ODE

Sample Midterm 1 Math 3331 (Summer 2014) June 16, 2014

$\boxed{\text{50 points}}$ 1. Find the solution of the following initial-value problem

1.
$$\frac{dy}{dt} = ty^2, \quad y(0) = 1.$$

Solution (S.O.V)

$$\frac{dy}{dt} = ty^2 \quad \Rightarrow \quad (1/y^2)dy = tdt \quad \Rightarrow \quad \int (1/y^2)dy = \int tdt$$
$$\Rightarrow \quad -1/y = t^2/2 + c \quad \Rightarrow \quad y(t) = -\frac{2}{t^2+k}$$

$$y(0) = 1 = -\frac{2}{0+k} \quad \Rightarrow \quad k = -2,$$
$$\Rightarrow \quad y(t) = \frac{2}{2-t^2}$$

2.
$$\frac{dy}{dt} = ry + a, \quad y(0) = y_0. \quad (r,\ a,\ y_0 \text{ parameters})$$

Solution (S.O.V)

$$\frac{dy}{dt} = ry + a \quad \Rightarrow \quad [1/(ry+a)]dy = dt \quad \Rightarrow \quad \int [1/(ry+a)]dy = \int dt$$
$$\Rightarrow \quad (\ln|ry+a|)/r = t + c \quad \Rightarrow \quad y(t) = ke^{rt} - a/r$$

$$y(0) = y_o = ke^0 - a/r \quad \Rightarrow \quad k = y_0 + a/r,$$
$$\Rightarrow \quad y(t) = (y_0 + a/r)e^{rt} - a/r$$

3.
$$\frac{dy}{dt} = \frac{y}{t}, \quad y(1) = -2.$$

Solution (S.O.V)

$$\frac{dy}{dt} = \frac{y}{t} \quad \Rightarrow \quad (1/y)dy = (1/t)dt \quad \Rightarrow \quad \int (1/y)dy = \int (1/t)dt$$
$$\Rightarrow \quad \ln|y| = \ln|t| + c \quad \Rightarrow \quad y(t) = kt$$

$$y(1) = -2 = k * 1 \quad \Rightarrow \quad k = -2,$$
$$\Rightarrow \quad y(t) = -2t$$

4.
$$\frac{dy}{dt} = \frac{\sin t}{y}, \quad y(\frac{\pi}{2}) = 1.$$

Solution (S.O.V)

$$\frac{dy}{dt} = \frac{\sin t}{y} \quad \Rightarrow \quad ydy = \sin t dt \quad \Rightarrow \quad \int ydy = \int \sin t dt$$
$$\Rightarrow \quad y^2/2 = -\cos t + c \quad \Rightarrow \quad y(t) = \pm\sqrt{k - 2\ cost}$$

$$y(\pi/2) = 1 = \sqrt{k - 2\cos(\pi/2)} \quad \Rightarrow \quad k = 1,$$
$$\Rightarrow \quad y(t) = \sqrt{1 - 2\cos t}$$

5.
$$\frac{dy}{dt} = 1 + y^2, \quad y(0) = 1.$$

Solution (S.O.V)

$$\frac{dy}{dt} = 1 + y^2 \quad \Rightarrow \quad [1/(1+y^2)]dy = dt \quad \Rightarrow \quad \int [1/(1+y^2)]dy = \int dt$$
$$\Rightarrow \quad arctany = t + c \quad \Rightarrow \quad y(t) = \tan(t + c) + n\pi \ (n \text{ integer})$$

$$y(0) = 1 = \tan(0 + c) + n\pi \quad \Rightarrow \quad c = \pi/4, n = 0,$$
$$\Rightarrow \quad y(t) = \tan(t + \pi/4)$$

6.
$$\frac{dy}{dt} = \frac{t}{y}, \quad y(0) = 1.$$

Solution (S.O.V)

$$\frac{dy}{dt} = \frac{t}{y} \quad \Rightarrow \quad ydy = tdt \quad \Rightarrow \quad \int ydy = \int tdt$$
$$\Rightarrow \quad y^2/2 = t^2/2 + c \quad \Rightarrow \quad y(t) = \pm\sqrt{t^2 + k}$$

$$y(0) = 1 = \sqrt{0^2 + k} \quad \Rightarrow \quad k = 1$$
$$\Rightarrow \quad y(t) = \sqrt{t^2 + 1}$$

