Math 2331 – Linear Algebra

1.9 The Matrix of a Linear Transformation

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math2331

1.9 The Matrix of a Linear Transformation

- Matrix Transformation: Identity Matrix
- Linear Transformation: Generalized Result
- Matrix of a Linear Transformation
  - Theorem
  - Examples
    - Geometric Linear Transformations of $\mathbf{R}^2$

Identity Matrix

Identity Matrix

I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

Example

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} + \_\_\_\_ \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \_\_\_\_.$$

Linear Transformation

Identity Matrix

In general, for $\mathbf{x}$ in $\mathbf{R}^n$, $I_n\mathbf{x} =$ ___

Linear Transformation

From Section 1.8, if $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T\,(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}\,(\mathbf{u}) + d\mathbf{T}\,(\mathbf{v})$.

Generalized Result

$$T\,(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T\,(\mathbf{v}_1) + \cdots + c_pT\,(\mathbf{v}_p).$$

Linear Transformation: Example

Example

The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
Suppose T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ where

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: A vector $\mathbf{x}$ in $\mathbf{R}^2$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \_\_\_\_\_ \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \_\_\_\_\_ \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \_\_\_\_\_\mathbf{e}_1 + \_\_\_\_\_\mathbf{e}_2$$

Linear Transformation: Example (cont.)

Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \_\_\_\_\_ T(\mathbf{e}_1) + \_\_\_\_\_ T(\mathbf{e}_2)$$

$$= \_\_\_\_\_ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \_\_\_\_\_ \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}.$$

Note that

$$T(\mathbf{x}) = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} \mathbf{x} = A\mathbf{x}$$

To get A, replace the identity matrix $[\mathbf{e}_1 \ \mathbf{e}_2]$ with $[T(\mathbf{e}_2) \ T(\mathbf{e}_2)]$.

Matrix of Linear Transformation: Theorem

Theorem

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ *be a linear transformation. Then there exists a unique matrix* A *such that*

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact, A *is the* $m \times n$ *matrix whose jth column is the vector* $T(\mathbf{e}_j)$*, where* $\mathbf{e}_j$ *is the jth column of the identity matrix in* $\mathbf{R}^n$*.*

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

$$\uparrow$$

(standard matrix for the linear transformation)T

Matrix of Linear Transformation: Example

Example

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

Solution:

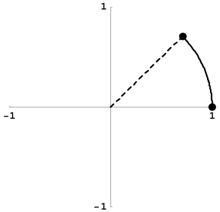
$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \text{standard matrix of the linear transformation } T$$

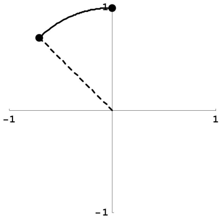
$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \qquad \text{(fill-in)}$$

Matrix of Linear Transformation: Example

Example

Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).





$$A = \begin{bmatrix} \downarrow & \end{bmatrix}$$

$$T(\mathbf{e}_1) = \begin{bmatrix} \quad \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} \quad \end{bmatrix}$$