Math 2331 – Linear Algebra

5.1-5.3 Eigenvalues & Eigenvectors
Key Exercises

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5.1 Eigenvectors & Eigenvalues
Key Exercises 21–27, 29–33

- Key Exercises: 21–27, 29–33.

A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

21. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$, then λ is an eigenvalue of A.

b. A matrix A is not invertible if and only if 0 is an eigenvalue of A.

c. A number c is an eigenvalue of A if and only if the equation $(A - cI)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

d. Finding an eigenvector of A may be difficult, but checking whether a given vector is in fact an eigenvector is easy.

e. To find the eigenvalues of A, reduce A to echelon form.

A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer.

22. a. If $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ, then $\mathbf{x}$ is an eigenvector of A.

b. If $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent eigenvectors, then they correspond to distinct eigenvalues.

c. A steady-state vector for a stochastic matrix is actually an eigenvector.

d. The eigenvalues of a matrix are on its main diagonal.

e. An eigenspace of A is a null space of a certain matrix.

23. Explain why a 2×2 matrix can have at most two distinct eigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.

24. Construct an example of a 2×2 matrix with only one distinct eigenvalue.

25. Let λ be an eigenvalue of an invertible matrix A. Show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint:* Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

26. Show that if A^2 is the zero matrix, then the only eigenvalue of A is 0.

27. Show that λ is an eigenvalue of A if and only if λ is an eigenvalue of A^T. [*Hint:* Find out how $A - \lambda I$ and $A^T - \lambda I$ are related.]

29. Consider an $n \times n$ matrix A with the property that the row sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Find an eigenvector.]

30. Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint:* Use Exercises 27 and 29.]

In Exercises 31 and 32, let A be the matrix of the linear transformation T. Without writing A, find an eigenvalue of A and describe the eigenspace.

31. T is the transformation on $\mathbb{R}^2$ that reflects points across some line through the origin.

32. T is the transformation on $\mathbb{R}^3$ that rotates points about some line through the origin.

5.2 The Characteristic Equation
Key Exercises 19–24

- Key Exercises: 19–24.

19. Let A be an $n \times n$ matrix, and suppose A has n real eigenvalues, $\lambda_1, \ldots, \lambda_n$, repeated according to multiplicities, so that

$$\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \cdots (\lambda_n - \lambda)$$

Explain why $\det A$ is the product of the n eigenvalues of A. (This result is true for any square matrix when complex eigenvalues are considered.)

20. Use a property of determinants to show that A and A^T have the same characteristic polynomial.

A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

21. a. The determinant of A is the product of the diagonal entries in A.

b. An elementary row operation on A does not change the determinant.

c. $(\det A)(\det B) = \det AB$

d. If $\lambda + 5$ is a factor of the characteristic polynomial of A, then 5 is an eigenvalue of A.

A is $n \times n$ matrix. Mark each statement True or False. Justify each answer.

22. a. If A is 3×3, with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then $\det A$ equals the volume of the parallelepiped determined by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.

b. $\det A^T = (-1) \det A$.

c. The multiplicity of a root r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.

d. A row replacement operation on A does not change the eigenvalues.

23. Show that if $A = QR$ with Q invertible, then A is similar to $A_1 = RQ$.

24. Show that if A and B are similar, then $\det A = \det B$.

5.3 Diagonalization
Key Exercises 21–28, 31–32

- Key Exercises: 21–28, 31–32.

A, P and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

21. a. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and some invertible matrix P.

b. If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

c. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.

d. If A is diagonalizable, then A is invertible.

A, P and D are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

22. a. A is diagonalizable if A has n eigenvectors.

b. If A is diagonalizable, then A has n distinct eigenvalues.

c. If $AP = PD$, with D diagonal, then the nonzero columns of P must be eigenvectors of A.

d. If A is invertible, then A is diagonalizable.

23. A is a 5×5 matrix with two eigenvalues. One eigenspace is three-dimensional, and the other eigenspace is two-dimensional. Is A diagonalizable? Why?

24. A is a 3×3 matrix with two eigenvalues. Each eigenspace is one-dimensional. Is A diagonalizable? Why?

25. A is a 4×4 matrix with three eigenvalues. One eigenspace is one-dimensional, and one of the other eigenspaces is two-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

26. A is a 7×7 matrix with three eigenvalues. One eigenspace is two-dimensional, and one of the other eigenspaces is three-dimensional. Is it possible that A is *not* diagonalizable? Justify your answer.

27. Show that if A is both diagonalizable and invertible, then so is A^{-1}.

28. Show that if A has n linearly independent eigenvectors, then so does A^T. [*Hint:* Use the Diagonalization Theorem.]

31. Construct a nonzero 2×2 matrix that is invertible but not diagonalizable.

32. Construct a nondiagonal 2×2 matrix that is diagonalizable but not invertible.