50 points 1. Find the solution of the following initial-value problems

a. $y' = ty^2, \quad y(0) = 2.$

Solution (S.O.V)

$$\frac{dy}{dt} = ty^2 \quad \Rightarrow \quad (1/y^2)dy = tdt \quad \Rightarrow \quad \int (1/y^2)dy = \int tdt$$
$$\Rightarrow \quad -1/y = t^2/2 + c \quad \Rightarrow \quad y(t) = -\frac{2}{t^2+k}$$

$$y(0) = 2 = -\frac{2}{0+k} \quad \Rightarrow \quad k = -1,$$
$$\Rightarrow \quad y(t) = \frac{2}{1-t^2}$$

b. $y' + 2ty = 2t^3, \quad y(0) = 1.$

Solution (method of undetermined coefficients)
The homogeneous solution is

$$y_h(x) = C\exp\left(\int (-2t)dt\right) = Ce^{-t^2}.$$

A particular solution to the inhomogeneous equation with polynomial forcing term $2t^3$ has a trial form

$$y_p(t) = at^2 + bt + c, \qquad \text{with coefficients } a,\, b,\, c \text{ to be determined.}$$

Substitue y_b in ODE yields

$$\begin{aligned}
&y_p' + 2ty_p = 2t^3,\\
&\Rightarrow \quad (2at+b) + 2t(at^2+bt+c) = 2at^3 + 2bt^2 + 2(a+c)t + b \equiv 2t^3\\
&\Rightarrow \quad b = 0, \quad a + c = 0, \quad 2a = 2, \quad \Rightarrow \quad a = 1, \quad b = 0, \quad c = -1.\\
&\Rightarrow \quad y_p(t) = t^2 - 1
\end{aligned}$$

The general solution to the inhomogeneous equation is

$$y(t) = Ce^{-t^2} + t^2 - 1$$

Solution (Variation of Parameter)

$$y_h(x) = \exp\left(\int (-2t)dt\right) = e^{-t^2} \quad \Rightarrow \quad v(t) = \int 2t^3 e^{t^2} dt = e^{t^2}(t^2-1)$$
$$\Rightarrow \quad y_p(t) = t^2 - 1 \quad \Rightarrow \quad y(t) = Ce^{-t^2} + t^2 - 1$$

Applying I.C. gives

$$y(0) = 1 = C - 1 \quad \Rightarrow \quad C = 2$$
$$\Rightarrow \quad y(t) = 2e^{-t^2} + t^2 - 1$$

c. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0.$

Solution The homogeneous equation, its Characteristic Equation and roots

$$y'' + 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -2, \lambda_2 = -1$$

The homogeneous solution is

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-2t} + C_2 e^{-t}$$

The particular solution $y_p = Ae^{-4t}$ has derivatives $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$, which when subsitituted into the equation provides

$$y_p'' + 3y_p' + 2y_p = 3e^{-4t} \quad \Rightarrow \quad 16Ae^{-4t} + 3(-4Ae^{-4t}) + 2(Ae^{-4t}) = 3e^{-4t} \quad \Rightarrow \quad A = \frac{1}{2}$$

Thus, a particular solution is $y_p = \frac{1}{2}e^{-4t}$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{2}e^{-4t}$$

ICs: $y(0) = 1 = C_1 + C_2 + \frac{1}{2}$ and $y'(0) = 0 = -2C_1 - C_2 - 2$ imply

$$C_1 = -\frac{5}{2}, C_2 = 3, \quad \Rightarrow y(t) = -\frac{5}{2}e^{-2t} + 3e^{-t} + \frac{1}{2}e^{-4t}$$

d.

$$\begin{aligned} x' &= 2x + 4y + 4z \\ y' &= x + 2y + 3z \\ z' &= -3x - 4y - 5z \end{aligned}$$

with $x(0) = 1$, $y(0) = -1$ and $z(0) = 0$.

Solution In matrix form, the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The eigen-pairs of $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix}$ are

$$\begin{aligned} &\lambda_1 = -1, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i \\ &v_1 = (0, -1, 1)^T, \quad v_2 = (-2, -1 - i, 2)^T, \quad v_3 = (-2, -1 + i, 2)^T. \end{aligned}$$

Using Euler's formula

$$w(t) = e^{2it}\begin{pmatrix}-2\\-1-i\\2\end{pmatrix} = (\cos 2t + i\sin 2t)\left(\begin{pmatrix}-2\\-1\\2\end{pmatrix} + i\begin{pmatrix}0\\-1\\0\end{pmatrix}\right)$$
$$= \left(\cos 2t\begin{pmatrix}-2\\-1\\2\end{pmatrix} - \sin 2t\begin{pmatrix}0\\-1\\0\end{pmatrix}\right) + i\left(\cos 2t\begin{pmatrix}0\\-1\\0\end{pmatrix} + \sin 2t\begin{pmatrix}-2\\-1\\2\end{pmatrix}\right)$$

