Lecture 27

11.7 Power Series

11.8 Differentiation and Integration of Power Series

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Geometric Series

Geometric Series: $\sum_{k=0}^{\infty} x^k$

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots \begin{cases} \dfrac{1}{1-x}, & \text{if } |x| < 1, \\ \text{diverges}, & \text{if } |x| \geq 1. \end{cases}$$

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Define a function f on the interval $(-1, 1)$

$$f(x) = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x} \quad \text{for } |x| < 1$$

As the Limit

f can be viewed as the limit of a sequence of polynomials:

$$f(x) = \lim_{n\to\infty} p_n(x),$$

where $p_n(x) = 1 + x + x^2 + x^3 + \cdots + x^n$.

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Closed forms for many power series can be found by relating the series to the geometric series

Examples

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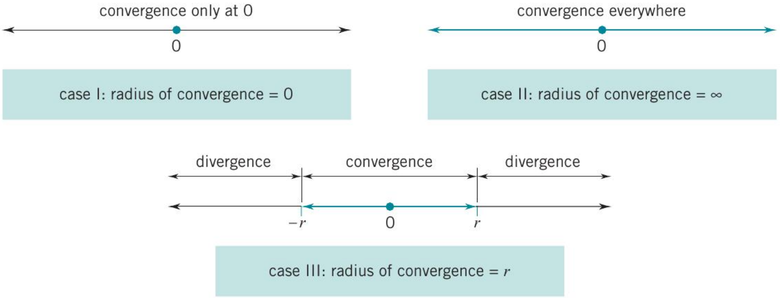
Radius of Convergence

There are exactly three possibilities for a power series: $\sum a_k x^k$.



Radius of Convergence: Ratio Test (I)

The radius of convergence of a power series can usually be found by applying the ratio test. In some cases the root test is easier.

Example

$$f(x) = \sum_{k=1}^{\infty} k^2 x^k = x + 4x^2 + 9x^3 + \cdots$$

$$\text{Ratio Test}: \left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{(k+1)^2 x^{k+1}}{k^2 x^k}\right|$$

$$= \frac{(k+1)^2}{k^2}|x| \to |x| \quad \text{as } k \to \infty$$

Thus the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$.

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Interval of Convergence

For a series with radius of convergence r, the interval of convergence can be $[-r, r]$, $(-r, r]$, $[-r, r)$, or $(-r, r)$.

Example

In general, the behavior of a power series at $-r$ and at r is not predictable. For example, the series

$$\sum x^k, \quad \sum \frac{(-1)^k}{k} x^k, \quad \sum \frac{1}{k} x^k, \quad \sum \frac{1}{k^2} x^k$$

all have radius of convergence 1, but the first series converges only on $(-1, 1)$, the second converges on $(-1, 1]$, but the third converges on $[-1, 1)$, the fourth on $[-1, 1]$.

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Differentiation and Integration

Theorem

Let $f(x) = \sum a_k x^k$ be a power series with a nonzero radius of convergence r. Then

$$f'(x) = \sum a_k\, k\, x^{k-1} \qquad \text{for } |x| < r$$

$$\int f(x)\, dx = \sum \frac{a_k}{k+1} x^{k+1} + C \qquad \text{for } |x| < r$$

Geometric series: $\displaystyle \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \qquad \text{for } |x| < 1$

Differentiation: $\displaystyle \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k\, x^{k-1} \sum_{k=0}^{\infty} (k+1)\, x^k \quad \text{for } |x| < 1$

Integration: $\displaystyle -\ln(1-x) = \sum_{k=0}^{\infty} \frac{1}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{1}{k} x^k \quad \text{for } |x| < 1$

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Power Series Expansion of $\ln(1+x)$

Note: $\dfrac{d}{dx}\ln(1+x) = \dfrac{1}{1+x} = \sum_{k=0}^{\infty}(-1)^k x^k$ for $|x| < 1$

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$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k} x^k = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots$$

The interval of convergence is $(-1, 1]$. At $x = 1$,

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Outline

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  - Geometric Series and Variations
  - Radius of Convergence
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  - Examples