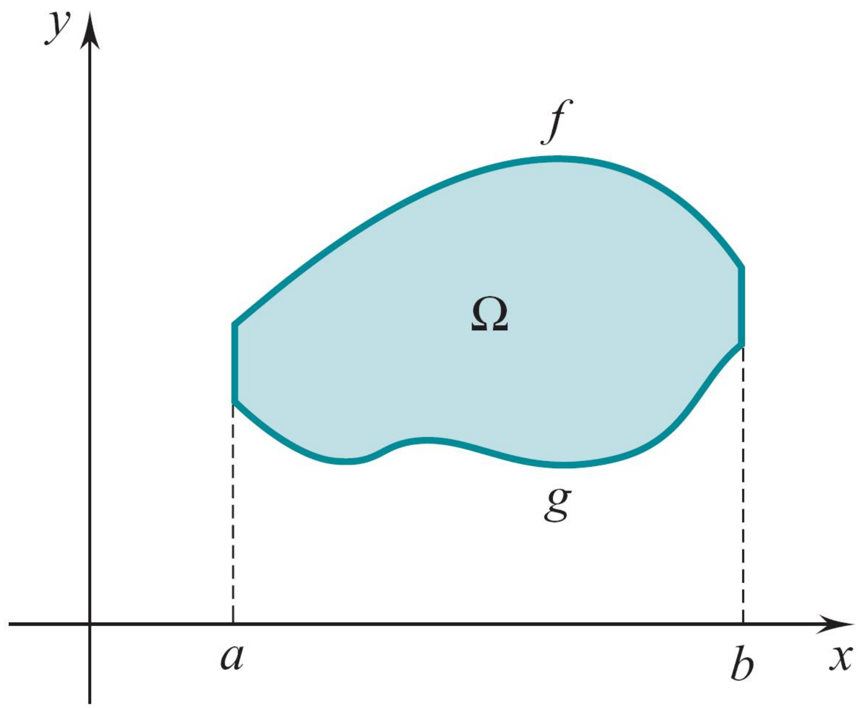
Example
Example 1. Find the centroid of the quarter-disc shown in the figure below.



Example
Example 2. Find the centroid of the right triangle shown in the figure below.



Centroid of a Region between the graphs of f and g



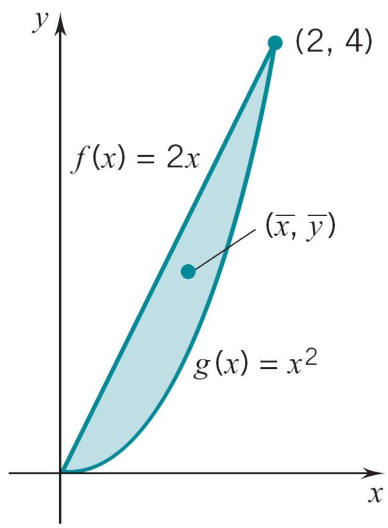
$$f(x) \geq g(x) \geq 0 \quad \text{for all } x \text{ in } [a,b].$$

$$\Omega = \text{ region between the graphs of } f \text{ (Top) and } g \text{ (Bottom).}$$

Let the region Ω between the graphs of f and g have an area A. The centroid $(\bar{x}, \bar{y})$ of Ω is given by

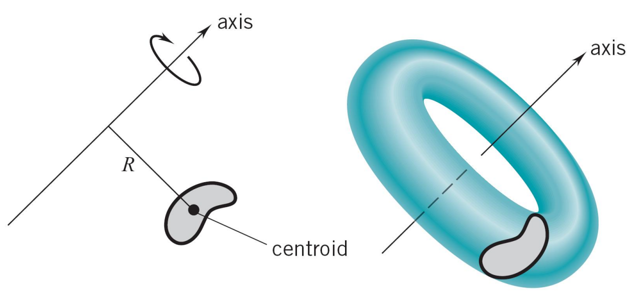
$$\bar{x}A = \int_a^b x\left[f(x) - g(x)\right] dx, \quad \bar{y}A = \int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right) dx.$$

Example
Example 3. Find the centroid of the region shown in the figure below.



1.2 Pappus' Theorem on Volumes

Pappus' Theorem on Volumes

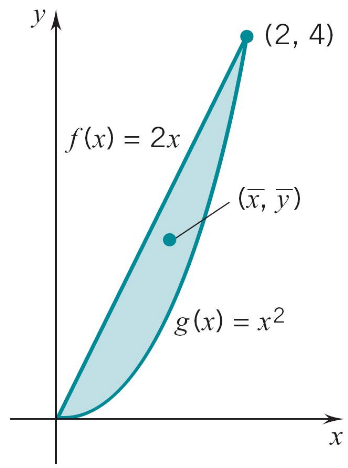


Pappus' Theorem on Volumes
A plane region is revolved about an axis that lies in its plane. If the region does not cross the axis, then the volume of the resulting solid of revolution is

$$V = 2\pi\,\bar{R}\,A = \text{ (area of the region)} \times \text{(circumference of the circle)}$$

where A is the area of the region and $\bar{R}$ is the distance from the axis to the centroid of the region.

Example
Example 4. Find the volume of the solids formed by revolving the region, shown in the figure below, (a) about the y-axis, (b) about the $y = 5$.

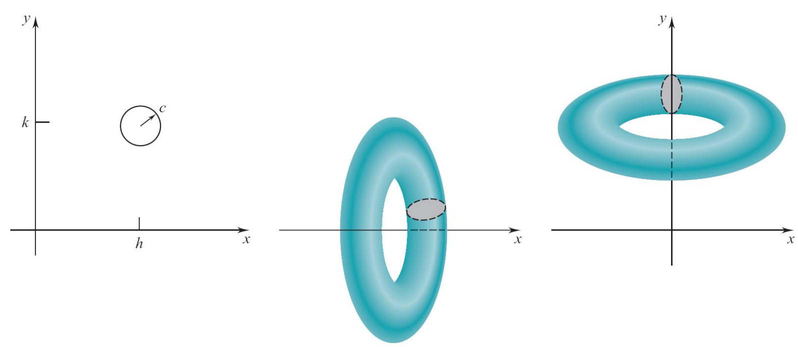


Example

Example 5. Find the volume of the torus generated by revolving the circular disc

$$(x-h)^2 + (y-k)^2 \leq c^2, \quad h, k \geq c > 0$$

(a) about the x-axis, (b) about the y-axis.



Example

Example 6. Find the centroid of the half-disc

$$x^2 + y^2 \leq r^2, \quad y \geq 0$$

by appealing to Pappus's theorem.