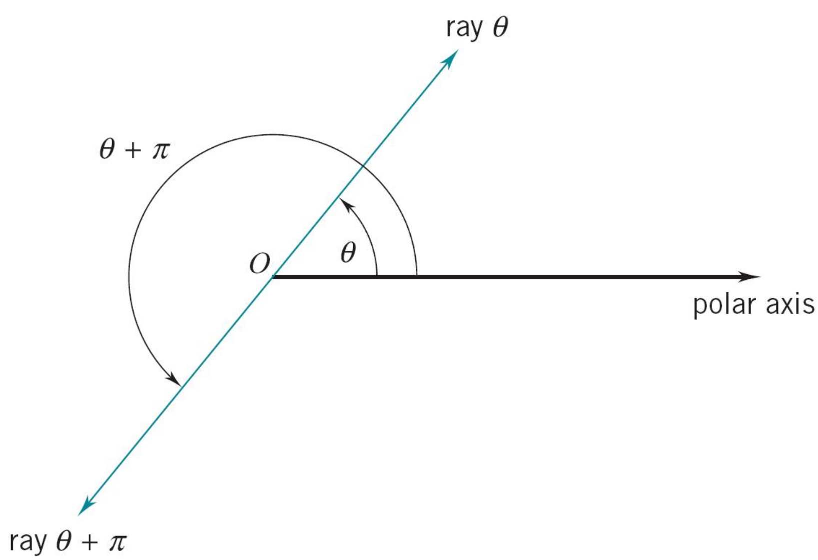
Lecture 12 Section 9.3 Polar Coordinates Section 9.4 Graphing in Polar Coordinates

Jiwen He

1 Polar Coordinates

1.1 Polar Coordinates

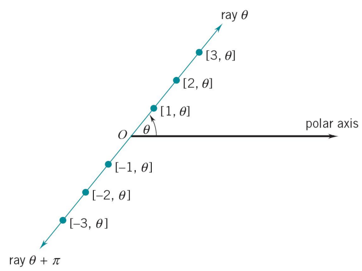
Polar Coordinate System

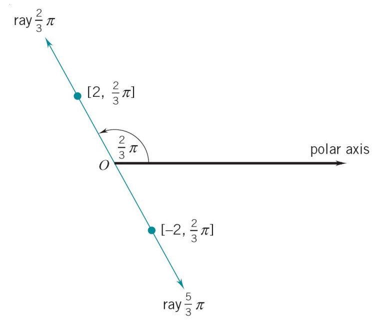


The purpose of the polar coordinates is to represent curves that have *symmetry* about a point or *spiral* about a point.

Frame of Reference
In the polar coordinate system, the frame of reference is a point O that we call the *pole* and a ray that emanates from it that we call the *polar axis*.

Polar Coordinates

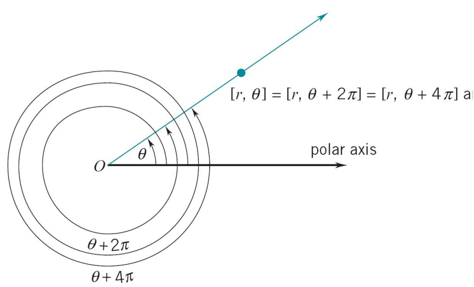


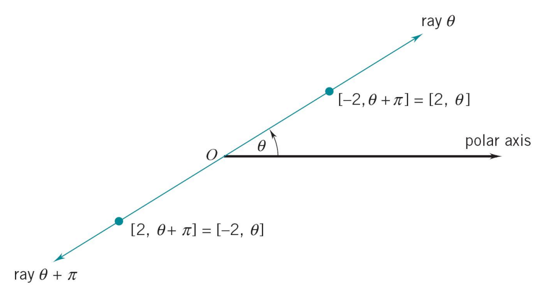


Definition

A point is given *polar coordinates* $[r, \theta]$ iff it lies at a distance $|r|$ from the pole a long the ray θ, if $r \geq 0$, and along the ray $\theta + \pi$, if $r < 0$.

