Math 2331 – Linear Algebra

1.1 Systems of Linear Equations

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1.1 Systems of Linear Equations

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Linear Equation

A Linear Equation

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Examples (Linear)

$4x_1 - 5x_2 + 2 = x_1$ and $x_2 = 2(\sqrt{6} - x_1) + x_3$

$\downarrow$ rearranged $\downarrow$ (each)

$$3x_1 - 5x_2 = -2 \qquad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

Examples (Not Linear)

$$4x_1 - 6x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 7$$

Linear System

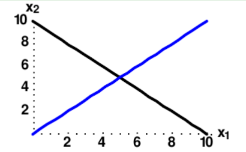
A solution of a System of Linear Equations

A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system true when the values $s_1, s_2, \ldots, s_n$ are substituted for $x_1, x_2, \ldots, x_n$, respectively.

Examples (Two Equations in Two Variables)

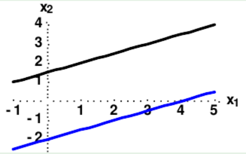
Each equation determines a line in 2-space.

$$\begin{array}{rcrcr} x_1 & + & x_2 & = & 10 \\ -x_1 & + & x_2 & = & 0 \end{array}$$



one unique solution

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -3 \\ 2x_1 & - & 4x_2 & = & 8 \end{array}$$



no solution

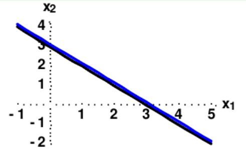
Basic Fact on Solution

Basic Fact on Solution of a Linear System

1. exactly one solution *(consistent)* or
2. infinitely many solutions *(consistent)* or
3. no solution *(inconsistent)*.

Examples (Two Equ. Two Var.)

$$\begin{aligned} x_1 + x_2 &= 3 \\ -2x_1 - 2x_2 &= -6 \end{aligned}$$



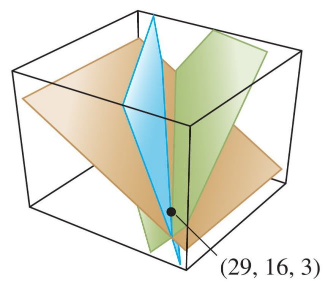
infinitely many solutions

Basic Fact on Solution (cont.)

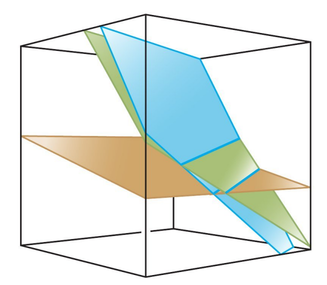
Examples (Three Equations in Three Variables)

Each equation determines a plane in 3-space.

i) The planes intersect in one point. *(one solution)*



ii) There is not point in common to all three planes. *(no solution)*



Equivalent Systems

Solution Set of a Linear System

The set of all possible solutions of a linear system.

Equivalent Systems

Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM

Replace one system with an equivalent system that is easier to solve.

Examples (Two Equ. Two Var.)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$

$$\downarrow$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array}$$

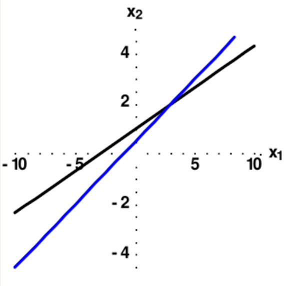
$$\downarrow$$

$$\begin{array}{rcrcr} x_1 & & & = & 3 \\ & & x_2 & = & 2 \end{array}$$

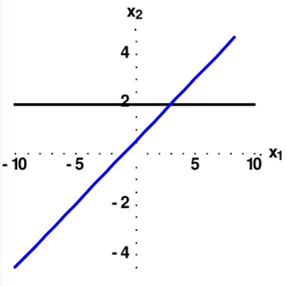
Equivalent Systems (cont.)

Examples (Two Equ. in Two Var. (cont.))

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array}$$



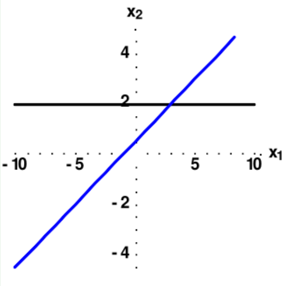
$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array}$$



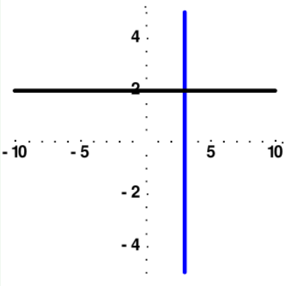
Equivalent Systems (cont.)

