Math 2331 – Linear Algebra

4.6 Rank

Jiwen He

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math2331

4.6 Rank

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Row Space

Row Space

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A and is denoted by Row A.

Example

Let

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \quad \text{and} \quad \begin{array}{l} \mathbf{r}_1 = (-1,\ 2,\ 3,\ 6) \\ \mathbf{r}_2 = (2, -5, -6, -12) \\ \mathbf{r}_3 = (1, -3, -3, -6) \end{array}.$$

$$\text{Row } A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} \quad (\text{a subspace of } \mathbf{R}^4)$$

While it is natural to express row vectors horizontally, they can also be written as column vectors if it is more convenient. Therefore

$$\boxed{\text{Col } A^T = \text{Row } A}.$$

Row Space and Row Equivalence

When we use row operations to reduce matrix A to matrix B, we are taking linear combinations of the rows of A to come up with B. We could reverse this process and use row operations on B to get back to A. Because of this, the row space of A equals the row space of B.

Theorem (13)

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B.

Row Space: Example

Example

The matrices

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent. Find a basis for row space, column space and null space of A. Also state the dimension of each.

Solution: Basis for Row A:

$$\left\{ \qquad\qquad\qquad \right\}$$

dim Row A :_____

Row Space: Example (cont.)

Basis for Col A:

$$\left\{ \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} \right\}$$

dim Col A :_____

To find Nul A, solve $A\mathbf{x} = \mathbf{0}$ first:

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row Space: Example (cont.)

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Basis for Nul } A : \quad \left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and dim Nul A = ______

Rank: Defintion

Note the following:

- $\dim \text{Col } A = \#$ of pivots of $A = \#$ of nonzero rows in B $= \dim \text{Row } A$.
- $\dim \text{Nul } A = \#$ of free variables $= \#$ of nonpivot columns of A.

Rank

The **rank** of A is the dimension of the column space of A.

$$\boxed{\text{rank } A = \dim \text{Col } A = \# \text{ of pivot columns of } A = \dim \text{Row } A}.$$

Rank Theorem

$$\underbrace{\text{rank } A}_{\displaystyle\updownarrow \atop \left\{\begin{array}{c}\#\text{ of pivot}\\ \text{columns}\\ \text{of } A\end{array}\right\}} + \underbrace{\text{dim Nul } A}_{\displaystyle\updownarrow \atop \left\{\begin{array}{c}\#\text{ of nonpivot}\\ \text{columns}\\ \text{of } A\end{array}\right\}} = \underbrace{n}_{\displaystyle\updownarrow \atop \left\{\begin{array}{c}\#\text{ of}\\ \text{columns}\\ \text{of } A\end{array}\right\}}$$

Theorem (14 Rank Theorem)

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$rank\ A + \dim\ Nul\ A = n.$$

Rank Theorem: Example

Since Row A = Col A^T, $\boxed{\text{rank } A = \text{rank } A^T}$.

Example

Suppose that a 5×8 matrix A has rank 5. Find dim Nul A, dim Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\updownarrow \atop 5} + \underbrace{\text{dim Nul } A}_{\downarrow \atop ?} = \underbrace{n}_{\updownarrow \atop 8}$$

$$5 + \text{dim Nul } A = 8 \quad \Rightarrow \quad \text{dim Nul } A = \_\_\_\_\_$$

$$\text{dim Row } A = \text{rank } A = \_\_\_\_\_$$

$$\Rightarrow \quad \text{rank } A^T = \text{rank } \_\_\_\_\_ = \_\_\_\_\_$$

Since rank A = # of pivots in A = 5, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbf{R}^5$.

Rank Theorem: Example

Example

For a 9×12 matrix A, find the smallest possible value of dim Nul A.

Solution:

$$\text{rank } A + \dim \text{Nul } A = 12$$

$$\dim \text{Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value}=\_\_\_\_\_}$$

$$\text{smallest possible value of dim Nul } A = \_\_\_\_\_$$

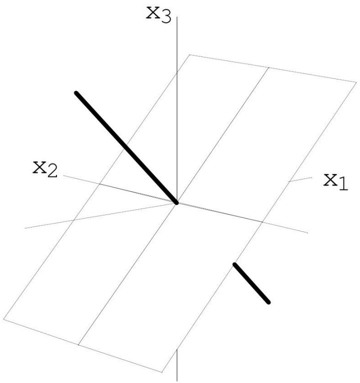
Visualizing Row A and Nul A

Example

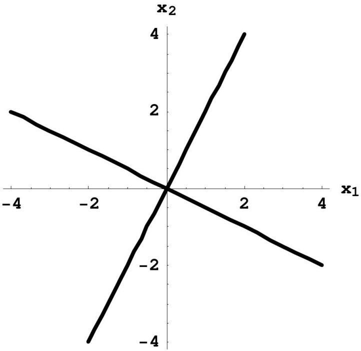
Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. One can easily verify the following:

1. Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and Nul A is a plane in $\mathbf{R}^3$.
2. Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and Row A is a line in $\mathbf{R}^3$.
3. Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and Col A is a line in $\mathbf{R}^2$.
4. Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and Nul A^T is a line in $\mathbf{R}^2$.

Rank Theorem: Example (cont.)



Subspaces Nul A and Row A



Subspaces Nul A^T and Col A

Rank Theorem: Example

The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

Example

A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

Solution: Recall that

- rank $A = \dim \text{Col } A = \#$ of pivot columns of A
- $\dim \text{Nul } A = \#$ of free variables

Rank Theorem: Example (cont.)

In this case $A\mathbf{x} = \mathbf{0}$ of where A is 50×54.

By the rank theorem,

$$\text{rank } A + \_\_\_\_\_\_ = \_\_\_\_\_\_$$

or

$$\text{rank } A = \_\_\_\_\_\_.$$

So any nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution because there is a pivot in every row.

The Invertible Matrix Theorem (continued)

Theorem (Invertible Matrix Theorem (continued))

Let A be a square $n \times n$ matrix. The the following statements are equivalent:

m. *The columns of A form a basis for $\mathbf{R}^n$*

n. $Col\ A = \mathbf{R}^n$

o. $dim\ Col\ A = n$

p. $rank\ A = n$

q. $Nul\ A = \{\mathbf{0}\}$

r. $dim\ Nul\ A = 0$