Points in Polar Coordinates



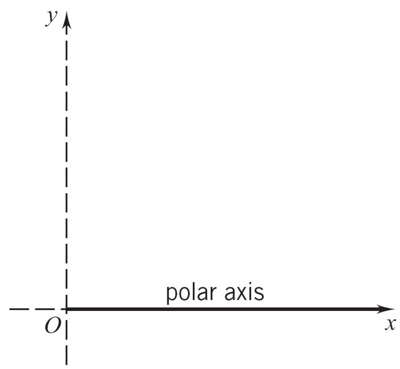


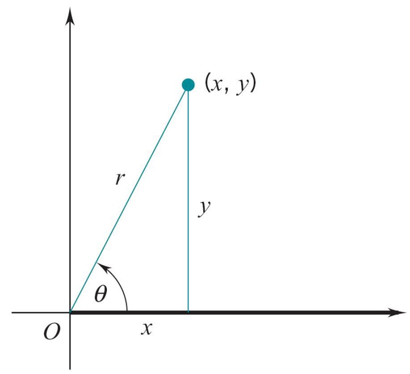
Points in Polar Coordinates

- $O = [0, \theta]$ for all θ.
- $[r, \theta] = [r, \theta + 2n\pi]$ for all integers n.
- $[r, -\theta] = [r, \theta + \pi]$.

1.2 Relation to Rectangular Coordinates

Relation to Rectangular Coordinates

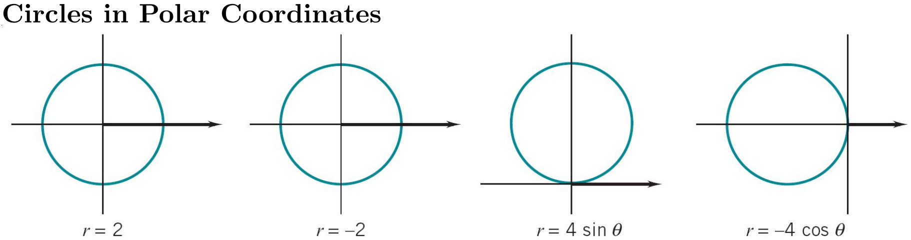




Relation to Rectangular Coordinates

- $x = r\cos\theta,\ y = r\sin\theta. \quad \Rightarrow \quad x^2 + y^2 = r^2, \quad \tan\theta = \dfrac{y}{x}$
- $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}\dfrac{y}{x}.$

Circles in Polar Coordinates

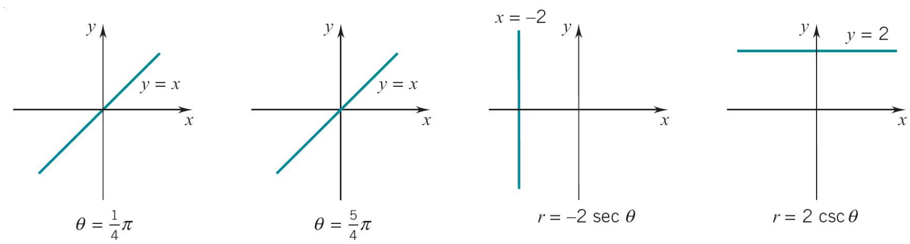


Circles in Polar Coordinates

In rectangular coordinates	In polar coordinates
$x^2 + y^2 = a^2$	$r = a$
$x^2 + (y - a)^2 = a^2$	$r = 2a\sin\theta$
$(x - a)^2 + y^2 = a^2$	$r = 2a\cos\theta$

$$x^2 + y^2 = a^2 \quad \Rightarrow \quad r^2 = a^2$$
$$x^2 + (y - a)^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 = 2ay \quad \Rightarrow \quad r^2 = 2ar\sin\theta$$
$$(x - a)^2 + y^2 = a^2 \quad \Rightarrow \quad x^2 + y^2 = 2ax \quad \Rightarrow \quad r^2 = 2ar\cos\theta$$