7.
$$\frac{dy}{dt} + \cos ty = \frac{1}{2}\sin 2t, \quad y(0) = 1.$$

Solution (Variation of Parameter)

$$y_h(x) = \exp\left(\int(-\cos t)dt\right) = e^{-\sin t} \quad \Rightarrow \quad v(t) = \int \frac{1}{2}\sin 2te^{\sin t}dt = e^{\sin t}(\sin t - 1)$$
$$\Rightarrow \quad y_p(t) = \sin t - 1 \quad \Rightarrow \quad y(t) = ke^{-\sin t} + \sin t - 1$$

$$y(0) = 1 = k - 1 \quad \Rightarrow \quad k = 2$$
$$\Rightarrow \quad y(t) = 2e^{-\sin t} + \sin t - 1$$

8.
$$\frac{dy}{dt} + 2ty = 2t^3, \quad y(0) = -1.$$

Solution (Variation of Parameter)

$$y_h(x) = \exp\left(\int (-2t)dt\right) = e^{-t^2} \quad \Rightarrow \quad v(t) = \int 2t^3 e^{t^2} dt = e^{t^2}(t^2 - 1)$$
$$\Rightarrow \quad y_p(t) = t^2 - 1 \quad \Rightarrow \quad y(t) = ke^{-t^2} + t^2 - 1$$

$$y(0) = -1 = k - 1 \quad \Rightarrow \quad k = 0$$
$$\Rightarrow \quad y(t) = t^2 - 1$$

9.
$$\frac{dy}{dt} + \frac{y}{1+t} = 2, \quad y(0) = 1.$$

Solution (Integrating Factor)

$$u(t) = \exp\left(\int [1/(1+t)]dt\right) = 1 + t$$

$$\text{multiply ODE by } u \quad \Rightarrow \quad [(1+t)y]' = 2(1+t) \quad \Rightarrow \quad (1+t)y = \int 2(1+t)dt$$
$$\Rightarrow \quad y(t) = (c + 2t + t^2)/(1+t)$$

$$y(0) = 1 = c \quad \Rightarrow \quad c = 1$$
$$\Rightarrow \quad y(t) = (1 + 2t + t^2)/(1+t) \quad \Rightarrow \quad y(t) = 1 + t$$

10.
$$\frac{dy}{dt} - \frac{n}{t}y = e^t t^n, \quad y(1) = 1.$$

Solution (Integrating Factor)

$$u(t) = \exp\left(\int (-n/t)dt\right) = t^{-n}$$

$$\text{multiply ODE by } u \quad \Rightarrow \quad [t^{-n}y]' = e^t \quad \Rightarrow \quad t^{-n}y = \int e^t dt$$
$$\Rightarrow \quad y(t) = t^n(c + e^t)$$

$$y(1) = 1 = c + e \quad \Rightarrow \quad c = 1 - e$$
$$\Rightarrow \quad y(t) = t^n(1 - e + e^t)$$

$\boxed{\text{50 points}}$ 2. Find the general solution of the following differential equations

- (Radiactive decay)
$$\frac{dN}{dt} = -\lambda N$$
where λ is a constant.
Solution
$$N(t) = N_0 e^{-\lambda t}$$
where $N_0 := N(0)$
- (Newton's law of cooling)
$$\frac{dT}{dt} = k(A - T)$$
where k and A are constants.
Solution
$$T(t) = A + e^{-kt}(T_0 - A)$$
where $T_0 := T(0)$.
- (Motion with air resistance)
$$\frac{dv}{dt} = -g - \frac{k}{m}v$$
where g, k and m are constants.
Solution
$$v(t) = (v_0 + gm/k)e^{-kt/m} - gm/k$$
where $v_0 := v(0)$.
- (Personal finance)
$$\frac{dP}{dt} = rP + Q(t),$$
where $Q(t)$ is a given function of t.
Solution
$$P(t) = P_0 e^{rt} + e^{rt}\int_0^t e^{-r\tau}Q(\tau)d\tau$$
- (Logistic equation)
$$\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P,$$
where k and N are constants.
Solution
$$P(t) = \frac{NP_0}{P_0 + (N - P_0)e^{-kt}}$$
where $P_0 := P(0)$.

