# Math 2331 – Linear Algebra

## 6.1 Inner Product, Length & Orthogonality

**Jiwen He**

Department of Mathematics, University of Houston

jiwenhe@math.uh.edu
math.uh.edu/∼jiwenhe/math2331

# 6.1 Inner Product, Length & Orthogonality

- Inner Product: Examples, Definition, Properties
- Length of a Vector: Examples, Definition, Properties
- Orthogonal
  - Orthogonal Vectors
  - The Pythagorean Theorem
  - Orthogonal Complements
- Row, Null and Columns Spaces

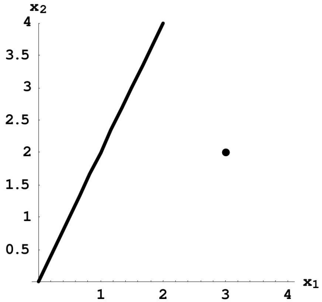
# Motivation: Example

Not all linear systems have solutions.

## Example

No solution to $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ exists. Why?

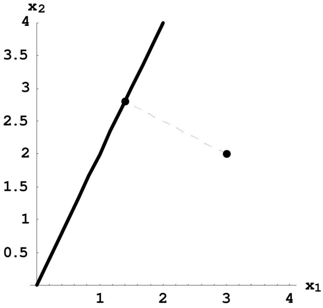
$A\mathbf{x}$ is a point on the line spanned by $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and $\mathbf{b}$ is not on the line. So $A\mathbf{x} \neq \mathbf{b}$ for all $\mathbf{x}$.



# Motivation: Example (cont.)

Instead find $\widehat{\mathbf{x}}$ so that $A\widehat{\mathbf{x}}$ lies "closest" to $\mathbf{b}$. Using information we will learn in this chapter, we will find that

$$\widehat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 0 \end{bmatrix}, \text{ so that } A\widehat{\mathbf{x}} = \begin{bmatrix} 1.4 \\ 2.8 \end{bmatrix}.$$




Segment joining $A\widehat{\mathbf{x}}$ and $\mathbf{b}$ is *perpendicular (*or *orthogonal)* to the set of solutions to $A\mathbf{x} = \mathbf{b}$.

Need to develop fundamental ideas of *length*, *orthogonality* and *orthogonal projections*.

# Inner Product

## Inner Product

**Inner product** or **dot product** of

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}:$$

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\mathbf{v} \cdot \mathbf{u} = \mathbf{v}^T \mathbf{u} = \mathbf{u}^T \mathbf{v} = \mathbf{u} \cdot \mathbf{v}$$

# Inner Product: Properties

## Theorem (1)

*Let* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *be vectors in* $\mathbf{R}^n$*, and let* $c$ *be any scalar. Then*

a. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

b. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

c. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$

d. $\mathbf{u} \cdot \mathbf{u} \geq \mathbf{0}$*, and* $\mathbf{u} \cdot \mathbf{u} = \mathbf{0}$ *if and only if* $\mathbf{u} = \mathbf{0}$*.*

Combining parts b and c, one can show

$$\boxed{(c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p) \cdot \mathbf{w} = c_1(\mathbf{u}_1 \cdot \mathbf{w}) + \cdots + c_p(\mathbf{u}_p \cdot \mathbf{w})}$$

# Length of a Vector

## Length of a Vector

For $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, the **length** or **norm of v** is the nonnegative scalar $\|\mathbf{v}\|$ defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} \quad \text{and} \quad \|\mathbf{v}\|^2 = \mathbf{v} \cdot \mathbf{v}.$$

For any scalar $c$,

$$\|c\mathbf{v}\| = |c| \, \|\mathbf{v}\|$$

# Length of a Vector: Example

## Example

If $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\|\mathbf{v}\| = \sqrt{a^2 + b^2}$ (distance between $\mathbf{0}$ and $\mathbf{v}$).