Lines in Polar Coordinates



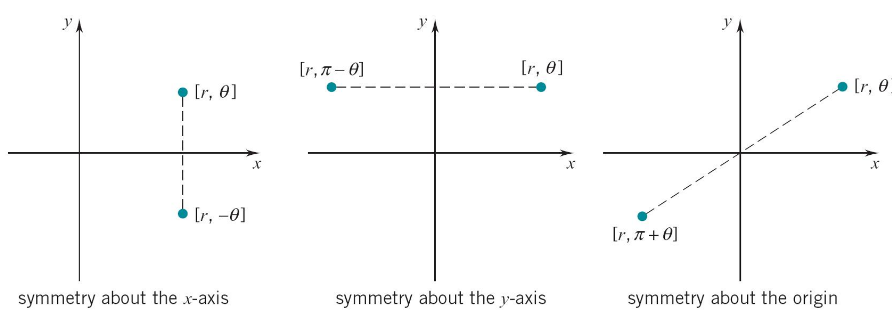
Lines in Polar Coordinates

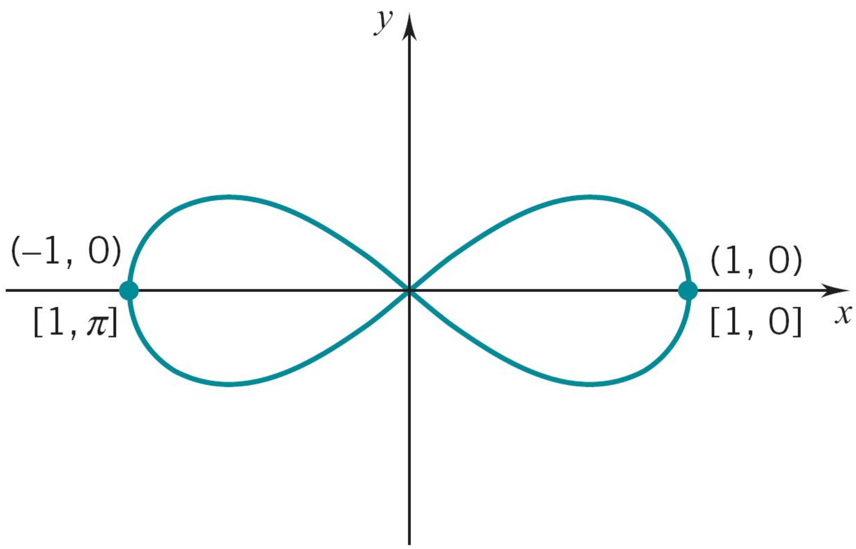
In rectangular coordinates	In polar coordinates
$y = mx$	$\theta = \alpha$ with $\alpha = \tan^{-1} m$
$x = a$	$r = a \sec\theta$
$y = a$	$r = a \csc\theta$

$$
\begin{aligned}
y = mx \quad &\Rightarrow \quad \frac{y}{x} = m \quad \Rightarrow \quad \tan\theta = m\\
x = a \quad &\Rightarrow \quad r\cos\theta = a \quad \Rightarrow \quad r = a\sec\theta\\
y = a \quad &\Rightarrow \quad r\sin\theta = a \quad \Rightarrow \quad r = a\csc\theta
\end{aligned}
$$

