Math 2331 – Linear Algebra

2.2 The Inverse of a Matrix
Key Exercises 11–24, 35

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- The proof of Theorem 5 is important; students need to know the ways that both uniqueness and existence are proved.
- Elementary matrices are also used in Section 2.5 and in Section 3.2
- The algorithm for finding A^{-1} is popular because it is so familar and leads to easy exam questions.
- Key Exercises: 11–24, 35
  - Exercise 12 is referenced in Section 2.3 after the proof of Theorem 8
  - Exercise 15 is useful and indicates how matrix products involving inverses are actually computed in practice. It will be used in Sections 4.7 and 5.4
  - Exercises 23 and 24 are cited in the proof of Theorem 8.

11. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Show that the equation $AX = B$ has a unique solution $A^{-1}B$.

12. Use matrix algebra to show that if A is invertible and D satisfies $AD = I$, then $D = A^{-1}$.

13. Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

14. Suppose $(B - C)D = 0$, where B and C are $m \times n$ matrices and D is invertible. Show that $B = C$.

15. Let A be an invertible $n \times n$ matrix, and let B be an $n \times p$ matrix. Explain why $A^{-1}B$ can be computed by row reduction:

If $[\,A \quad B\,] \sim \cdots \sim [\,I \quad X\,]$, then $X = A^{-1}B$.

If A is larger than 2×2, then row reduction of $[A \quad B]$ is much faster than computing both A^{-1} and $A^{-1}B$.

16. Suppose A and B are $n \times n$ matrices, B is invertible, and AB is invertible. Show that A is invertible. [*Hint:* Let $C = AB$, and solve this equation for A.]

17. Suppose A, B, and C are invertible $n \times n$ matrices. Show that ABC is also invertible by producing a matrix D such that $(ABC)D = I$ and $D(ABC) = I$.

18. Solve the equation $AB = BC$ for A, assuming that A, B, and C are square and B is invertible.

19. If A, B, and C are $n \times n$ invertible matrices, does the equation $C^{-1}(A + X)B^{-1} = I_n$ have a solution, X? If so, find it.

20. Suppose A, B, and X are $n \times n$ matrices with A, X, and $A - AX$ invertible, and suppose

$$(A - AX)^{-1} = X^{-1}B \tag{3}$$

a. Explain why B is invertible.

b. Solve equation (3) for X. If a matrix needs to be inverted, explain why that matrix is invertible.

21. Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

22. Explain why the columns of an $n \times n$ matrix A span $\mathbb{R}^n$ when A is invertible. [*Hint:* Review Theorem 4 in Section 1.4.]

23. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why A has n pivot columns and A is row equivalent to I_n. By Theorem 7, this shows that A must be invertible. (This exercise and Exercise 24 will be cited in Section 2.3.)

24. Suppose A is $n \times n$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why A must be invertible. [*Hint:* Is A row equivalent to I_n?]

35. Let $A = \begin{bmatrix} -1 & -7 & -3 \\ 2 & 15 & 6 \\ 1 & 3 & 2 \end{bmatrix}$. Find the third column of A^{-1} without computing the other columns.