Math 2331 – Linear Algebra

6.2 Orthogonal Sets

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6.2 Orthogonal Sets

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Orthogonal Sets

Orthogonal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthogonal set** if $\mathbf{u}_i \cdot \mathbf{u}_j = 0$ whenever $i \neq j$.

Example

Is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ an orthogonal set?

Solution: Label the vectors $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ respectively. Then

$\mathbf{u}_1 \cdot \mathbf{u}_2 =$

$\mathbf{u}_1 \cdot \mathbf{u}_3 =$

$\mathbf{u}_2 \cdot \mathbf{u}_3 =$

Therefore, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.

Orthogonal Sets: Theorem

Theorem (4)

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbf{R}^n$ and $W =$span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$. Then S is a linearly independent set and is therefore a basis for W.

Partial Proof: Suppose

$$\begin{aligned}
c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p &= \mathbf{0} \\
(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p) \cdot \qquad &= \mathbf{0} \cdot \\
(c_1\mathbf{u}_1) \cdot \mathbf{u}_1 + (c_2\mathbf{u}_2) \cdot \mathbf{u}_1 + \cdots + (c_p\mathbf{u}_p) \cdot \mathbf{u}_1 &= \mathbf{0} \\
c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) + c_2 (\mathbf{u}_2 \cdot \mathbf{u}_1) + \cdots + c_p (\mathbf{u}_p \cdot \mathbf{u}_1) &= \mathbf{0} \\
c_1 (\mathbf{u}_1 \cdot \mathbf{u}_1) &= \mathbf{0}
\end{aligned}$$

Since $\mathbf{u}_1 \neq \mathbf{0}$, $\mathbf{u}_1 \cdot \mathbf{u}_1 > 0$ which means $c_1 =$ ____.
In a similar manner, $c_2, \ldots, c_p$ can be shown to by all 0. So S is a linearly independent set.∎

Orthogonal Basis

Orthogonal Basis: Example

An **orthogonal basis** for a subspace W of $\mathbf{R}^n$ is a basis for W that is also an orthogonal set.

Example

Suppose $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthogonal basis for a subspace W of $\mathbf{R}^n$ and suppose $\mathbf{y}$ is in W. Find $c_1,\ldots,c_p$ so that

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p.$$

Solution:

$$\begin{aligned}
\mathbf{y}\cdot \quad &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot \\
\mathbf{y}\cdot\mathbf{u}_1 &= (c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p)\cdot\mathbf{u}_1 \\
\mathbf{y}\cdot\mathbf{u}_1 &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) + c_2(\mathbf{u}_2\cdot\mathbf{u}_1) + \cdots + c_p(\mathbf{u}_p\cdot\mathbf{u}_1) \\
\mathbf{y}\cdot\mathbf{u}_1 &= c_1(\mathbf{u}_1\cdot\mathbf{u}_1) \quad \Longrightarrow \quad c_1 = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}
\end{aligned}$$

Similarly, $c_2 = \qquad , \ldots, \qquad c_p =$

Orthogonal Basis: Theorem

Theorem (5)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ *be an orthogonal basis for a subspace* W *of* $\mathbf{R}^n$. *Then each* $\mathbf{y}$ *in* W *has a unique representation as a linear combination of* $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p$. *In fact, if*

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p$$

then

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j} \qquad (j = 1, \ldots, p)$$

Orthogonal Basis: Example

Example

Express $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Solution:

$$\frac{\mathbf{y} \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} = \qquad \frac{\mathbf{y} \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2} = \qquad \frac{\mathbf{y} \cdot \mathbf{u}_3}{\mathbf{u}_3 \cdot \mathbf{u}_3} =$$

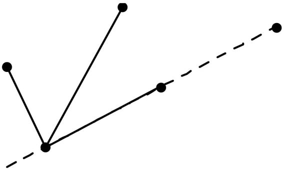
Hence

$$\mathbf{y} = \_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\_\mathbf{u}_2 + \_\_\_\_\_\_\mathbf{u}_3$$

Orthogonal Projections

For a nonzero vector $\mathbf{u}$ in $\mathbf{R}^n$, suppose we want to write $\mathbf{y}$ in $\mathbf{R}^n$ as the the following

$$\mathbf{y} = (\text{multiple of } \mathbf{u}) + (\text{multiple a vector } \perp \text{ to } \mathbf{u})$$



$$(\mathbf{y} - \alpha\mathbf{u}) \cdot \mathbf{u} = 0 \quad \Longrightarrow \quad \mathbf{y} \cdot \mathbf{u} - \alpha(\mathbf{u} \cdot \mathbf{u}) = 0 \quad \Longrightarrow \quad \alpha =$$

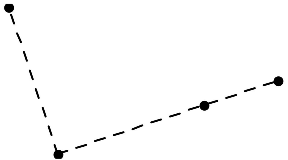
$$\widehat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \qquad (\textbf{orthogonal projection of y onto u})$$

$$\mathbf{z} = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} \qquad (\textbf{component of y orthogonal to u})$$

Orthogonal Projections: Example

Example

Let $\mathbf{y} = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Compute the distance from $\mathbf{y}$ to the line through $\mathbf{0}$ and $\mathbf{u}$.



Solution:

$$\widehat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} =$$

Distance from $\mathbf{y}$ to the line through $\mathbf{0}$ and $\mathbf{u}$ = distance from $\widehat{\mathbf{y}}$ to $\mathbf{y}$

$= \|\widehat{\mathbf{y}} - \mathbf{y}\| =$

Orthonormal Sets

Orthonormal Sets

A set of vectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ in $\mathbf{R}^n$ is called an **orthonormal set** if it is an orthogonal set of unit vectors.

Orthonormal Basis

If W =span$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$, then $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for W.

Recall that $\mathbf{v}$ is a unit vector if $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathbf{v}^T\mathbf{v}} = 1$.

Orthonormal Matrix: Example

Example

Suppose $U = [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3]$ where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set.

$$U^T U = \begin{bmatrix} \mathbf{u}_1^T \\ \mathbf{u}_2^T \\ \mathbf{u}_3^T \end{bmatrix} [\mathbf{u}_1 \ \ \mathbf{u}_2 \ \ \mathbf{u}_3] = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$$

Orthogonal Matrix

It can be shown that

$$UU^T = I.$$

So

$$U^{-1} = U^T$$

(such a matrix is called an **orthogonal matrix**).

Orthonormal Matrix: Theorems

Theorem (6)

An $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I$.

Theorem (7)

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbf{R}^n$. Then

a. $\|U\mathbf{x}\| = \|\mathbf{x}\|$

b. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$

c. $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ *if and only if* $\mathbf{x} \cdot \mathbf{y} = 0$.

Proof of part b: $(U\mathbf{x}) \cdot (U\mathbf{y}) =$