1.3 Symmetry

Symmetry

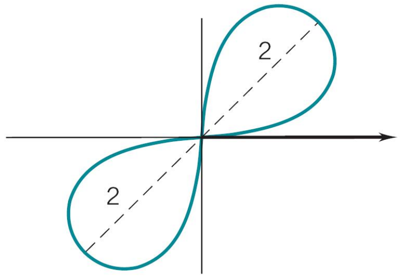




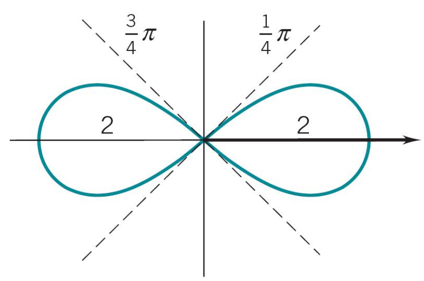
$r^2 = \cos 2\theta$
lemniscate

Lemniscate (ribbon) $r^2 = \cos 2\theta$
$\cos[2(-\theta)] = \cos(-2\theta) = \cos 2\theta$ [1ex]$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, -\theta] \in$ graph [1ex]$\Rightarrow$ symmetric about the x-axis. $\cos[2(\pi - \theta)] = \cos(2\pi - 2\theta) = \cos 2\theta$ [1ex]$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi - \theta] \in$ graph [1ex]$\Rightarrow$ symmetric about the y-axis. $\cos[2(\pi + \theta)] = \cos(2\pi + 2\theta) = \cos 2\theta$ [1ex]$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi + \theta] \in$ graph [1ex]$\Rightarrow$ symmetric about the origin.

Lemniscates (Ribbons) $r^2 = a \sin 2\theta$, $r^2 = a \cos 2\theta$



$r^2 = 4 \sin 2\theta$



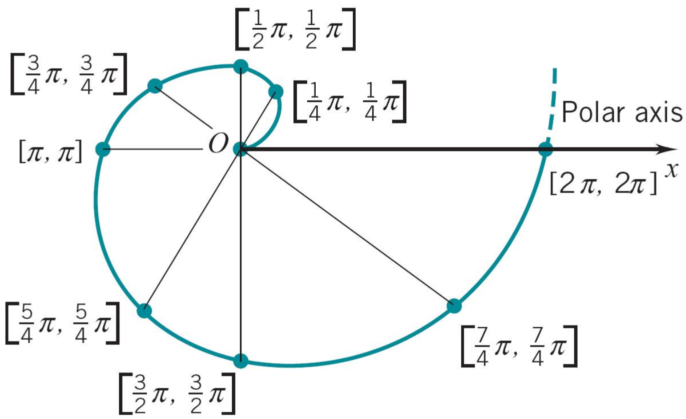
$r^2 = 4 \cos 2\theta$

Lemniscate $r^2 = a \sin 2\theta$
$\sin[2(\pi + \theta)] = \sin(2\pi + 2\theta) = \sin 2\theta$ [2ex]$\Rightarrow$ if $[r, \theta] \in$ graph, then $[r, \pi + \theta] \in$ graph [2ex]$\Rightarrow$ symmetric about the origin.

2 Graphing in Polar Coordinates

2.1 Spiral

Spiral of Archimedes $r = \theta$**,** $\theta \geq 0$



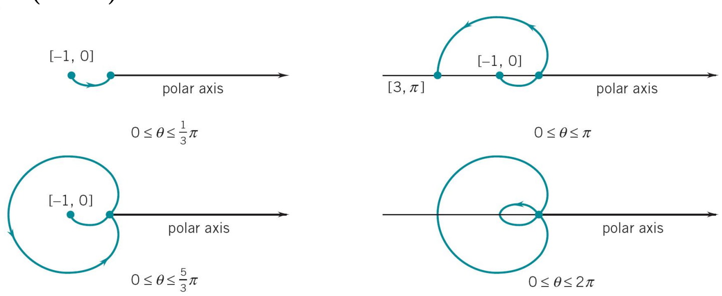
$r = \theta, \quad \theta \geq 0$

spiral of Archimedes

The curve is a nonending spiral. Here it is shown in detail from $\theta = 0$ to $\theta = 2\pi$.