20 points 3. Suppose a population is growing according to the logistic equation

$$\frac{dP}{dt} = f(P) \quad \text{where } f(P) = r_0\left(1 - \frac{P}{K}\right)P$$

with r_0 being the natural reproductive rate and K being the carrying capacity. Perform each of the following tasks without the aid of technology.

(i) Sketch a graph of $f(P)$

(ii) Use the graph of f to develop a phase line for the autonomous equation. Classify each equilibrium point as either unstable or asymptotically stable.

(iii) Sketch the equilibrium solutions in the t-P plans. These equilibrium solutions divide the t-P plane into regions. Sketch at least one solution trajectory in each of these regions.

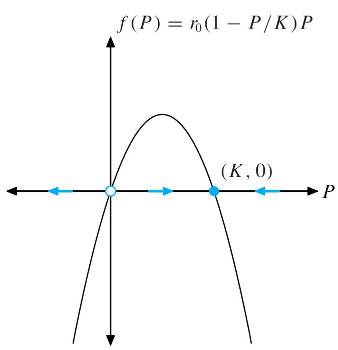
Solution (i) The graph of $f(P)$ and (ii) the associated phase line are shown in Figure (left). (ii) The equilibrium points are where $f(P) = 0$, or at $P_1 = 0$ and $P_2 = K$. Note that

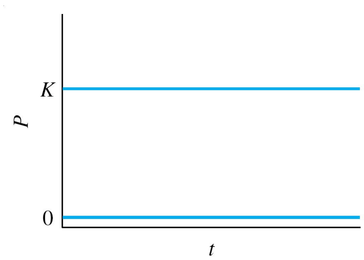
$$f(P) > 0 \text{ if } 0 < P < K \quad \text{and} \quad f(P) < 0 \text{ if } P < 0 \text{ or } P > K$$

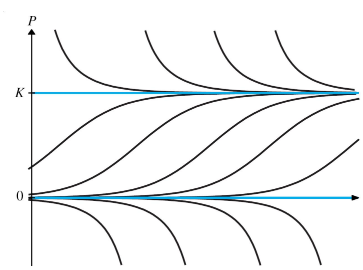
Hence $P_1 = 0$ is un unstable equilibrium point and $P_2 = K$ is stable. (iii) The equilibrium solutions are

$$P_1(t) = 0 \quad \text{and} \quad P_2(t) = K.$$

The graph of the equilibrium solutions are shown in Figure (center). The solution curves are sketched in Figure (right), where any solution with a positive population must stay positive and must tend to K as $t \mapsto \infty$.







20 points

4. A population is observed to obey the logistic equation with eventual population 20,000. The initial population is 1,000, and 8 hour later, the observed population is 1,200. Find the reproductive rate and the time required for the population to reach 75% of its carrying capacity.

Solution The logistic equation can be solved analytically by S.O.V,

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P(0) = P_0 \quad \text{(S.O.V)} \;\Rightarrow\; P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

Given carrying capacity K =20,000, the initial population P_0 =1,000, and at time t_1 =8hrs, the population $P_1 = P(t_1)$ =1,200, we can find the reproductive rate r by

$$r = \frac{1}{t_1}\ln\left(\frac{P_1(K - P_0)}{P_0(K - P_1)}\right) = \frac{1}{8}\ln\left(\frac{1200(20000 - 1000)}{1000(20000 - 1200)}\right) \approx 0.0241/\text{hr}$$

We can compute the time t^* required for the population to reach $P^* = P(t^*) = 0.75K$ =15,000 by

$$t^* = \frac{1}{r}\ln\left(\frac{P^*(K-P_0)}{P_0(K-P^*)}\right) = \frac{1}{0.0241}\ln\left(\frac{15000(20000-1000)}{1000(20000-15000)}\right) \approx 167.76\text{hrs}$$

20 points

5. On the day of his birth, Jason's grandmother pledges to make available \$50,000 on his eighteenth birthday for his college education. She negotiate an account paying 6.25% annual interest, compounded continuously, with no initial deposit, but agrees to deposit a fixed amount each year. What annual deposit should be made to reach her goal?

