Name and ID: ______________________________

50 points 1. Find the solution of the following initial-value problems

a. $y' = ty^2$
with $y(0) = 2$.

b. $y' + 2ty = 2t^3$
with $y(0) = 1$.

c. $y'' + 3y' + 2y = 3e^{-4t}$
with $y(0) = 1$ and $y'(0) = 0$.

d. $x' = 2x + 4y + 4z, \quad y' = x + 2y + 3z, \quad z' = -3x - 4y - 5z$
with $x(0) = 1$, $y(0) = -1$ and $z(0) = 0$.

e. $y''' - y'' + y' - y = 0$
with $y(0) = 2$, $y'(0) = 1$ and $y''(0) = 0$.

15 points 2. An undamped spring-mass system with external driving force is modeled with

$$x'' + 4x = 4\cos 2t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).

20 points 3. Consider the initial value problem

$$x' = -x + t, \quad x(0) = \frac{1}{2}. \tag{1}$$

Carry out *one step* calculation of the Euler and RK2 methods with step size $h = \frac{1}{2}$ to approximate the value of $x(\frac{1}{2})$ and compute the error of your numerical solution (Use the fact that $e^{-\frac{1}{2}} \approx \frac{3}{5}$).

15 points 4. Classify the equilibrium point of the system

$$x' = -4x + 10y$$
$$y' = -2x + 4y$$

Sketch the phase portrait by hand.

20 points 5. (**BONUS PROBLEM**) Cindy and Richard would like to buy a home. They've examined their budget and determined that they can afford monthly payments of \$1,000. If the annual interest is 3%, and the term of the loan is 30 years, what amount can they afford to borrow? (Use the fact that $e^{-0.9} \approx 0.4$).

You can use the following results to facilitate your calculation

The eigen-pairs of $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix}$ are

$$\lambda_1 = -1, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i$$
$$v_1 = (0, -1, 1)^T, \quad v_2 = (-2, -1 - i, 2)^T, \quad v_3 = (-2, -1 + i, 2)^T.$$

Name and ID: ____________________

Problem 1.a

Name and ID: ____________________

Problem 1.b

Name and ID: ____________________

Problem 1.c

Name and ID: ______________________________

Problem 1.d

Name and ID: ______________________________

Problem 1.e

Name and ID: ____________________

Problem 2.

Name and ID: ____________________

Problem 3.

Name and ID: ______________________

Problem 4.

Name and ID: ____________________

Problem 5. (BONUS PROBLEM)

When you finish this exam, you should go back and reexamine your work for any errors that you may have made.