The real and imaginary parts of w are solutions and we can write the general solution

$$\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} = c_1e^{-t}\begin{pmatrix}0\\-1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\cos 2t\\-\cos 2t+\sin 2t\\2\cos 2t\end{pmatrix} + c_3\begin{pmatrix}-2\sin 2t\\-\cos 2t-\sin 2t\\2\sin 2t\end{pmatrix}$$

If $x(0) = 1$, $y(0) = -1$ and $z(0) = 0$, then

$$\begin{pmatrix}1\\-1\\0\end{pmatrix} = c_1\begin{pmatrix}0\\-1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\-1\\2\end{pmatrix} + c_3\begin{pmatrix}0\\-1\\0\end{pmatrix}$$

We find that $c_1 = 1$, $c_2 = -1/2$, and $c_3 = 1/2$. Hence the solution is

$$\begin{pmatrix}x(t)\\y(t)\\z(t)\end{pmatrix} = e^{-t}\begin{pmatrix}0\\-1\\1\end{pmatrix} - \frac{1}{2}\begin{pmatrix}-2\cos 2t\\-\cos 2t+\sin 2t\\2\cos 2t\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-2\sin 2t\\-\cos 2t-\sin 2t\\2\sin 2t\end{pmatrix}$$
$$= \begin{pmatrix}\cos 2t - \sin 2t\\-e^{-t}-\sin 2t\\e^{-t}-\cos 2t+\sin 2t\end{pmatrix}$$

e. $y''' - y'' + y' - y = 0$ with $y(0) = 2$, $y'(0) = 1$ and $y''(0) = 0$.

Solution The characteristic polynomial is

$$\lambda^3 - \lambda^2 + \lambda - 1 = (\lambda - 1)(\lambda^2 + 1).$$

Consequently, we have roots 1 and $\pm i$. Thus we have three linear independent solutions

$$y_1(t) = \cos t, \quad y_2(t) = \sin t, \quad y_3(t) = e^t.$$

The general solution is

$$y(t) = C_1\cos t + C_2\sin t + C_3e^t.$$

To apply I.C., we first differentiate $y(t)$:

$$y'(t) = -C_1\sin t + C_2\cos t + C_3e^t, \quad y''(t) = -C_1\cos t - C_2\sin t + C_3e^t.$$

Evaluating at $t = 0$, we get

$$\begin{aligned} 2 &= y(0) = C_1 + C_3, \\ 1 &= y'(0) = C_2 + C_3, \\ 0 &= y''(0) = -C_1 + C_3 \end{aligned}$$

Solving these equations, we find that

$$C_1 = 1, \quad C_2 = 0, \quad C_3 = 1.$$

Hence, the solution is

$$y(t) = \cos t + e^t.$$

15 points 2. An undamped spring-mass system with external driving force is modeled with

$$x'' + 4x = 4\cos 2t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).

Solution (a) The general solution of the homogeneous equation is

$$x_h(t) = C_1 \cos 2t + C_2 \sin 2t$$

A particular solution is

$$x_p(t) = t \sin 2t$$

So the general solution of the inhomogeneous equation has the form

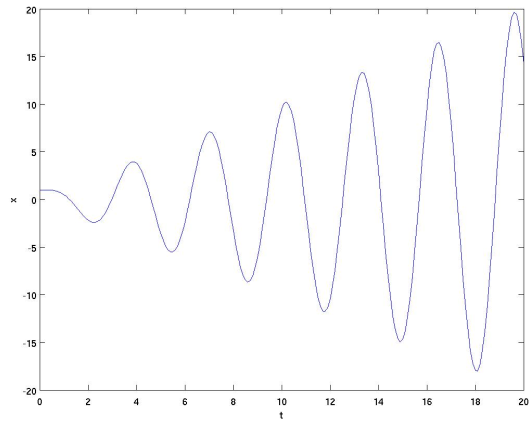
$$x(t) = x_h(t) + x_p(t) = C_1 \cos 2t + C_2 \sin 2t + t \sin 2t$$

Apply I.C.s yields $1 = x(0) = C_1$ and $0 = x'(0) = 2C_2$. So

$$x(t) = \cos 2t + t \sin 2t$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.

(b)



15 points

3. Consider the initial value problem

$$x' = -x + t, \quad x(0) = \frac{1}{2}. \tag{1}$$

Carry out *one step* calculation of the Euler and RK2 methods with step size $h = \frac{1}{2}$ to approximate the value of $x(\frac{1}{2})$ and compute the error of your numerical solution (Use the fact that $e^{-\frac{1}{2}} \approx \frac{3}{5}$).