Solution Let $P(t)$ be the balance in the account t years after Jason's day of birth, $r = 0.0625$ (6.25%) be the annual interest rate, P_0 the initial deposit, d be the fixed annual deposit. Then we have

$$\frac{dP}{dt} = rP + d,\ P(0) = P_0 \quad \text{(S.O.V)} \ \Rightarrow \quad P(t) = e^{rt}\left(P_0 + d/r\right) - d/r$$

With $P_0 = 0$, i.e., no initial deposit, we have

$$P(t) = (d/r)\left(e^{rt} - 1\right)$$

The goal is to have $P^* = P(t^*)$ =50,000 at t^* =18yrs. The annual deposit d must be

$$d = \frac{P^* r}{e^{r^* t} - 1} = \frac{50000 \times 0.0625}{e^{0.0625\times 18} - 1} = \$\ 1{,}502.25$$

20 points

6. Don and Heidi would like to buy a home. They've examined their budget and determined that they can afford monthly payments of \$1,000. If the annual interest is 7.25%, and the term of the loan is 30 years, what amount can they afford to borrow?

Solution Let $P(t)$ be the loan balance after t years, $r = 0.0725$ (7.25%) be the annual interest rate, P_0 the amount of the loan, w be the annual payment. Then we have

$$\frac{dP}{dt} = rP - w,\ P(0) = P_0 \quad \text{(S.O.V)} \ \Rightarrow \quad P(t) = e^{rt}\left(P_0 - w/r\right) + w/r$$

A monthly payments of \$1,000 makes \$12,000 per year, so $w = 12000$. Furthermore, the term of the loan is t^* =30 years, so $P(t^*) = 0$, we have

$$0 = e^{rt^*}\left(P_0 - w/r\right) + w/r \quad \Rightarrow \quad P_0 = (w/r)\left(1 - 1/e^{rt^*}\right)$$

The amount of the loan they afford to borrow is

$$P_0 = (12000/0.0725)\left(1 - 1/e^{0.0725\times 30}\right) \approx \ \$146{,}713$$

20 points

7. Find the solution of the following initial-value problem

a. $y'' - 3y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 1.$

Solution DE, its Characteristic Equation and roots

$$y'' - 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 - 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = 2, \lambda_2 = 1$$

The general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{2t} + C_2 e^t \quad \Rightarrow y'(t) = 2C_1 e^{2t} + C_2 e^t$$

ICs: $y(0) = 2 = C_1 + C_2$ and $y'(0) = 1 = 2C_1 + C_2$ imply

$$C_1 = -1, C_2 = 3 \quad \Rightarrow y(t) = -e^{2t} + 3e^t$$

b. $y'' + 2y' + 2y = 0, \quad y(0) = 2, \quad y'(0) = 3.$

Solution DE, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_{1,2} = -1 \pm i$$

The general solution is

$$y(t) = e^{at}(C_1 \cos(bt) + C_2 \sin(bt)) = e^{-t}(C_1 \cos(t) + C_2 \sin(t))$$
$$\Rightarrow y'(t) = -e^{-t}(C_1 \cos(t) + C_2 \sin(t)) + e^{-t}(-C_1 \sin(t) + C_2 \cos(t))$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = 3 = -C_1 + C_2$ imply

$$C_1 = 2, C_2 = 5 \quad \Rightarrow y(t) = e^{-t}(2\cos(t) + 5\sin(t))$$

c. $y'' - 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = -1.$

Solution DE, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_{1,2} = 1$$

The general solution is

$$y(t) = (C_1 + C_2 t)e^{\lambda_1 t} = (C_1 + C_2 t)e^t$$
$$\Rightarrow y'(t) = (C_1 + C_2 t)e^t + C_2 e^t$$

ICs: $y(0) = 2 = C_1$ and $y'(0) = -1 = C_1 + C_2$ imply

$$C_1 = 2, C_2 = -3 \quad \Rightarrow y(t) = (2 - 3t)e^t.$$

20 points

8. A 0.1 kg mass is attached to a spring having a spring constant 3.6 kg/s. The system is allowed to come to rest. Then the mass is given a sharp tap, imparting an instantaneous downward velocity of 0.4 m/s. If there is no damping present, find the amplitude,

frequency, and phase of the resulting motion. Plot the solution.