Examples (Two Equ. in Two Var. (cont.))

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$



$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$



Matrix Notation

Example (Coefficient Matrix: Two Row and Two Columns)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

(coefficient matrix)

Example (Augmented Matrix: Two Row and Three Columns)

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$$

(augmented matrix)

Solving a Linear System

Example

Solving a System in Matrix Form

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ -x_1 & + & 3x_2 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix}$$

(augmented matrix)

$$\downarrow$$

$$\begin{array}{rcrcr} x_1 & - & 2x_2 & = & -1 \\ & & x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcr} x_1 & = & 3 \\ x_2 & = & 2 \end{array} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Row Operations

Elementary Row Operations

1. *(Replacement)* Add one row to a multiple of another row.
2. *(Interchange)* Interchange two rows.
3. *(Scaling)* Multiply all entries in a row by a nonzero constant.

Row Equivalent Matrices

Two matrices where one matrix can be transformed into the other matrix by a sequence of elementary row operations.

Fact about Row Equivalence

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Solving a System by Row Eliminations: Example

Example (Row Eliminations to a Triangular Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & - & 3x_2 & + & 13x_3 & = & -9 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solving a System by Row Eliminations: Example (cont.)

Example (Row Eliminations to a Diagonal Form)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & & & = & -3 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{array}{rcrcrcr} x_1 & & & & & = & 29 \\ & & x_2 & & & = & 16 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution: $(29, 16, 3)$

Solving a System by Row Eliminations: Example (cont.)

Example (Check the Answer)

Is $(29, 16, 3)$ a solution of the **original** system?

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

$$\begin{array}{rcll} (29) - 2(16) + (3) & = 29 - 32 + 3 & = & 0 \\ 2(16) - 8(3) & = 32 - 24 & = & 8 \\ -4(29) + 5(16) + 9(3) & = -116 + 80 + 27 & = & -9 \end{array}$$

Existence and Uniqueness

Two Fundamental Questions (Existence and Uniqueness)

1. Is the system consistent; (i.e. does a solution **exist**?)
2. If a solution exists, is it **unique**? (i.e. is there one & only one solution?)

Existence: Examples

Example (Is this system consistent?)

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & 2x_2 & - & 8x_3 & = & 8 \\ -4x_1 & + & 5x_2 & + & 9x_3 & = & -9 \end{array}$$

In the last example, this system was reduced to the triangular form:

$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & x_3 & = & 0 \\ & & x_2 & - & 4x_3 & = & 4 \\ & & & & x_3 & = & 3 \end{array} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This is sufficient to see that the system is consistent and unique. Why?

Existence: Examples (cont.)

Example (Is this system consistent?)

$$\begin{array}{rcrcrcr} & & 3x_2 & - & 6x_3 & = & 8 \\ x_1 & - & 2x_2 & + & 3x_3 & = & -1 \\ 5x_1 & - & 7x_2 & + & 9x_3 & = & 0 \end{array} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution: Row operations produce:

$$\begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Equation notation of triangular form:

$$\begin{array}{rcrcrcrl} x_1 & - & 2x_2 & + & 3x_3 & = & -1 & \\ & & 3x_2 & - & 6x_3 & = & 8 & \\ & & & & 0x_3 & = & -3 & \leftarrow \textit{Never true} \end{array}$$

The original system is inconsistent!

Existence: Examples (cont.)

Example (For what values of h will the system be consistent?)

$$\begin{array}{rcrcl} 3x_1 & - & 9x_2 & = & 4 \\ -2x_1 & + & 6x_2 & = & h \end{array}$$

Solution: Reduce to triangular form.

$$\begin{bmatrix} 3 & -9 & 4 \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ -2 & 6 & h \end{bmatrix} \to \begin{bmatrix} 1 & -3 & \frac{4}{3} \\ 0 & 0 & h+\frac{8}{3} \end{bmatrix}$$

The second equation is $0x_1 + 0x_2 = h + \frac{8}{3}$. System is consistent only if $h + \frac{8}{3} = 0$ or $h = \frac{-8}{3}$.