2.2 Limaçons

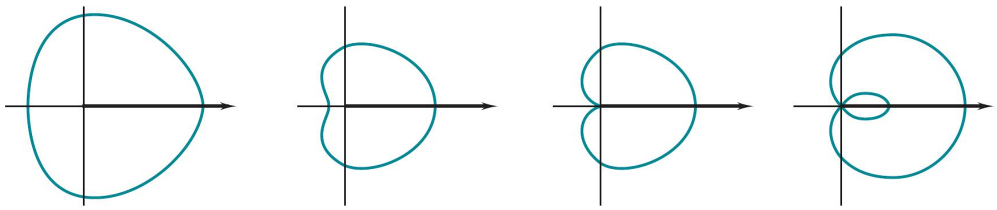
Limaçon (Snail): $r = 1 - 2\cos\theta$



θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π	$5\pi/4$	$4\pi/3$	$3\pi/2$	$5\pi/3$	$7\pi/4$	2π
r	-1	-0.41	0	1	2	2.41	3	2.41	2	1	0	-0.41	-1

- $r = 0$ at $\theta = \frac{1}{3}\pi, \frac{5}{3}\pi$; $\quad |r|$ is a local maximum at $\theta = 0, \pi, 2\pi$.
- Sketch in 4 stages: $[0, \frac{1}{3}\pi]$, $[\frac{1}{3}\pi, \pi]$, $[\pi, \frac{5}{3}\pi]$, $[\frac{5}{3}\pi, 2\pi]$.
- $\cos(-\theta) = \cos\theta \Rightarrow$ if $[r, \theta] \in$ graph, then $[r, -\theta] \in$ graph $\Rightarrow$ symmetric about the x-axis.

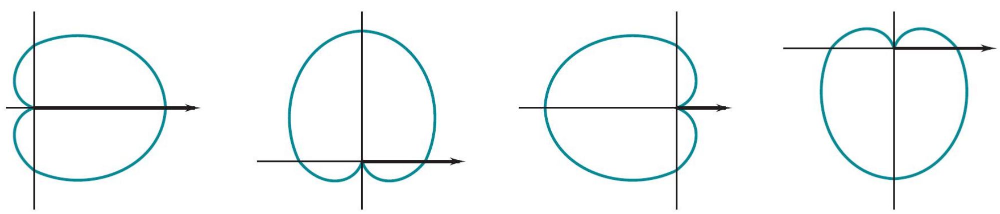
Limaçons (Snails): $r = a + b\cos\theta$



$r = 3 + \cos\theta$ convex limaçon — $r = \frac{3}{2} + \cos\theta$ limaçon with a dimple — $r = 1 + \cos\theta$ cardioid — $r = \frac{1}{2} + \cos\theta$ limaçon with an inner loop

The general shape of the curve depends on the relative magnitudes of $|a|$ and $|b|$.

Cardioids (Heart-Shaped): $r = 1 \pm \cos\theta$**,** $r = 1 \pm \sin\theta$



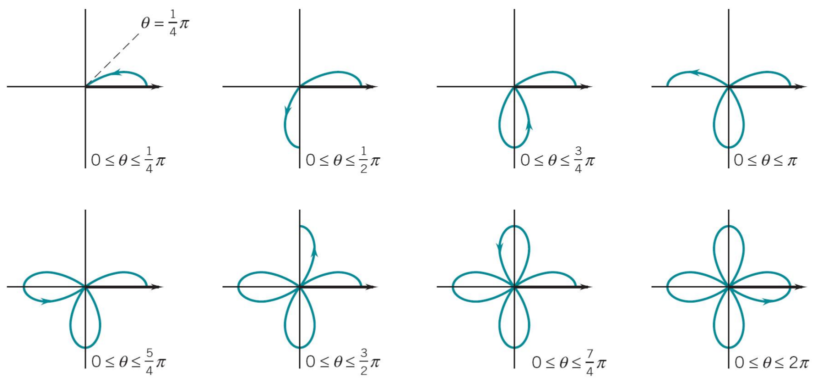
$r = 1 + \cos\theta$ — $r = 1 + \sin\theta$ — $r = 1 - \cos\theta$ — $r = 1 - \sin\theta$

Each change $\cos\theta \to \sin\theta \to -\cos\theta \to -\sin\theta$ represents a *counterclockwise rotation* by $\frac{1}{2}\pi$ radians.