Solution Subsitute m =0.1 kg and k =3.6 kg/s in $my''+ky=0$ to obtain $0.1y''+3.6y=0$, or

$$y'' + 36y = 0$$

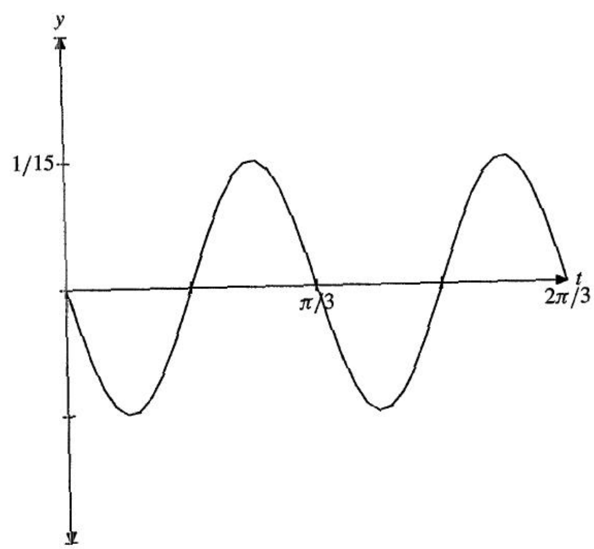
The general solution is

$$y(t) = C_1 \cos 6t + C_2 \sin 6t.$$

The initial displacement is zero, so $y(0) = 0 = C_1$. The initial velocity is -0.4 m/s, so $y'(0) = -0.4 = 6C_2$, leading to $C_2 = -1/15$. Thus the solution is

$$y(t) = -\frac{1}{15}\sin 6t = \frac{1}{15}\cos\left(6t + \pi/2\right)$$

which has amplitude 1/15, frequency 6 rad/s, and $-\pi/2$ phase.



20 points 9. Find the solution of the following initial-value problems

a. $y'' + 3y' + 2y = 3e^{-4t}, \quad y(0) = 1, \quad y'(0) = 0.$

Solution The homogeneous equation, its Characteristic Equation and roots

$$y'' + 3y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -2, \lambda_2 = -1$$

The homogeneous solution is

$$y_h(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-2t} + C_2 e^{-t}$$

The particular solution $y_p = Ae^{-4t}$ has derivatives $y_p' = -4Ae^{-4t}$ and $y_p'' = 16Ae^{-4t}$, which when subsitituted into the equation provides

$$y_p'' + 3y_p' + 2y_p = 3e^{-4t} \quad \Rightarrow \quad 16Ae^{-4t} + 3(-4Ae^{-4t}) + 2(Ae^{-4t}) = 3e^{-4t} \quad \Rightarrow \quad A = \frac{1}{2}$$

Thus, a particular solution is $y_p = \frac{1}{2}e^{-4t}$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = C_1 e^{-2t} + C_2 e^{-t} + \frac{1}{2}e^{-4t}$$

ICs: $y(0) = 1 = C_1 + C_2 + \frac{1}{2}$ and $y'(0) = 0 = -C_1 - 2C_2 - 2$ imply

$$C_1 = 3, C_2 = -\frac{5}{2} \quad \Rightarrow y(t) = 3e^{-2t} - \frac{5}{2}e^{-t} + \frac{1}{2}e^{-4t}$$

b. $y'' + 2y' + 2y = 2\cos 2t, \quad y(0) = -2, \quad y'(0) = 0.$

Solution The homogeneous equation, its Characteristic Equation and roots

$$y'' + 2y' + 2y = 0 \quad \Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0 \quad \Rightarrow \quad \lambda_1 = -1 + i, \lambda_2 = -1 - i$$

The homogeneous solution is

$$y_h(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) = e^{-t}(C_1 \cos t + C_2 \sin t)$$