Picture:

# Distance

## Distance

The **distance between u and v** in $\mathbf{R}^n$:

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

## Example

This agrees with the usual formulas for $\mathbf{R}^2$ and $\mathbf{R}^3$. Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

Then $\mathbf{u} - \mathbf{v} = (u_1 - v_1, u_2 - v_2)$ and

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \|(u_1 - v_1, u_2 - v_2)\|$$

$$= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

# Orthogonal Vectors

$$[\text{dist}\,(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}-\mathbf{v}\|^2 = (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

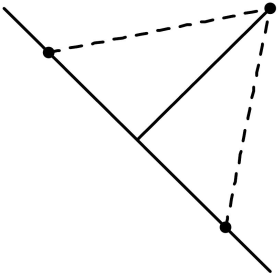
$$= (\mathbf{u})\cdot(\mathbf{u}-\mathbf{v}) + (-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) =$$

$$= \mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} + -\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}$$

$$= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

$$\Rightarrow \qquad [\text{dist}\,(\mathbf{u},\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v}$$

$$\Rightarrow \qquad [\text{dist}\,(\mathbf{u},-\mathbf{v})]^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\mathbf{u}\cdot\mathbf{v}$$



Since $[\text{dist}\,(\mathbf{u},-\mathbf{v})]^2 = [\text{dist}\,(\mathbf{u},\mathbf{v})]^2$, $\quad \mathbf{u}\cdot\mathbf{v} =$ _____.

**Orthogonal**

Two vectors $\mathbf{u}$ and $\mathbf{v}$ are said to be **orthogonal** (to each other) if $\mathbf{u}\cdot\mathbf{v} = \mathbf{0}$.

# The Pythagorean Theorem

**Theorem (2 The Pythagorean Theorem)**

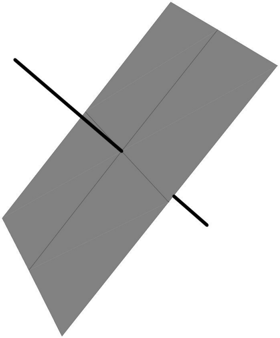
*Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if*

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$$

# Orthogonal Complements

## Orthogonal Complements

If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace $W$ of $\mathbf{R}^n$, then $\mathbf{z}$ is said to be **orthogonal to** $W$. The set of vectors $\mathbf{z}$ that are orthogonal to $W$ is called the **orthogonal complement** of $W$ and is denoted by $W^\perp$ (read as "W perp").



# Row, Null and Columns Spaces

## Theorem (3)

*Let $A$ be an $m \times n$ matrix. Then the orthogonal complement of the row space of $A$ is the nullspace of $A$, and the orthogonal complement of the column space of $A$ is the nullspace of $A^T$:*

$$(Row\ A)^{\perp} = Nul\ A, \qquad (Col\ A)^{\perp} = Nul\ A^T.$$

**Why? (See complete proof in the text)** Note that

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1 \cdot \mathbf{x} \\ \mathbf{r}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_m \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and so $\mathbf{x}$ is orthogonal to the row $A$ since $\mathbf{x}$ is orthogonal to $\mathbf{r}_1, \ldots, \mathbf{r}_m$.

# Row, Null and Columns Spaces: Example

## Example

Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$.

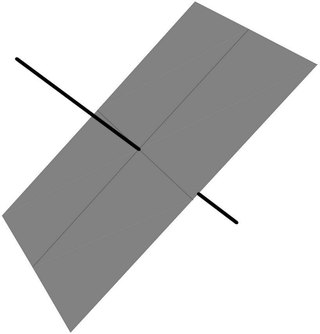
Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and Nul $A$ is a plane in $\mathbf{R}^3$.

Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and Row $A$ is a line in $\mathbf{R}^3$.

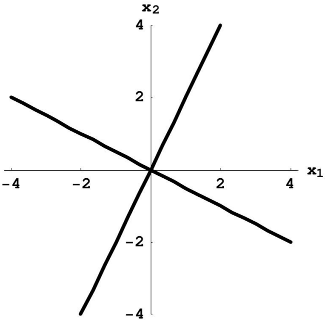
Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and Col $A$ is a line in $\mathbf{R}^2$.

Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and Nul $A^T$ is a line in $\mathbf{R}^2$.

# Row, Null and Columns Spaces: Example (cont.)



Subspaces Nul $A$ and Row $A$




Subspaces Nul $A^T$ and Col $A$