Solution We have $t_0 = 0$, $x_0 = \frac{1}{2}$, $h = \frac{1}{2}$, and $f(t, y) = t - x$.

a. The first step of Euler's method is completed as follows

$$x_1 = x_0 + hf(t_0, x_0) = \frac{1}{2} + \frac{1}{2} * (0 - \frac{1}{2}) = \frac{1}{4}$$
$$t_1 = t_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$$

b. The first step of RK2 method follows. First we compute the slopes

$$s_1 = f(t_0, x_0) = f(0, \frac{1}{2}) = 0 - \frac{1}{2} = -\frac{1}{2}$$
$$s_2 = f(t_0 + h, x_0 + hs_1) = f(\frac{1}{2}, \frac{1}{4}) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

Youn can now update x and t

$$x_1 = x_0 + h\frac{1}{2}(s_1 + s_2) = \frac{1}{2} + \frac{1}{2} * \frac{1}{2} * (-\frac{1}{2} + \frac{1}{4}) = \frac{7}{16}$$
$$t_1 = t_0 + h = 0 + \frac{1}{2} = \frac{1}{2}$$

c. The equation is linear and inhomogeneous. We have

$$x_h(t) = ce^{-t}, \quad x_p(t) = t - 1$$

The general solution is

$$x(t) = x_h(t) + x_p(t) = ce^{-t} + t - 1.$$

Applying I.C., i.e., $\frac{1}{2} = x(0) = c - 1$, provides $c = \frac{3}{2}$. We find the solution

$$x(t) = \frac{3}{2}e^{-t} + t - 1$$

We can compute the true value:

$$x(\frac{1}{2}) = \frac{3}{2}e^{-\frac{1}{2}} + \frac{1}{2} - 1 \approx \frac{3}{2} * \frac{3}{5} + \frac{1}{2} - 1 = \frac{4}{10} = \frac{2}{5}.$$

d. We can complete the following table

	time	approx.	"true value"	error
Euler	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$
RK2	$\frac{1}{2}$	$\frac{7}{16}$	$\frac{2}{5}$	$\frac{3}{80}$

10 points 4. Classify the equilibrium point of the system

$$\begin{aligned} x' &= -4x + 10y \\ y' &= -2x + 4y \end{aligned}$$

Sketch the phase portrait by hand.

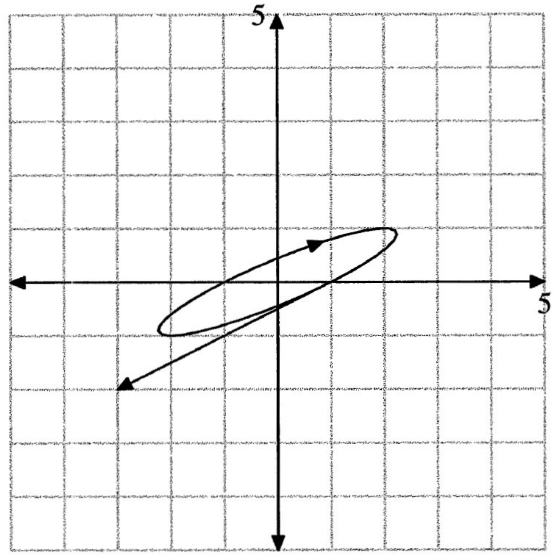
Solution The coefficient matrix of the system is

$$A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$$

with the trace $T = 0$ and the deternimant $D = 4$. Then the equilibrium point at the origin is a center. At $(1, 0)$,

$$\begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

so the rotation is clockwise. A hand sketch follows



20 points

5. (**BONUS PROBLEM**) Cindy and Richard would like to buy a home. They've examined their budget and determined that they can afford monthly payments of \$1,000. If the annual interest is 3%, and the term of the loan is 30 years, what amount can they afford to borrow? (Use the fact that $e^{-0.9} \approx 0.4$).

Solution Let $P(t)$ be the loan balance after t years, $r = 0.03$ (3%) be the annual interest rate, P_0 the amount of the loan, w be the annual payment. Then we have

$$\frac{dP}{dt} = rP - w,\ P(0) = P_0 \quad \text{(S.O.V)} \ \Rightarrow \quad P(t) = e^{rt}\left(P_0 - w/r\right) + w/r$$

A monthly payments of \$1,000 makes \$12,000 per year, so $w = 12000$. Furthermore, the term of the loan is $t^* =$30 years, so $P(t^*) = 0$, we have

$$0 = e^{rt^*}\left(P_0 - w/r\right) + w/r \quad \Rightarrow \quad P_0 = (w/r)\left(1 - e^{-rt^*}\right)$$

The amount of the loan they afford to borrow is

$$P_0 = (12000/0.03)\left(1 - e^{-0.03\times 30}\right) = 400,000*(1-e^{-0.9}) \approx 400,000*(1-0.4) = \ \$240,000$$

You can use some of the following results to facilitate your calculation

The eigen-pairs of $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 2 & 3 \\ -3 & -4 & -5 \end{pmatrix}$ are

$$\lambda_1 = -1, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i$$
$$v_1 = (0, -1, 1)^T, \quad v_2 = (-2, -1-i, 2)^T, \quad v_3 = (-2, -1+i, 2)^T.$$