The particular solution $y_p = A_1 \cos 2t + A_2 \sin 2t$ has derivatives

$$y_p' = -2A_1 \sin 2t + 2A_2 \cos 2t, \quad y_p'' = -4A_1 \cos 2t - 4A_2 \sin 2t,$$

which when subsitituted into the equation provides

$$\begin{aligned} y_p'' + 2y_p' + 2y_p = 2\cos 2t \quad &\Rightarrow \quad (-4A_1 \cos 2t - 4A_2 \sin 2t) \\ &+ 2(-2A_1 \sin 2t + 2A_2 \cos 2t) + 2(A_1 \cos 2t + A_2 \sin 2t) = 2\cos 2t \\ \Rightarrow \quad -A_1 + 2A_2 = 1, \quad 2A_1 + A_2 = 0 \quad &\Rightarrow \quad A_1 = -\frac{1}{5}, \quad A_2 = \frac{2}{5} \end{aligned}$$

Thus, a particular solution is $y_p = -\frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = e^{-t}(C_1 \cos t + C_2 \sin t) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

ICs: $y(0) = -2 = C_1 - \frac{1}{5}$ and $y'(0) = 0 = C_2 - C_1 + \frac{4}{5}$ imply

$$C_1 = -\frac{9}{5}, C_2 = -\frac{13}{5} \quad \Rightarrow y(t) = e^{-t}\left(-\frac{9}{5}\cos t - \frac{13}{5}\sin t\right) - \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t$$

c. $y'' - 2y' + y = t^3, \quad y(0) = 1, \quad y'(0) = 0.$

Solution The homogeneous equation, its Characteristic Equation and roots

$$y'' - 2y' + y = 0 \quad \Rightarrow \quad \lambda^2 - 2\lambda + 1 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = 1$$

The homogeneous solution is

$$y_h(t) = (C_1 + C_2 t)e^{\lambda t} = (C_1 + C_2 t)e^{t}$$

The particular solution $y_p = at^3 + bt^2 + ct + d$ has derivatives

$$y_p' = 3at^2 + 2bt + c, \quad y_p'' = 6at + 2b,$$

which when subsitituted into the equation provides

$$\begin{aligned} &y_p'' - 2y_p' + y_p = t^3 \quad \Rightarrow \quad (6at + 2b) - 2(3at^2 + 2bt + c) + (at^3 + bt^2 + ct + d) = t^3 \\ &\quad \Rightarrow \quad a = 1, \quad -6a + b = 0, \quad 6a - 4b + c = 0, \quad 2b - 2c + d = 0 \\ &\quad \Rightarrow \quad a = 1, b = 6, c = 18, d = 24 \end{aligned}$$

Thus, a particular solution is $y_p = t^3 + 6t^2 + 18t + 24$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = (C_1 + C_2 t)e^{t} + t^3 + 6t^2 + 18t + 24$$

ICs: $y(0) = 1 = C_1 + 24$ and $y'(0) = 0 = C_2 + C_1 + 18$ imply

$$C_1 = -23, C_2 = 5 \quad \Rightarrow \quad y(t) = (-23 + 5t)e^{t} + t^3 + 6t^2 + 18t + 24$$

d. $y'' + 4y' + 4y = 2e^{-2t}, \quad y(0) = 0, \quad y'(0) = 1.$

Solution The homogeneous equation, its Characteristic Equation and roots

$$y'' + 4y' + 4y = 0 \quad \Rightarrow \quad \lambda^2 + 4\lambda + 4 = 0 \quad \Rightarrow \quad \lambda_1 = \lambda_2 = -2$$

The homogeneous solution is

$$y_h(t) = (C_1 + C_2 t)e^{\lambda t} = (C_1 + C_2 t)e^{-2t}$$

The forcing term $f(t) = 2e^{-2t}$ and Ate^{-2t} are solutions of the homogeneous equations, so multiply by another factor of t and try the particular solution $y_p = At^2e^{-2t}$. The derivatives of y_p are

$$y_p' = 2Ae^{-2t}(t - t^2), \quad y_p'' = 2Ae^{-2t}(2t^2 - 4t + 1),$$

which when subsitituted into the equation provides

$$\begin{aligned} &y_p'' + 4y_p' + 4y_p = 2e^{-2t} \quad \Rightarrow \quad 2Ae^{-2t}(2t^2 - 4t + 1) + 4(2Ae^{-2t}(t - t^2)) + 4(At^2e^{-2t}) = 2e^{-2t} \\ &\quad \Rightarrow \quad [2(2t^2 - 4t + 1) + 8(t - t^2) + 4t^2]A = 2 \\ &\quad \Rightarrow \quad A = 1 \end{aligned}$$