- Rotation by $\frac{1}{2}\pi$: $r = 1 + \cos(\theta - \frac{1}{2}\pi) = 1 + \sin\theta$.
- Rotation by $\frac{1}{2}\pi$: $r = 1 + \sin(\theta - \frac{1}{2}\pi) = 1 - \cos\theta$.
- Rotation by $\frac{1}{2}\pi$: $r = 1 - \cos(\theta - \frac{1}{2}\pi) = 1 - \sin\theta$.

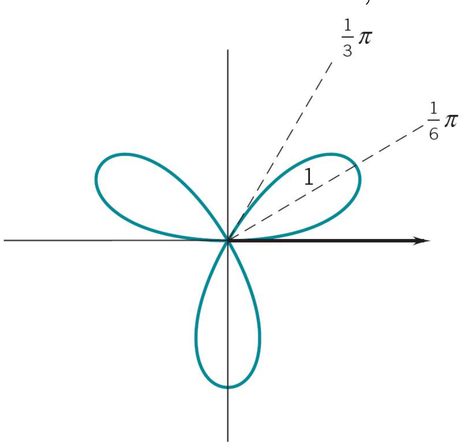
2.3 Flowers

Petal Curve: $r = \cos 2\theta$

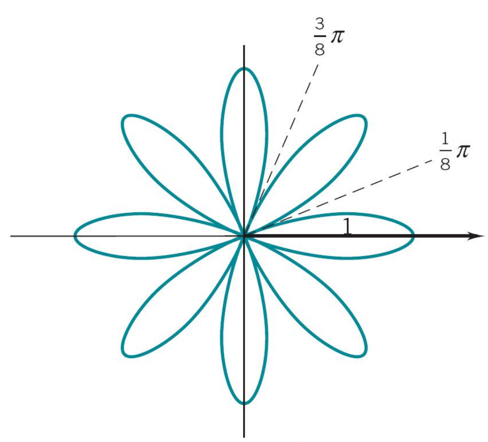


- $r = 0$ at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$; $|r|$ is a local maximum at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.
- Sketch the curve in 8 stages.
- $\cos[2(-\theta)] = \cos 2\theta$, $\cos[2(\pi \pm \theta)] = \cos 2\theta \Rightarrow$ symmetric about the x-axis, the y-axis, and the origin.

Petal Curves: $r = a \cos n\theta$**,** $r = a \sin n\theta$



$r = \sin 3\theta$

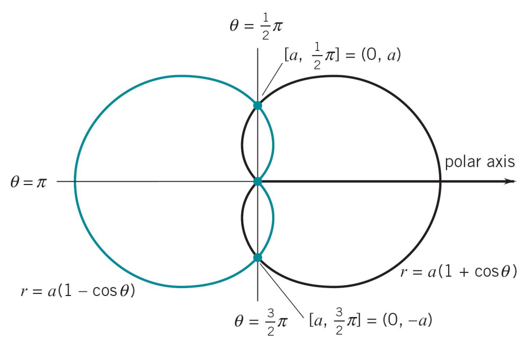


$r = \cos 4\theta$

- If n is *odd*, there are n petals.
- If n is *even*, there are $2n$ petals.

2.4 Intersections

Intersections: $r = a(1 - \cos\theta)$ **and** $r = a(1 + \cos\theta)$



- $r = a(1 - \cos\theta)$ and $r = a(1 + \cos\theta) \Rightarrow r = a$ and $\cos\theta = 0 \Rightarrow r = a$ and $\theta = \frac{\pi}{2} + n\pi \Rightarrow [a, \frac{\pi}{2} + n\pi] \in$ intersection $\Rightarrow n$ even, $[a, \frac{\pi}{2} + n\pi] = [a, \frac{\pi}{2}]$; n odd, $[a, \frac{\pi}{2} + n\pi] = [a, \frac{3\pi}{2}]$
- Two intersection points: $[a, \frac{\pi}{2}] = (0, a)$ and $[a, \frac{3\pi}{2}] = (0, -a)$.
- The intersection third point: the origin; but the two cardioids pass through the origin at *different* times (θ).

Outline

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