Thus, a particular solution is $y_p = t^2e^{-2t}$. This leads to the general solution

$$y(t) = y_h(t) + y_p(t) = (C_1 + C_2 t)e^{-2t} + t^2e^{-2t}$$

ICs: $y(0) = 0 = C_1$ and $y'(0) = 1 = C_2 - 2C_1$ imply

$$C_1 = 0, C_2 = 1 \quad \Rightarrow \quad y(t) = te^{-2t} + t^2e^{-2t}$$

$\boxed{\text{20 points}}$ 10. Verify that $y_1(t) = t$ and $y_2(t) = t^{-3}$ are solutions to the homogeneous equation

$$t^2y'' + 3ty' - 3y = 0, \quad \text{for } t > 0.$$

Use the variation of parameters to find the general solution to

$$t^2y'' + 3ty' - 3y = \frac{1}{t}, \quad \text{for } t > 0.$$

Solution We start by rewriting the equation

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = \frac{1}{t^3}$$

First, check $y_1(t) = t$ is a solution

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (0) + \frac{3}{t}(1) - \frac{3}{t^2}(t) = 0$$

Check $y_2(t) = t^{-3}$ is a solution

$$y'' + \frac{3}{t}y' - \frac{3}{t^2}y = (12t^{-5}) + \frac{3}{t}(-3t^{-4}) - \frac{3}{t^2}(t^{-3}) = 0$$

Calculate the Wronskian

$$W(t, t^{-3}) = \begin{vmatrix} t & t^{-3} \\ 1 & -3t^{-4} \end{vmatrix} = -4t^{-3}$$

Next

$$\begin{aligned} v_1' &= \frac{-y_2 g(t)}{W} = \frac{-t^{-3}t^{-3}}{-4t^{-3}} = \frac{1}{4}t^{-3} \\ v_2' &= \frac{y_1 g(t)}{W} = \frac{tt^{-3}}{-4t^{-3}} = -\frac{1}{4}t \end{aligned}$$

Thus

$$v_1 = -\frac{1}{8}t^{-2}, \quad v_1 = -\frac{1}{8}t^2$$

Form

$$y_p = v_1y_1 + v_2y_2 = \left(-\frac{1}{8}t^{-2}\right)t + \left(-\frac{1}{8}t^2\right)t^{-3} = -\frac{1}{4t}$$

Thus, the general solution is

$$y(t) = y_h(t) + y_p(t) = C_1 t + \frac{C_2}{t^3} - \frac{1}{4t}$$

$\boxed{\text{20 points}}$ 11. An undamped spring-mass system with external driving force is modeled with

$$x'' + 25x = 4\cos 5t.$$

The parameters of this equation are "tuned" so that the frequency of the driving force equals the natural frequency of the undriven system. Suppose that the mass is displaced one positive unit and released from rest.

(a) Find the position of the mass as a function of time. What part of the solution guarantees that this solution resonates?

(b) Sketch the solution found in part (a).

Solution (a) As in the notes, the particular solution is

$$x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t = \frac{2}{5} t \sin 5t$$

The general solution of the homogeneous equation is

$$x_h(t) = C_1 \cos 5t + C_2 \sin 5t$$

So the solution has the form

$$x(t) = x_h(t) + x_p(t) = C_1 \cos 5t + C_2 \sin 5t + \frac{2}{5} t \sin 5t$$

Apply I.C.s yields $1 = x(0) = C_1$ and $0 = x'(0) = 5C_2$. So

$$x(t) = \cos 5t + \frac{2}{5} t \sin 5t$$

The particular solution $x_p(t)$ has a factor of t so its amplitude will grow, indicating a resonant solution.

(b)